A NEW APPROACH TO MIXED $H_2$/$H_\infty$ OPTIMAL PI/PID CONTROLLER DESIGN

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Abstract: This paper proposes a new approach to solve the problem of designing optimal proportional-integral-derivative (PID) controllers that minimize an $H_2$-norm associated with the set-point response while subjecting to an $H_\infty$-norm on the load disturbance rejection. The proposed design approach consists of constructing the feasible domain in the controller gain space and searching over the domain the optimal gain values of minimizing the $H_2$-norm objective. The construction of the feasible domain that satisfies the requirements of closed-loop stability and $H_\infty$-norm constraint on disturbance-rejection is achieved through analytically characterizing the domain boundary with the notion of principal points associated with the value set of differentiable mapping. The feasible domain boundary construction greatly save the computational effort required to search for the $H_2$-optimal controller gain values that satisfy both the stability and disturbance rejection requirements.

Keywords: PID controllers, $H_\infty$-norm on disturbance rejection, $H_2$-optimal tracking, path-following algorithm, value set

1. Introduction

The PID control algorithm has played a dominant role in industrial process control systems for over sixty years (Åström and Hägglund, 1994). Consequently, many different methods have been proposed for determining the three controller parameters, i.e., proportional, integral, and derivative gains, to meet different requirements of various control applications. The proposed PID controller design methods can broadly grouped into three categories. The first category consists of methods which make use of frequency-response information. The early Ziegler-Nichols (Z-N) tuning method (Ziegler and Nichols, 1942) is based on the knowledge of only one point (critical point) on the process Nyquist curve. Recently developed Z-N like methods (Blickley, 1990; Bobal, 1995) try to automate and improve the PID controller tuning using a slight increase amount of process frequency response data. The second category
includes model-based controller design methods. These methods require the process transfer function model and they derive PID controller settings with the goals such as pole placement, internal model control. The third category consists of design methods (Graham and Lathrop, 1953) which determine the PID controller parameters by minimizing some integral of feedback error performance criteria. This class of methods can be applied to general transfer function models and typically requires a numerical optimization method to search the optimal PID controller parameters.

The purpose of this paper is to present a new approach to design optimal PID controllers of minimizing a $H_2$-norm performance index subject to an $H_\infty$-norm constraint on external disturbance attenuation. This mixed $H_2/H_\infty$ optimal PID controller design problem has been previously studied by Chen and Lee (1995) and Krohling (1997). Chen and Lee (1995) proposed a hybrid solution approach consisting of a genetic algorithm with binary codification for minimization of the integral of squared error performance index together with a numerical algorithm for evaluating a disturbance rejection constraint. Instead of using ISE as nominal $H_2$-norm performance index, Krohling (1997) solved with two real-coded GAs the mixed $H_2/H_\infty$ optimal PID control design problem for the minimization of the integral of time-weighted squared-error (ITSE) in conjunction with a $H_\infty$ disturbance attenuation constraint. Our improvements upon this previous attempt lie in three aspects. Firstly, we consider the general case where the process can be described by a rational transfer function with a pure delay or by an irrational transfer function. Hence, our result is applicable to a larger class of plants, especially to most chemical processes in which a transportation delay or a distributed delay always exists. Secondly, we employ a generalized quadratic error functional associated with the step response of setpoint input as the performance index to be minimized. The generalized quadratic cost functional includes the commonly used ISE and ITSE performance criteria as special cases. Lastly and most importantly, we propose in the paper to identify explicitly the feasible domain in the controller gain space with the notion of principal points in characterizing the value set boundary of a differentiable mapping. With the controller parameters selecting from the feasible domain, the constraints on internal stability and disturbance attenuation $H_\infty$-norm are automatically guaranteed. The explicit construction of feasible gain domain can greatly reduce the computational effort required to search for the optimal PID controller gain values.

