STABILITY ANALYSIS OF MULTIRATE MODEL PREDICTIVE CONTROL

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Abstract: This paper proposes a multirate Model Predictive Control (MPC) algorithm for constrained systems where the control effort is adjusted several times during one sampling interval. Stability criterion for this new proposed algorithm is established. The test of the stability condition, the estimation of stability regions and the choice of the terminal weighting term are discussed.

Keywords: Model predictive control; stability; saturation; optimisation

1. INTRODUCTION

Model predictive control (MPC) is a powerful control strategy for engineering systems. It has two main features. One is that an optimisation solver is involved in feedback loop at each sampling instant. The other is that input/state constraints can be dealt with explicitly. These features enable MPC to fully use available control authority to achieve best possible performance under constraints. However, on the other hand, these features also make the analysis of behaviours of MPC much difficult. Earlier examples showing the possible instability of MPC algorithms trigger considerable research interest in the analysis of stability of MPC; for example see (Bitmead et al., 1990) and (Soeterboek, 1992). Due to the efforts over last two decades, stability analysis of MPC is now reaching a preliminarily mature stage (Mayne et al., 2000). Various stability results of different MPC schemes for different kinds of systems (linear/nonlinear, unconstrained/constrained, continuous-time/discrete-time) have been developed; for example see (De Nicolao et al., 1997), (Scokaert et al., 1999), (Mosca and Zhang, 1992), (Gyurkovics, 1998), (Magni and Sepulchre, 1997), (Chen and Allgöwer, 1998), (Chen et al., 2000). Almost all the stability results are established based on employing the predictive performance cost under the optimal control profile yielded by an on-line optimizer as an Lyapunov function. For state of the art of stability analysis of MPC, reader can refer to (Mayne et al., 2000).

This paper proposes a new multi-rate MPC algorithm for constrained discrete-time systems with the aim to increase stability region. It is allowed that the control effort can be adjusted several times during one sampling interval. It is expected that the extra freedom gained from this scheme will increase the stability region.

This paper is organised as follows: An multirate MPC scheme for constrained linear systems is described and necessary preliminaries are introduced in Section 2. Section 3 devotes to stability analysis of the MPC scheme. It is shown that stability of MPC for constrained linear systems is guaranteed if the terminal state arrives in a set that is defined by an inequality. How to test this stability condition, how to estimate the stability region and how to choose a terminal weighting term off-line according to stability requirements are addressed in Section 4. Finally this paper is ended with brief conclusion in Section 5. Due to the limitation of the space, the proof of Theorem 2 and 3 is omitted.

2. PRELIMINARIES

Consider a continuous time system
\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t) \\
x(0) &= x_0
\end{align*}
\]  

(1)

with control constraints

\[
\begin{align*}
u \in \mathcal{U} & \triangleq \{ u = [u_1, \ldots, u_m]^T : |u_i| \leq \bar{u}_i, \\
i &= 1, \ldots, m \}
\end{align*}
\]  

(2)

where \( x \in \mathbb{R}^n \) and \( u \in \mathbb{R}^m \) are the state and control vectors, respectively. Since on-line optimisation is required for MPC, in general this algorithm is implemented by digital computer. Suppose that the sampling interval is chosen as \( h \). Discretisation of the continuous time system (1) gives a sampled-data system D1. Now we consider to use a fast sampling rate, that is, the sampling interval is chosen as \( h/L \) where \( L \geq 1 \). The fast sampled data system is referred to as D2, given by

\[
\begin{align*}
x(k+1) &= Ax(k) + Bu(k) \\
x(0) &= x_0
\end{align*}
\]  

(3)

where the time instant \( k \) denotes \( k \times \frac{h}{L} \).

