REGULATING CONCENTRATION PROFILE IN FLUID FLOW USING PROCESS TOMOGRAPHY

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Abstract: Process tomography is a sensing technique that can be used to measure distributed flows inside pipes. As a result, it is suitable for use as the sensor in a feedback control system for regulating the concentration of a substance in a fluid flow along the pipe. This paper describes such a control system and starts by developing a model based on the convection-diffusion equation, which describes the diffusion of a substance within the fluid flow, and then develops a state space description of the system. The state space model is used as the basis of a state feedback design. Estimates of the states are obtained from an observer that uses the measurements obtained by the tomographic sensor. A simulation of a feedback system based upon proportional plus integral control is presented.

Keywords: Distributed parameter control, process tomography, fluid flow, modelling.

1. INTRODUCTION

Although tomography has been used extensively in medical applications, it is only relatively recently that tomographic sensors have been developed for use in industrial processes (Williams and Beck, 1995). The main benefit of tomographic sensors is that they provide a non-invasive technique for measuring distributions such as flow patterns, concentration profiles or phase profiles, from within pipes and process vessels. A number of different tomographic techniques have been developed, including nucleonic, microwave and acoustic methods (Williams and Beck, 1995), but the commonest approach in industrial applications is electrical methods. In these sensors a set of electrodes is arranged around the pipe or vessel, as shown in figure 1. In electrical impedance tomography (EIT) a current is applied to each pair of electrodes in turn and the voltage difference between the electrodes measured. Combining the voltages measurements from each current injection, a map of the impedance (or conductivity) profile across the pipe can be obtained. Provided that the impedance is related to the variable of interest (for example, the concentration profile), then this provides a measurement of the variation over the pipe. In electrical capacitance tomography (ECT), a voltage is applied to one of the electrodes and the charge on the other electrodes is measured, from which a measurement of the permittivity profile can be obtained, giving a measurement of the variable of interest.
The development of electrical tomographic sensors has now reached the point where successful implementations are being reported (Grieve et al., 2001). The sensing technique provides a non-invasive measurement of a distributed variable, so these sensors can be used in distributed parameter control systems. This paper describes the use of an electrical impedance tomographic sensor as part of a system for regulating the concentration of a fluid flow in a pipe. Section 2 sets out the process model, which is described by a convection-diffusion, partial differential equation (PDE). By applying finite differences in the direction of the flow, section 3 shows how the PDE is reduced to a form that can be approximated by a finite dimensional state space model. Section 4 describes how the states can be estimated from the measurements obtained by the tomographic sensor and section 5 shows simulation results from a simple control strategy.

2. PROCESS MODEL

In the system described here, the purpose of the tomographic sensor is to provide a measurement of the concentration profile of a substance in a flow along a straight pipe with a circular cross-section using measurements taken by a tomographic sensing system. These estimates can then be used in a state feedback controller that regulates the concentration of the substance in the fluid flow at one or more injection points in the pipe. In a typical arrangement, the sensor will be downstream from the point of injection, as shown in figure 2.

The concentration, $c(r,t)$, of the substance in the flow, is a function of both position, $r$, and time, $t$. For flow along a circular pipe, it is convenient to use cylindrical polar coordinates, so that $r = r(\theta, z)$, where $\{r : 0 \leq r \leq a\}$ denotes the radial distance from the centre of the pipe, where $a$ is the radius of the pipe, $\theta$ is the angle around the pipe and $z$ is the distance along the pipe. If it is assumed that the point of injection is sufficiently far from the entry of the pipe, then it is reasonable to assume that the velocity profile for the fluid will have been established. For a straight pipe with circular symmetry, the fluid velocity will be in the $z$ direction and the velocity profile, $v_z(r)$, will be a function of $r$ only. If it is also assumed that upon the effect of the injection of the substance on the underlying flow of the fluid is negligible, then the concentration at any point downstream of the injection point is described by the convection-diffusion equation (Ockendon et al., 1999)

$$\frac{\partial c}{\partial t} + v_z(r) \frac{\partial c}{\partial z} = D \left\{ \frac{\partial^2 c}{\partial r^2} + \frac{1}{r} \frac{\partial c}{\partial r} + \frac{1}{r^2} \frac{\partial^2 c}{\partial \theta^2} \right\}$$