### 2. PROBLEM DESCRIPTION

Consider the feedback control system shown in Figure 1. In this block diagram, $G_c(s)$ and $G_p(s)$ are transfer functions of the controller and the plant to be controlled, respectively, and $R(s)$, $D(s)$ and $Y(s)$ denote the Laplace transforms of the setpoint input variable $r(t)$, disturbance input variable $d(t)$, and process output variable $y(t)$, respectively. For PID feedback control, the controller transfer function is given by:

$$G_c(s) = k_p + \frac{k_i}{s} + k_ds$$ (1)

where $k_p$, $k_i$, and $k_d$ are the gains of the proportional, integral, and derivative control actions, respectively. The tuning of PID controller $G_c(s)$ is to determine the values of its three gains. Usually, the first requirement for the controller gain settings is that the feedback control system in Figure 1 is internally stable. Note that a transfer function $G(s)$ is BIBO stable if and only if it is proper and analytic on the closed right half of the $s$ plane. Hence, the three parameters of the PID controller $G_c(s)$ must satisfy the following stability constraint:

$$1 + G_c(\sigma + j\omega)G_p(\sigma + j\omega) \neq 0 \quad \text{for} \quad \sigma \geq 0$$ (2)

where $j = \sqrt{-1}$.

Let $L^2$ denote the space of signals with finite energy and define the $L^2$-norm

$$\|f\|_2 = \sqrt{\int_0^\infty |f(t)|^2 dt}$$ (3)

for $f(t) \in L^2$. Let $y_d(t)$ denote the output response due to the presence of external disturbance $d(t) \in L^2$. Then under the assumption that the feedback control system is internal stable, the desired disturbance attenuation level $\gamma < 1$ is specified by

$$\sup_{d(t) \in L^2} \frac{\|y_d\|_2}{\|d\|_2} = \frac{\|G_p(s)\|_\infty}{1 + G_c(s)G_p(s)} \leq \gamma$$ (4)

where $\|G(s)\|_\infty$ denotes the $H_\infty$ norm of a BIBO stable transfer function and is defined as

$$\|G(s)\|_\infty = \sup_{\omega \in [0, \infty)} |G(j\omega)|$$ (5)

Hence, in order to achieve the specified disturbance attenuation level $\gamma$, the gain parameters of the PID controller $G_c(s)$ must be chosen to satisfy the following constraint:

$$\sup_{\omega \in [0, \infty)} \frac{|G_p(j\omega)|}{|1 + G_c(j\omega)G_p(j\omega)|} \leq \gamma$$ (6)
Now, we consider the set-point tracking ability of the feedback control system in Figure 1. The error $e(t)$ due to a unit step setpoint input $R(s) = 1/s$ has its Laplace transform as follows:

$$E(s) = \mathcal{L}\{e(t)\} = \frac{R(s)}{1 + G_c(s)G_p(s)}$$

(7)

An analytical measure of the tracking ability for the feedback control system in Figure 1 can be defined as follows:

$$J_m = \int_0^\infty t^m e^2(t)dt$$

(8)

where $m$ is a nonnegative integer. It is noted that if the feedback control system satisfies the internal stability requirement (2), the Laplace transform of time-weighted error variable $t^m e^2(t)$, denoted by $E_{m/2}(s)$, belongs in the $H_2$ Hardy space consisting of all functions $G(s)$ which are analytical in the closed right-half plane of the $s$ plane and have finite $H_2$ norm, defined as

$$\|G\|_2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} |G(j\omega)|^2 d\omega$$

(9)

where the $G$ denotes the complex conjugate of $G$ and $G(j\omega) = G(-j\omega)$ if $G(s)$ is a function with real coefficients. Hence, the performance index $J_m$ is an $H_2$ norm. Also noted is that the performance index $J_0$ and $J_1$ are usually referred to as ISE and ITSE performance indices, respectively. For the generalized quadratic performance index $J_m$, the time weighting factor $t^m$ provides the measure with a flexibility of placing less emphasis on the error during initial transient period, which can consequently lead to a less oscillatory step response than those based on minimizing ISE performance index.

Since the setpoint tracking error of the feedback control in Figure 1 can only be reduced by adjusting the three PID gain parameters, the PID controller tuning for achieving a good tracking ability is to minimize the performance index $J_m$. Hence, the optimal PID control design problem taking both tracking and disturbance rejection into account can be formulated as the following mixed $H_2/H_\infty$ optimization problem:

$$\min_k J_m(k)$$

subject to the constraints (2) and (4)

$$k = (k_p, k_i, k_d).$$

To solve this constrained optimization problem, we suggest first to construct the feasible gain domain $K_f$ and then to search the optimal gain values from the constructed feasible domain $K_f$ that minimizes $H_2$-norm $J_m$. In the next section, effort will be devoted to the construction of the feasible gain domain $K_f$.