The predictive performance index is defined by

\[
J(k) = x(k + NL|k)^T P x(k + NL|k) + \\
+ \sum_{i=0}^{NL-1} (x(k+i|k)^T Q x(k+i|k) + \\
+ u(k+i|k)^T R u(k+i|k))
\]  

(4)

where \( NL = N \times L \) is the predictive length, and \( Q > 0 \) (or \( Q \geq 0 \) and \( [A, Q^{1/2}] \) is detectable), \( R > 0 \) and \( P > 0 \) are the state, control and terminal weighting matrices respectively. \( x(k+i|k) \) denotes the predicted state at time instant \( k+i \) based on the state measurement at time instant \( k \), i.e., \( x(k) \), and the control sequence \( u(k|k), \ldots, u(k+i-1|k) \). When an MPC algorithm is applied in control of the system (3), at time instant \( k \), the minimisation problem

\[
J(k)^* = \min_{u(k|k), \ldots, u(k+NL-1|k)} J(k)
\]  

(5)

subject to

\[
\begin{align*}
x(k+i+1|k) &= Ax(k+i|k) + Bu(k+i|k) \\
x(k|k) &= x(k)
\end{align*}
\]  

(6)

and

\[
u(k+i|k) \in \mathcal{U}, \quad 0 \leq i \leq N - 1.
\]  

(7)

is solved on-line using an optimisation solver, for example, quadratic programming (QP), and a control sequence \( u(k|k)^*, \ldots, u(k+NL-1|k)^* \) is yielded. The real input imposed on the plant during the period \( k \) to \( k + NL \) is given by

\[
\begin{align*}
u(k) &= u(k|k)^*, \quad u(k+1) = u(k+1|k)^*, \ldots, \\
u(k+NL-1) &= u(k+NL-1|k)^*.
\end{align*}
\]  

(8)

Then after executing the above control sequence, at the time instant \( k+L \), the state \( x(k+L) \) is measured and the above process repeats in MPC as time goes.

The main difference between the above algorithm with conventional MPC algorithms lies that a control sequence consisting of several piece-wise constants, instead of a constant is employed during the sampling interval. This provides more degree of freedom for optimisation and as a result, the stability region and performance could be increased. Obviously, a large computational burden is required since there are more variables to be optimised on-line. This paper first addresses stability of the above MPC scheme for constrained linear systems and then the developed results will be extended to general nonlinear systems with a general performance index.

Let the control effort in each sampling interval be given by

\[
\begin{align*}
u(k+i|k) &= K(k+i)x(k) \\
i &= 0, \ldots, NL - 1
\end{align*}
\]  

(9)

where \( K(k+i) \) is the control gain at time instant \( k+i \). Putting the control efforts from \( i = 0 \) to \( NL - 1 \) in a vector form yields

\[
\bar{U}_{NL}(k|k) = \bar{K}_{NL}(k)x(k)
\]  

(10)

where

\[
\begin{align*}
\bar{U}_{NL}(k|k) &= \begin{bmatrix} u(k|k) \\
\vdots \\
u(k + NL - 1|k) \end{bmatrix} \\
\bar{K}_{NL}(k) &= \begin{bmatrix} K(k) \\
\vdots \\
K(k + NL - 1) \end{bmatrix}
\end{align*}
\]  

(11)

and the subscript, \( NL \), denotes the number of sampling instants. Thus the state at time instant \( k+i \) driven by the control sequence \( u(k|k), \ldots, u(k + NL - 1|k) \) from the state \( x(k) \) can be predicted by

\[
\begin{align*}
x(k+i|k) &= A^i x(k) + A^{i-1} Bu(k|k) + \ldots + \\
&\quad + ABu(k+i-2|k) + Bu(k+i-1|k)
\end{align*}
\]  

(12)

Similarly, putting the predicted state from time instant \( k \) to \( k + NL \) in a vector format obtains

\[
\bar{X}(k) = \Phi_{NL} x(k) + \Gamma_{NL} \bar{U}_{NL}(k|k)
\]

\[
= \Phi_{NL} x(k) + \Gamma_{NL} \bar{K}_{NL}(k)x(k)
\]

\[
= (\Phi_{NL} + \Gamma_{NL} \bar{K}_{NL}(k))x(k)
\]  

(13)