(1)

where the diffusion in the $z$-direction is ignored and $D$ is the diffusivity of the substance within the fluid, together with the boundary conditions

$$\frac{\partial c}{\partial r} \bigg|_{r=a} = 0$$

(2)

$$c(r, \theta, 0, t) = c_0(r, \theta, t)$$

(3)

It is assumed that the injectors are placed at $z = 0$ and $c_0(r, \theta, t)$ is the concentration profile generated by the injection of the substance. In practice, the injection of the substance is likely to be incorporated as a condition on the boundary of the pipe at $z = 0$, but the analysis can adapted to include this case.

Aside. The form of the equation in (1) assumes that $c(r, \theta, z, t)$ can take negative values, which is not realistic. In practice, the concentration, $c(r, \theta, z, t)$, will be measured as a deviation from some fixed level, $\bar{c}$. At a point a long way downstream from the point of injection and/or at a long time after a change in the rate of injection, the concentration of the substance will tend to $\bar{c}$, giving two further boundary conditions

$$\lim_{z \to \infty} c(r, \theta, z, t) = 0$$

and

$$\lim_{t \to \infty} c(r, \theta, z, t) = 0.$$  

Any control actions, $u(t)$, can be expressed relative to $\bar{u}$, the steady rate of injection required to maintain a concentration $\bar{c}$.

It is difficult to obtain an analytic solution to the convection-diffusion equation in (1), primarily because the spatial partial derivatives are not self-adjoint (Ockendon et al., 1999) and it is common to use numerical techniques, such as finite element analysis to solve for $c(r, \theta, z, t)$ (Seppanen et al., 2000) or to identify the spatial modes using techniques such as the Karhunen-Loeve expansion. In this paper, a different approach is adopted, where a finite difference approximation is used in the $z$-direction, which makes the problem self-adjoint with respect to $r$, $\theta$ and $t$. This leads to a state-space formulation of the model that is particularly suited for control and state estimation, although a similar model could also be derived from a finite element formulation.

Let the concentration profile across the pipe at discrete points along the pipe, $\{z = kh : k = 0, 1, 2, \ldots\}$, be denoted by $c^{(k)}(r, \theta, t)$, where $h$ is the distance between discrete samples in the $z$-direction and $c^{(k)}(r, \theta, t) = c(r, \theta, kh, t)$. Using this discretization.

Fig. 2. Diagram of fluid flow in pipe with EIT sensor

Tomographic Sensor

Substance A

Substance B
\[
\frac{\partial c}{\partial z} \bigg|_{z=kh} \approx \frac{c^{(k)}(r, \theta, t) - c^{(k-1)}(r, \theta, t)}{h}
\]  

(4)

and this approximation can be applied in the partial differential equation in (1), to give,

\[
\frac{\partial c^{(k)}}{\partial t} = D \left\{ \frac{\partial^2 c^{(k)}}{\partial r^2} + \frac{1}{r} \frac{\partial c^{(k)}}{\partial r} + \frac{1}{r^2} \frac{\partial^2 c^{(k)}}{\partial \theta^2} \right\} + \frac{V_0}{h} r^2 c^{(k)} - \frac{V_0}{h} a^2 c^{(k-1)} + \frac{V_0(r)}{h} a^2 c^{(k-1)} - \frac{V_0(r)}{h} r^2 c^{(k-1)}
\]

(5)

This PDE holds for each of concentration profiles, \( \{c^{(k)}(r, \theta, t) : k = 0, 1, 2, \ldots \} \).