3. CONSTRUCTION OF FEASIBLE GAIN DOMAIN

The purpose of this section is to construct the feasible gain domain $K_f$ described by (11). As it will be shown later, the construction of the feasible gain domain $K_f$ is the same as the construction of the $H_\infty$ constraint domain

$$K_{\infty}(\gamma) = \{ k : k_p \geq 0, k_i \geq 0, k_d \geq 0, \parallel G_p(s) \parallel_{\infty} \leq \gamma \}$$

(12)

Hence, effort will be first devoted to construct the $H_\infty$ constraint domain $K_{\infty}(\gamma)$.

3.1 Construction of $H_\infty$ constraint domain $K_{\infty}(\gamma)$

For a specified disturbance attenuation level $\gamma$, we have from (1) and (5) the following equality constraint

$$\phi(k_p, k_i, \omega; k_d, \gamma) = \frac{1}{\gamma^2} \left| G_p(j\omega) \right|^2$$

$$- |1 + \left(k_p - j\left(\frac{k_i}{\omega} - k_d\omega\right)\right) G_p(j\omega)|^2 = 0 \quad (13)$$

where $k_d$ and $\gamma$ are regarded as parameters. For a fixed $\omega$ and a given pair of $(k_d, \gamma)$, the above quadratic equation defines an ellipsoidal curve in the $k_p$-$k_i$ plane. It is obvious that the union of those curves corresponding to every $\omega \in [0, \infty)$ constitutes a connected region in the $k_p$-$k_i$ plane. We will refer to such a region as an equality set. Mathematically, it is described by

$$E(k_d, \gamma) = \{(k_p, k_i) : \phi(k_p, k_i, \omega; k_d, \gamma) = 0, \forall \omega \in [0, \infty)\}$$

(14)

In the following we shall characterize the boundary of the equality constraint set $E(k_d, \gamma)$, denoted by $\partial E(k_d, \gamma)$.

For notational simplicity, we omit temporarily the parameters $k_d$ and $\gamma$ from $\phi$ and $E$. Hence, the constraint

$$\phi(k_p, k_i, \omega) = 0$$

(15)
defines a two-dimensional (2-D) surface \( S \) in the three-dimensional gain-frequency space. The projection of this 2-D surface onto the \( k_p-k_i \) plane is exactly the equality constraint set \( E \). By regarding \( k_p-k_i \) plane as the complex plane, the equality constraint set \( E \) can be viewed as the image of the surface \( S \) under the complex-valued mapping

\[
g(k_p, k_i, \omega) = k_p + jk_i
\]

That is

\[
E = g(S) = \{k_p + jk_i : \forall (k_p, k_i, \omega) \in S\}
\]

(17)

To characterize the boundary of the value set \( g(S) \), we first note that the differential of the mapping \( g \) is given by

\[
dg = \frac{\partial g}{\partial k_p} dk_p + \frac{\partial g}{\partial k_i} dk_i + \frac{\partial g}{\partial \omega} d\omega
\]

(18)

Since \( (k_p, k_i, \omega) \in S \), the differential changes \( dk_p, dk_i, \) and \( d\omega \) are not independent but satisfy the following relation

\[
d\phi(k_p, k_i, \omega) = \frac{\partial \phi}{\partial k_p} dk_p + \frac{\partial \phi}{\partial k_i} dk_i + \frac{\partial \phi}{\partial \omega} d\omega = 0
\]

(19)

Eliminating \( dk_p \) from the above two equations, we obtain

\[
dg = (-\frac{\partial \phi}{\partial \omega} + j)dk_i + \frac{\partial \phi}{\partial k_i} dk_i + \frac{\partial \phi}{\partial k_p} dk_p
\]