(14)
For the sake of notational simplicity, let

$$\tilde{X}(k) = \begin{bmatrix} x(k|k) \\ x(k+1|k) \\ \vdots \\ x(k + NL|k) \end{bmatrix}, \quad \Phi_{NL} = \begin{bmatrix} I \\ A \\ \vdots \\ A^{NL} \end{bmatrix}$$

and

$$\Gamma_{NL} = \begin{bmatrix} 0 & 0 & \ldots & 0 \\ B & 0 & \ldots & 0 \\ AB & B & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A^{NL-1} B & A^{NL-2} B & \ldots & B \end{bmatrix}$$

Therefore using notations in (10-16), the performance index (4) can be re-written as

$$J(k) = X(k)^T \tilde{D}_{NL} X(k) + \tilde{U}_{NL}(k|k) \tilde{R}_{NL} \tilde{U}_{NL}(k|k)$$

$$= x(k)^T ((\Phi_{NL} + \Gamma_{NL} \tilde{K}_{NL}(k))^T \tilde{D}_{NL}$$

$$+ \tilde{K}_{NL}(k)^T \tilde{R}_{NL} \tilde{K}_{NL}(k)) x(k)$$

where

$$\tilde{Q}_{NL} = \text{diag}\{Q, \ldots, Q\}_{NL}$$

$$\tilde{R}_{NL} = \text{diag}\{R, \ldots, R\}_{NL}$$

$$\tilde{D}_{NL} = \text{diag}\{Q_{NL}.P\}$$

### 3. STABILITY OF MPC

Let $J(k)^*$ denote the optimal performance index yielded by solving the optimisation problem and the associated control sequence be given by $u(k|k)^*, u(k+1|k)^*, \ldots, u(k + NL - 1|k)^*$. Choose an Lyapunov function candidate $V(x(k))$ for the system (3) under the MPC scheme as $J(k)^*$, i.e.,

$$V(x(k)) \triangleq J(k)^* = \min_{u(k|k), \ldots, u(k + NL - 1|k)} J(k)$$

For the sake of notational simplicity, let

$$F(x(i)) = x(i)^T P x(i),$$

$$Y(x(i), u(i)) = x(i)^T Q x(i) + u(i)^T R u(i)$$

$$S_L(x(k)) \triangleq F(x(k + L)) - F(x(k)) +$$

$$\sum_{i=0}^{L-1} Y(x(k + i), u(k + i))$$

and $\mathcal{N}$ denote the set of all $x(k)$ such that there exists $\tilde{U}_L(k) = [u(k), u(k+1), \ldots, u(k + L - 1)]^T$ satisfying $S_L \leq 0$ and the control constraints, i.e.,

$$\mathcal{N} \triangleq \{ x(k) \in \mathbb{R}^n : \exists u(k+i) \in U, i = 0, \ldots, L - 1, \text{ such that } S_L(x(k)) \leq 0 \}$$

Then one of the main results of this paper is given in Theorem 1.

**Theorem 1:** For a given integer $L$, the MPC with the performance index (4) can asymptotically stabilise the constrained linear systems (3) about the origin if the optimal control sequence yielded by solving the optimisation problem (5)-(7) at time instant $k = 0$ renders the state at time instant $NL$ into the set $\mathcal{N}$, i.e., $x(NL|0) \in \mathcal{N}$.

**Proof:** At time instant $k$, the optimisation problem is solved to yield the optimal control sequence $u(k|k)^*, u(k+1|k)^*, \ldots, u(k + NL - 1|k)^*$ and associated optimal performance index $J(k)^*$. Suppose that $x(k + NL|k)^*$ belongs to the set $\mathcal{N}$. Then there exists a control sequence $\tilde{u}(k + NL), \ldots, \tilde{u}(k + NL + L - 1)$ satisfying the control constraints (2) such that $S_L(x(k + NL)|k)^* \leq 0$. At time instant $k + L$, choose the control sequence $\tilde{U}_{NL}(k + L|k + L) = [\tilde{u}(k + L + i|k + L), \ldots, \tilde{u}(k + NL + L - 1|k + L)]^T$ as

$$\tilde{u}(k + L + i|k + L) = \begin{cases} u(k + L + i|k) \text{ for } i = 0, \ldots, NL - L - 1 \\ u(k + i) \text{ for } i = NL - L, \ldots, NL - 1 \end{cases}$$