Consider the example where the dimensions of the pipe and the fluid velocity mean that the flow is laminar. Under these circumstances (Tritton, 1988),

\[ v_z(r) = V_0(a^2 - r^2) \]

(6)

where \( V_0 \) is the maximum fluid velocity at the center of the pipe. Inserting this velocity profile into (5) gives

\[
\frac{\partial c^{(k)}}{\partial t} = D \left\{ \frac{\partial^2 c^{(k)}}{\partial r^2} + \frac{1}{r} \frac{\partial c^{(k)}}{\partial r} + \frac{1}{r^2} \frac{\partial^2 c^{(k)}}{\partial \theta^2} \right\} + \frac{V_0}{h} r^2 c^{(k)} - \frac{V_0}{h} a^2 c^{(k-1)} + \frac{V_0(r)}{h} a^2 c^{(k-1)} - \frac{V_0(r)}{h} r^2 c^{(k-1)}
\]

(7)

The analysis can be extended to other velocity profiles, including the 1/7th power law associated with turbulent flow (Tritton, 1988). If the term \( V_0 a^2 c^{(k)}/h \) is considered as part of the inhomogeneous part of the equation (Sneddon, 1957), then the homogeneous part is

\[
\frac{\partial c^{(k)}}{\partial t} = D \left\{ \frac{\partial^2 c^{(k)}}{\partial r^2} + \frac{1}{r} \frac{\partial c^{(k)}}{\partial r} + \frac{1}{r^2} \frac{\partial^2 c^{(k)}}{\partial \theta^2} \right\} + \frac{V_0}{h} r^2 c^{(k)}
\]

(8)

Substituting a solution of the form

\[ c^{(k)}(r, \theta, t) = R(t) \Theta(\theta) T(t) \]

(9)

leads to

\[
\frac{1}{D} \frac{dT}{dt} = \frac{1}{D} \frac{d^2 R}{dr^2} + \frac{1}{D} \frac{1}{r} \frac{dR}{dr} + \mu^2 + \frac{1}{D} \frac{1}{r^2} \frac{d^2 \Theta}{d\theta^2} = -\lambda
\]

(10)

where \( \mu = \frac{V_0}{D} \) and \( \lambda > 0 \) is a constant. Because the angular component, \( \Theta(\theta) \), is periodic with period \( 2\pi \),

\[ \Theta(\theta) = C \cos n\theta + D \sin n\theta \]

(11)

for \( n = 0, 1, 2, \ldots \), so that

\[
\frac{d}{dr} \left( r \frac{dR}{dr} \right) + \left[ \mu - \frac{n^2}{r^2} + \mu r^2 \right] r R = 0
\]

(12)

which is a Sturm-Liouville problem with a set of eigenfunctions that form an orthonormal basis with respect to the weighted scalar product

\[ < f(r), g(r) >_r = \int_0^a f(r) g(r) r dr \]

(13)

The ordinary differential equation in (12) has a regular singular point at \( r = 0 \), and its power series solution is

\[ R(r) = r^\alpha \sum_{\ell=0}^{\infty} \alpha(\ell) r^\ell \]

(14)

where the coefficients, \( \alpha(\ell) \), satisfy the recurrence relationship

\[ \alpha(0) = \frac{1}{2n!} \]

(15)

\[ \alpha(2) = -\frac{\lambda \alpha(0)}{(n+2)^2 - n^2} \]

(16)

\[ \alpha(\ell) = -\frac{\lambda \ell(\ell - 2) + \mu \ell(\ell - 4)}{(n+\ell)^2 - n^2} \] for \( \ell = 4, 6, 8, \ldots \)

(17)

and the coefficients associated with \( \ell \) odd are zero. The value of the coefficient for \( \alpha(0) \) is chosen so that \( R(r) \) coincides with \( n \)th Bessel function of the first kind, in the limit \( \mu \to 0 \).