(20)

where the signs of \( dk_i \) and \( d\omega \) can be made arbitrarily. If the plane vectors \( g_i \) and \( g_\omega \) are not colinear, then for any point \( v \) in the neighborhood of \( g(k_p, k_i, \omega) \) with \( |g - v| < 0^+ \), we can choose \( dk_i \) and \( d\omega \) such that \( g + dg = v \). In this case, the image \( g \) is not on the boundary of the value set \( g(S) \). Hence, the necessary condition for a point \( (k_p, k_i, \omega) \) on the surface \( \phi(k_p, k_i, \omega) = 0 \) with its image under the mapping \( g \) lying on the boundary of the value set \( g(S) \) is that the plane vectors \( g_i \) and \( g_\omega \) are colinear. Mathematically, it is represented by

\[
\psi(k_p, k_i, \omega) = \begin{vmatrix} \frac{\partial \phi}{\partial k_i} & \frac{\partial \phi}{\partial \omega} \\ \frac{\partial \phi}{\partial k_i} & \frac{\partial \phi}{\partial \omega} \\ \frac{\partial \phi}{\partial k_p} & \frac{\partial \phi}{\partial k_p} \end{vmatrix} = 0
\]

(21)

Let \( P \) be the set of points on the surface \( \phi(k_p, k_i, \omega) = 0 \) while satisfying the relation \( \psi(k_p, k_i, \omega) = 0 \) for \( \omega \in [0, \infty) \). Then, it is clear that the image of \( P \) under the mapping \( g \) includes the boundary of the equality constraint set \( g(S) \), that is

\[
\partial g(S) \subset g(P)
\]

(22)

It is noted that \( P \) is one-dimensional manifold in the \( (k_p, k_i, \omega) \) space. Hence, by tracing this manifold with a path-following algorithm, we can construct the boundary of the equality constraint set \( E = g(S) \).

### 3.2 The Stability Domain \( K_s \)

The primary concern in tuning the PID controller \( G_c(s) \) is that the \( G_c(s) \) must be a stabilizing controller. The stability domain \( K_s \) is defined as follows:

\[
K_s = \{k : k_p \geq 0, k_i \geq 0, k_d \geq 0, 1 + G_c(\sigma + j\omega)G_p(\sigma + j\omega) \neq 0 \quad \forall \sigma, \omega \in [0, \infty]\}
\]

(23)

The critical stability boundaries of \( K_s \) satisfy the equation

\[
1 + G_c(j\omega)G_p(j\omega) = 0
\]

(24)

which is equivalent to

\[
|1 + (k_p - j\frac{k_i}{\omega} - k_d\omega)G_p(j\omega)|^2 = 0
\]

(25)

Since the equation \( \phi(k_p, k_i, \omega; k_d, \gamma) = 0 \) defined in (13) reduces to the above equation as \( \gamma \to \infty \), it can be easily shown that the equality constraint set \( E(k_d, \gamma) \) includes the critical boundaries of the stability domain \( K_s \) described by (23). As a result, we have the following inclusion relation

\[
K_{\infty}(\gamma) \subset K_s
\]

(26)

Hence, a PID controller with its three gains locating in the \( H_{\infty} \) constraint domain \( K_{\infty}(\gamma) \) is guaranteed to be a stabilizing controller.

### 4. AN ILLUSTRATIVE EXAMPLE

This section presents an illustrative example to demonstrate the proposed approach to solve the mixed \( H_2/H_{\infty} \) optimal PID control design problem stated in (10). The solution procedure includes the followig steps: (i) applying the boundary generation method described in Section 3 to obtain the boundary of the equality constraint sets \( E(k_d, \gamma) \) for various values of \( k_d \) so as to construct the feasible gain domain \( K_{\infty}(\gamma) \); (ii) defining the cost functional to be minimized as
\[ I_m(k) = \begin{cases} J_m(k) & \text{if } k \in K_{\infty}(\gamma) \\ J_m(k) + a |G(k)|^2 & \text{if } k \not\in K_{\infty}(\gamma) \end{cases} \]  

where \( a \) is a arbitrary big number, say, \( 10^6 \) and \( G(k) \) is the penalty function; (iii) applying the differential evolution algorithm described in Section 5 to search for the optimal controller gain \( k \) that minimizes the cost functional \( I_m(k) \).