Let $\tilde{x}(k + L + i|k + L)$ denote the state at time instant $k + L + i$ driven by the control sequence $\tilde{U}_{NL}(k + L|k + L)$ from $x(k + L)$. The performance index (4) at time instant $k + L$ under this control sequence $\tilde{U}_{NL}(k + L|k + L)$ is denoted by $\tilde{J}(k + L)$, given by

$$\tilde{J}(k + L)$$

$$= \tilde{x}(k + NL + L + i|k + L)^T P \tilde{x}(k + NL + L + i|k + L) +$$

$$+ \sum_{i=0}^{NL-1} \{[\tilde{x}(k + L + i|k + L)^T Q \tilde{x}(k + L + i|k + L)] +$$

$$+ \tilde{u}(k + L + i|k + L)^T \tilde{R} \tilde{u}(k + L + i|k + L)\}$$

Let

$$\delta J(k, L) \triangleq \tilde{J}(k + L) - V(x(k))$$

Using the notation (20) and (21), substituting (25) and (4) into (26) gives

$$\Delta J(k, L) \triangleq \tilde{J}(k + L) - J(k)^*$$

$$= F(\tilde{x}(k + L + NL|k + L)) +$$

$$+ \sum_{i=0}^{NL-1} Y(\tilde{x}(k + L + i|k + L), \tilde{u}(k + L + i|k + L)) -$$
According to the definition of the sequence $\bar{U}_N(k + L[k + L])$ in (24), the control efforts at $k + L, \ldots, k + NL - 1$ in $\bar{U}_N(k + L[k + L])$ are the same as that in the optimal control sequence at time instant $k$. This implies that $x(k + L) = x^* + L[k]$ under the assumption that there is no uncertainty and disturbance. Then (27) reduces to

\[
\Delta J(k, L) = F(\dot{x}(k + L + NL[k + L]) - F(x(k + NL[k + L])) + \\
+ \sum_{i=NL-L}^{NL-1} Y(\dot{x}(k + L + i[k + L], u(k + L + i[k + L]) + \\
- \sum_{i=0}^{L-1} Y(x(k + i[k]^*, u(k + i[k]^*)
\]

Since the control efforts at $k + NL, \ldots, k + NL + L - 1$ in $\bar{U}_N(k + L[k + L])$ are formed by the control sequence satisfying the condition $S_L(x(k + NL)) \leq 0$, it implies that

\[
F(\dot{x}(k + L + NL[k + L]) - F(x(k + NL[k + L])) + \\
+ \sum_{i=NL-L}^{NL-1} Y(\dot{x}(k + L + i[k + L], u(k + L + i[k + L]) + \\
= F(\dot{x}(k + L + NL)) - F(x(k + NL)) + \\
+ \sum_{i=0}^{L-1} Y(\dot{x}(k + NL + i), u(k + NL + i)) + \\
= S_L(x(k + NL)) \leq 0
\]

Substituting (29) into (28) obtains

\[
\Delta J(k, L) \leq - \sum_{i=0}^{L-1} Y(x(k + i[k]^*, u(k + i[k]^*)
\]

Combining (30) with (32) yields

\[
\Delta V(k, L) \leq V(x(k + L)) - V(x(k)) + \\
\leq (V(x(k + L)) - \bar{J}(x(k + L))) + \\
\leq \delta J(k, L)
\]

Since $V(x(k)) = J(x(k)^* is positive definite for all nonzero $x(k)$ and it is non-increasing for every $L$ sampling intervals, stability for MPC is ready to prove using Lyapunov theory. Following (33), it can be shown that

\[
\sum_{i=0}^{L-1} Y(x(k + i[k]^*, u(k + i[k]^*) \rightarrow 0
\]

When $Q$ is positive definite, this implies

\[
x(k) \rightarrow 0 \text{ as } k \rightarrow \infty
\]

When $Q$ is positive semi-definite, the detectability of $[A, Q^{1/2}]$ results in the same conclusion as in (35). Hence the result is obtained by letting $k = 0$.