For a given value of \( n \), the set of eigenvalues \( \lambda_{m,n} : m = 0, 1, 2, \ldots \), which ensure that \( R(r) \) satisfies the boundary condition in (2) and the corresponding functions whose coefficients are determined from these eigenvalues, will be denoted by \( P_{m,n}(r) \). The polynomials are closely related to the corresponding Bessel functions but they deviate from the Bessel functions close to the edge, where the fluid is flowing slowly. This is illustrated in figure 3, which shows examples of \( P_{m,0}(r) \) plotted against \( r \).
Combining the radial and angular parts of the solution gives a set of basis functions

\[
\phi_{m,n}(r, \theta) = \begin{cases} 
P_{m,0}(r) & \text{for } n = 0 \\
P_{m/2,n}(r) \cos n\theta & \text{for } m \text{ even} \\
P_{(m-1)/2,n}(r) \sin n\theta & \text{for } m \text{ odd}
\end{cases}
\]

(18)

The concentration profile at \( z = kh \), as an expansion in terms of these basis functions

\[
c^{(k)}(r, \theta, t) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} q^{(k)}_{m,n}(t) \phi_{m,n}(r, \theta)
\]

(19)

where \( q^{(k)}_{m,n}(t) \) are the time varying coefficients of the expansion. If \( c^{(k-1)}(r, \theta, t) \), the concentration profile at \( z = (k - 1)h \), is expressed in terms of the same basis functions, where the coefficients of the expansion are denoted by \( q^{(k-1)}_{m,n}(t) \), then substituting these expressions into the partial differential equation in (7) gives

\[
\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{dq^{(k)}_{m,n}}{dt} \phi_{m,n} =
\]

\[
- \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left( D\lambda_{m,n} + \frac{V_0 a^2}{h} \right) q^{(k)}_{m,n} \phi_{m,n}
\]

\[
+ \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} q^{(k-1)}_{m,n} \frac{V_0}{h} (a^2 - r^2) \phi_{m,n}
\]

(20)

Multiplying through by \( r \phi_{m',n'} \), integrating over the cross section and then exploiting the orthogonality of \( \phi_{m,n} \), leads to a set of ordinary differential equations

\[
\frac{dq^{(k)}_{m,n}}{dt} = - \left( D\lambda_{m,n} + \frac{V_0 a^2}{h} \right) q^{(k)}_{m,n} + \frac{V_0}{h} \left( a^2 q^{(k-1)}_{m,n} \right)
\]

\[
- \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} q^{(k-1)}_{m,n} \int_0^{2\pi} \phi_{m,n} \int_0^a r^2 \phi_{m',n'} r \, dr \, d\theta
\]

\[
\int_0^{2\pi} \int_0^a \phi_{m,n} r^2 \phi_{m',n'} r \, dr \, d\theta
\]

(21)

The second term on the right hand side of this expression can be simplified using

\[
\int_0^{2\pi} \int_0^a \phi_{m,n} r^2 \phi_{m',n'} r \, dr \, d\theta =
\]

\[
\begin{cases} 
2\pi \int_0^a \int_0^{r_m} P_{m,n}(r)^2 P_{m',n}(r) r \, dr \, d\theta & \text{if } n \neq n' \\
0 & \text{if } n = n' = 0 \\
\pi \int_0^a \int_0^{r_m} P_{m,0}(r)^2 P_{m',0}(r) r \, dr \, d\theta & \text{if } n = n' \neq 0 - m - m' \text{ even} \\
0 & \text{if } n = n' \neq 0 - m - m' \text{ odd}
\end{cases}
\]

(22)

3. STATE SPACE MODEL

If the summations in the basis function expansion of \( c^{(k)}(r, \theta, t) \) in (19) are truncated to \( M \) and \( N \) terms respectively, then \( c^{(k)}(r, \theta, t) \) can be described by \( M \times N \) terms, each of which satisfies the ordinary differential equation in (21)). Considering the coefficients, \( q_{m,n}(t) \), as the states of the system and stacking these states into a vector

\[
[q^{(k)}(t)]^T = \left[ q^{(k)}_{1,1}(t) \ q^{(k)}_{1,2}(t) \ \ldots \ q^{(k)}_{m,n}(t) \ \ldots q^{(k)}_{M,N}(t) \right]
\]