In working out the following example, the one-dimensional manifold \( M \) described by \( \phi(k_p, k_i, \omega; k_d, \gamma) = 0 \) and \( \psi(k_p, k_i, \omega; k_d, \gamma) = 0 \) is traced by the spherical method (Yamamura, 1995). In general, the boundaries of the feasible gain domain \( K_{\infty}(\gamma) \) cannot be described by simple mathematical expressions. In practice, an approximate gain domain \( K_{\infty}(\gamma) \) with piecewise linear boundary is used.

Consider the feedback control system shown in Figure 1, in which the plant transfer function is given by

\[ G_p(s) = \frac{e^{-1.5s}}{(s + 1)^3} \]  

It is desired to find the settings of a PI/PID controller \( G_c(s) \) such that the \( H_2 \) performance \( J_0 \) is minimized while subject to the following \( H_\infty \)-constraint:

\[ \|G_p(s)/1 + G_c(s)G_p(s)\|_\infty \leq \gamma = 0.8 \]  

To solve this mixed \( H_2/H_\infty \) optimal PID controller design problem, we first construct the gain feasible domain \( K_{\infty}(\gamma) \). Using the spherical algorithm, we traced the boundary curves of the feasible domains in the \( k_p-k_i \) plane various values of \( k_d \). The traced boundary curves are shown in Figure 2. Figures 3a-d show the boundaries of the equality sets \( E_\alpha(k_d, 0.8) \) for \( k_d = 1.0, 1.5, 2.0 \) and 2.5 along with the curves defined by \( \phi = 0 \) for various values of \( \omega \). Having constructed the feasible domain \( K_{\infty} \), we applied the differential evolution algorithm with control variables \( F = 0.6 \) and \( C_r = 0.67 \) to search the optimal controller gain in the feasible domain \( K_{\infty}(0.8) \) that minimizes the \( H_2 \)-norm \( J_0 \). The obtained settings for PID controller are given by \((k_p^*, k_i^*, k_d^*) = (0.988, 0.297, 1.5)\), respectively. The minimum \( H_2 \)-norm attained by optimal PID control is \( J_0 = 3.306 \). The Bode diagrams of closed-loop transfer function with the designed controller is shown in Figures 4 for the optimal PID control, respectively. It can be seen that the maximum magnitude of the closed-loop frequency-responses are indeed less than the specified level \( \gamma = 0.8 \). The time responses of the closed-loop system subject to unit step reference input are shown in Figures 5.

5. CONCLUSIONS

A parameter space design method has been proposed for PID controllers which minimize the \( H_2 \)-norm associated with the set-point response while satisfying an \( H_\infty \)-norm on the load disturbance rejection. The method is based on presenting an effectiveness approach to construct the feasible control parameter domain within which the specified \( H_\infty \)-norm constraint is automatically satisfied. The construction of feasible domain is achieved through using the notion of principal points to derive an analytic expression for describing the boundaries of an equality constraint set defined by the maximum \( H_\infty \)-norm. By tracing the equality constraint set boundary using a path-following algorithm, the feasible domain boundaries in the \( k_p-k_i \) parameter plane can be easily obtained for each value of \( k_d \). The availability of feasible controller parameter domain and parametric formula for evaluating quadratic cost functional greatly facilitates searching of optimal controller parameters that minimizes the \( H_2 \)-norm associated with the set-point response. In order to obtain the global optimal solution, the search work has been done by the simple heuristic differential evolution algorithm. The effective of the proposed PID design method has demonstrated with an numerical example.

6. REFERENCES


IEICE Transactions on Fundamentals E78-A, 1233-1238.


\[ R(s) G_c(s) = k_p + k_i/s + k_d s \]

Fig. 1. Block diagram of the PID feedback control system.

Fig. 2. Boundaries of feasible domains and stability domains for \( k_d = 0.1 \) to 2.2 with \( \omega = 0.8 \).

Fig. 3. Boundaries of equality set \( E_s(k_d; \omega) \) of the example with \( k_d = 1.0, 1.5, 2.0 \) and 2.5.

Fig. 4. The magnitude frequency responses of \( G_d(s) \) for optimal \( H_2 \)-PID controllers with \( \omega = 0.8 \) and \( \omega = 1 \).

Fig. 5. Unit-step responses of optimal \( H_2 \)-PID control systems for \( \omega = 0.8 \) and \( \omega = 1 \).