QED

Remark 1: When $L = 1$, the result presented in this paper reduces to the existing method for stability analysis of MPC, for example, Lee et al. (1998), Lee (2000). In this case the MPC algorithm is exactly the same as conventional MPC algorithms. When $L > 1$, this implies in the measurement sampling interval $h$, a control sequence consisting of piecewise constants, instead of a constant, is imposed on plants. This increases the stability region for MPC at the price of increasing the computational burden.

4. STABILITY REGIONS

Theorem 1 establishes the stability condition of MPC using a multi-sampling approach. However, several problems remain unsolved. The first problem is how to test this condition, that is, how to determine whether a terminal state $x(NL)$ is within the set $\mathcal{N}$. The second problem is how to determine the set of initial state where all state trajectories starting, under the control sequence yielded by solving the optimisation problem, from arrives at the set $\mathcal{N}$. Moreover, it is easy to see that the optimal control sequence yielded by solving the optimisation problem (5)-(7) is affected by the terminal weighing matrix $P$ in the performance index (4). For a given initial state, the choice of $P$ affects whether the state at time instant $NL$ under the MPC scheme enters the terminal set $\mathcal{N}$. From the
definition of the set $\mathcal{N}$, the choice of $P$ also affects the size of this stability set. The third problem is how to choose the matrix $P$ such that the stability region is as large as possible. These problems are answered by Theorem 2 and 3 in this section.

Let
\[ \Phi^Q_L = \begin{bmatrix} (\bar{Q}_L)^{1/2} & 0 \\ 0 & I \end{bmatrix} \Phi_L, \]
and
\[ \Gamma^Q_L = \begin{bmatrix} (\bar{Q}_L)^{1/2} & 0 \\ 0 & I \end{bmatrix} \Gamma_L, \]
where $\Phi_L, \Gamma_L$ are obtained by replacing the order $NL$ with $L$ in (15) and (16), and $\bar{Q}_L$ and $\bar{R}_L$ are given in (18) with $L$ diagonal matrix elements $Q$ and $R$ respectively. The set $\mathcal{N}$ can be estimated by Theorem 2 using linear matrix inequalities (LMI’s).

**Theorem 2**: For a constrained linear system (3), suppose that there exist $W^\mu > 0, H^\mu$ and $\mu > 0$ such that
\[
\begin{bmatrix}
-W^\mu & (\Phi^Q_L W^\mu + \Gamma^Q_L \bar{R}^\mu)^T \\
\Phi^Q_L W^\mu + \Gamma^Q_L \bar{R}^\mu & -W^\mu \\
\bar{R}^\mu & -W^\mu \\
\end{bmatrix} \leq 0,
\]
(38)
and
\[
Y = \begin{bmatrix} H^\mu_t^T & W^\mu \\
\end{bmatrix} \geq 0, \quad Y_{ii} \leq \bar{a}^2_i, \\
i = 0, \ldots, L - 1
\]
where
\[
\bar{R}^\mu = \begin{bmatrix} H^\mu_0 \\
\vdots \\
H^\mu_{L-1} \\
\end{bmatrix}.
\]
If the control sequence yielded by solving the optimisation problem (5)-(7) renders the state at time instant $NL$ into the set $\mathcal{V}$, defined by
\[
\mathcal{V} = \{ x \in \mathbb{R}^n : x^TPx \leq \mu \},
\]
then the MPC scheme with the performance index (4) asymptotically stabilises the system (3) about the origin where $P = (W^\mu)^{-1}\mu$.

Theorem 2 gives the estimation of the set $\mathcal{N}$ for the terminal state $x(NL)$. But it does not give the information about under what initial state the state at time instant $NL$ under the MPC enters the set $\mathcal{N}$, which is more important form control system designers.