(23)

the system can be described by the state space model

\[
\frac{dq^{(k)}}{dt} = Aq^{(k)} + Bq^{(k-1)}
\]

(24)

where \( A \) is an \( MN \times MN \) diagonal matrix, with the entry on the diagonal corresponding to the \( m, n \) state is

\[
- \left( D\lambda_{m,n} + \frac{V_0 a^2}{h} \right)
\]

(25)

The \( B \) matrix also has dimension \( MN \times MN \) and the row associated with the \( m, n \) state is

\[
\frac{V_0}{h} \left( a^2 \delta_{m,m'} \delta_{n,n'} - \int_0^{2\pi} \int_0^a \phi_{m,n} r^2 \phi_{m',n'} r \, dr \, d\theta \right)
\]

\[
\int_0^{2\pi} \int_0^a \phi_{m,n}^2 \, r \, dr \, d\theta
\]

(26)

where the integral in the numerator of the second term inside the brackets can be evaluated from (22). Both of these matrices are independent of the states, \( q^{(k)}_{m,n}(t) \), and can be calculated directly from the maximum fluid velocity, \( V_0 \), and the dimensions of the system.

The state space model in (24) describes the evolution of the states associated with the concentration profile at \( z = Kh \) and the inputs to this state space model are the states associated with the profile at \( z = (k - 1)h \). Hence, the evolution of the whole system can be modelled by concatenating the state space models, so that the states associated with concentration profile at \( z = Kh \) can be related to the states associated with the profile at the start of the pipe, \( z = 0 \). The initial state, \( q^{(0)}_{m,n} \), is obtained from the concentration profile at \( z = 0 \), using

\[
q^{(0)}_{m,n}(t) = \frac{\int_0^{2\pi} \phi_{m,n}(r, \theta) c(r, \theta, 0, t) r \, dr \, d\theta}{\int_0^{2\pi} \int_0^a \phi_{m,n}^2 \, r \, dr \, d\theta}
\]

(27)

The concentration at \( z = 0 \) can be considered as

\[
c(r, \theta, 0, t) = d(r, \theta, t) + p(r, \theta) u(t)
\]

(28)

where \( d(r, \theta, t) \) represents any random disturbances in the concentration coming down the pipe,
while \( p(r, \theta) \) denotes the spatial shape of the mechanism for injecting the substance and \( u(t) \) represents the rate of injection of the substance. The model can be extended to incorporate multiple injectors, in which case, the input, \( u(t) \) becomes a vector whose dimension equals the number of injectors.

4. MEASUREMENT EQUATION

In EIT, the concentration distribution, \( c(r, \theta, Kh, t) \) at the point of measurement, \( z = Kh \), is determined from the conductivity distribution, \( \sigma(r, \theta, Kh, t) \), at this position and it is assumed that the relationship between concentration and conductivity is known. The conductivity distribution is measured at a series of discrete times, \( \{t = mT : m = 0, 1, 2, \ldots\} \), where \( T \) is the sampling interval, by applying a current, \( I_\ell \), to each pair of electrodes in turn and measuring the voltage, \( E_\ell(mT) \), between the electrodes. Here, \( \ell \) is used to index the possible combinations of electrode pairs; for \( N \) electrodes, there are \( N(N-1)/2 \) such combinations. Strictly, because the current is applied to each electrode pair in sequence, the voltages between the pairs are not measured simultaneously at times \( t = mT \), but the time between each current injection is short (typically 20\( \text{ms} \)), so provided that the concentration profile is changing slowly relative to this time scale, it is reasonable to assume that they are taken at the same time.