There are two ways to formulate the on-line optimisation problem when the terminal constraint (23) is used to guarantee stability. One is that in addition to the constraints (6) and (7), the constraint $x(NL + k|k) \in \mathcal{N}$ is added in the optimisation problem (5), which is solved on-line at each step. The other is that no explicit constraint on the terminal state is considered in the on-line optimisation but the optimal control sequence minimising the problem (5)-(7) automatically steers the state at time instant $NL$ into the set $\mathcal{N}$ under certain condition. Different allowable initial state sets are resulted for these two ways. A large allowable initial state set, i.e., the stability region of MPC, is achieved by the former. But the main disadvantage of the former is its large on-line numerical computational burden since the condition for the terminal state being within the set $\mathcal{N}$ is tested on-line in each routine of optimisation algorithm. Hence the latter is adopted in this paper. Theorem 3 gives the set where all initial state starting from automatically arrives in the set $\mathcal{N}$ under the control sequence yielded by solving the optimisation problem (5)-(7).

**Theorem 3**: Suppose that there exist matrices $R^{n\times n} \ni W^\mu > 0, R^{n\times n} \ni S^\mu > 0, \bar{G}^\mu \in R^{NLM \times n}$, $\bar{R}^\mu \in R^{LM \times n}$ and a scalar $\mu > 0$ such that
\[
\begin{bmatrix}
-W^\mu & (\Phi^Q_{NL})^T S^\mu + \Gamma^Q_{NL} (\bar{G}^\mu)^T \\
\Phi^Q_{NL} S^\mu + \Gamma^Q_{NL} \bar{G}^\mu & -\mu I \\
\end{bmatrix} \leq 0,
\]
(42)
and the conditions (38),(39) hold where
\[
\bar{G}^\mu_{NL} = [\bar{G}^0_{NL}, \ldots, \bar{G}^0_{NL - 1}]^T, \quad \Phi^Q_{NL} = [(\bar{Q}_{NL})^{1/2} / 0] \Phi_{NL}, \quad \Gamma^Q_{NL} = [(\bar{Q}_{NL})^{1/2} / 0] \Gamma_{NL},
\]
$\Phi_{NL}, \Gamma_{NL}, \bar{Q}_{NL},$ and $\bar{R}_{NL}$ are given in (15), (16) and (18).

MPC with the predictive performance index (4) asymptotically stabilises the constrained linear system (3) for all initial state within the set
\[
\mathcal{M} = \{ x \in \mathbb{R}^n : x^TZX \leq \mu \},
\]
(47)
where $Z = (S^\mu)^{-1}\mu$.

The results in Theorem 2 and 3 can be used in two ways. One is for analysis and the other is for synthesis. In analysis, all the weighting matrices in the performance cost has been determined and Theorem 2 and 3 are used to analyse stability of the resultant
MPC scheme. Since the conditions in (38), (39), (42) and (43) are LMI’s, to find such that these conditions are satisfied is a problem of feasibility test of LMI’s, which can be solved by existing LMI toolboxes. In synthesis, the weighting matrices, in particular the terminal weighting matrix $P_r$, are determined such that the stability region of the MPC scheme is as large as possible. For example, according to Theorem 3, the maximisation of the stability region in terms of the weighting matrices is formulated as
\[
\min_{S^\mu, W^\mu, R^\mu, \mu} \log(\det((S^\mu)^{-1}))
\]
subject to (38), (39), (42), (43) and
\[
S^\mu > 0, \mu > 0, W^\mu > 0
\]

5. CONCLUSION

This paper proposes a multirate MPC of constrained systems. It allows in each sampling interval the control effort is adjusted several times. Compared with conventional MPC algorithms, a larger computational burden is required to implement the proposed algorithm. The benefit of this scheme is that a larger stability region and better performance can be achieved. For a linear system, the test of the stability condition, the estimation of the stability region and maximisation of the stability region in terms of the weighting matrices are formulated as a convex optimisation problem.

6. REFERENCES