The estimation of the conductivity profile from a set of voltage measurements, \( \{E_{\ell}(mT) : \ell = 1, 2, \ldots N(N-1)/2\} \), can be regarded as an “inverse” problem (Vauhkonen et al., 1998). The voltage measurements enter the model as boundary conditions on the partial differential equation describing the electric field, \( E(r, \theta, Kh, mT) \), within the pipe over the cross-section at \( z = z_m \)

\[
\nabla (\sigma \nabla E) = 0 \tag{29}
\]

\[
E + \mu_t \sigma \frac{\partial E}{\partial r} \bigg|_{r=a} = E_\ell(mT) \quad \text{for} \quad \theta \in \theta_\ell \tag{30}
\]

\[
\int \sigma \frac{\partial E}{\partial r} \bigg|_{r=a} rd\theta = I_\ell \quad \text{for} \quad \theta \in \theta_\ell \tag{31}
\]

\[
\sigma \frac{\partial E}{\partial r} \bigg|_{r=a} = \quad \text{for} \quad \theta \notin \theta_\ell \tag{32}
\]

where \( \theta_\ell \) represents the angular segments on the boundary occupied by the electrodes and \( \mu_t \) is the contact impedance between the \( \ell \)th electrode pair and the wall of the pipe. To simplify the notation, the arguments on the variables have been omitted. For the existence and uniqueness of the solutions, it is also necessary to impose the following conditions

\[
\sum_{\ell=1}^{L} I_\ell = 0 \quad \sum_{\ell=1}^{L} E_\ell = 0 \tag{33}
\]

To find the conductivity distribution \( \sigma(r, \theta, Kh, mT) \), from the measured voltages, \( E_\ell(mT) \), it is necessary to discretize both distributions. The standard approach is to discretize over a finite-element grid, but an equivalent method is to expand the distributions in terms of the basis functions, \( \phi_{m,n}(r, \theta) \). By multiplying the boundary conditions through by \( \phi_{m,n}(r, \theta) \) and integrating over the cross-section at \( z = Kh \), a finite-dimensional description is obtained. Whichever basis function is used, the conductivity distribution, and hence the concentration distribution, can be estimated (Seppanen et al., 2000; Vauhkonen et al., 1998). These lead directly to estimates of \( q^K(mT) \), which can be used to generate a control signal via state feedback.

5. SIMULATION OF A CONTROL STRATEGY

The rate of injection, \( u(t) \), can be regarded as a control input to the process for regulating the concentration of the substance at a given point downstream. There are a number of control strategies that could be adopted, including minimising the variance over the cross-section at a particular point along the pipe or regulating the radial distribution at a particular point (Duncan, 2001). However, one of the most straightforward is to regulate the average concentration at a particular point. The average concentration over the cross-section of the pipe, at (for example) \( z = Kh \), is

\[
\frac{1}{\pi a^2} \int_{0}^{2\pi} \int_{0}^{a} c^{(K)}(r, \theta, t) r dr d\theta = \sum_{m=0}^{M} \sum_{n=0}^{N} q_{m,n}(t) \frac{1}{\pi a^2} \int_{0}^{2\pi} \int_{0}^{a} \phi_{m,n}(r, \theta) r dr d\theta = \mathbf{g}^T \mathbf{q}^{(K)}(t) \tag{34}
\]

where \( \mathbf{g} \) is a state feedback vector of length \( M \times N \), whose \( m, n \)th element is

\[
g_{m,n} = \frac{1}{\pi a^2} \int_{0}^{2\pi} \int_{0}^{a} \phi_{m,n}(r, \theta) r dr d\theta \tag{35}
\]

The value of the average concentration can be compared with the desired value and the error can be used to generate an input, \( u(t) \), via a simple proportional plus integral (P+I) controller to regulate the average flow.

Figure 4 shows a simulation of the concentration at a point 10\( m \) downstream of an injector in a pipe.
Fig. 4. Response of average concentration at end of pipe to step change in reference signal when under P+I control. Solid line is response when injector in centre of pipe, dashed line is response when injector at edge of pipe of cross-section 1m, where the peak flow velocity is $1 \text{ms}^{-1}$. The diffusivity is 0.1, so that neither convection, nor diffusion dominates the behaviour. The system was modelled using 6 discrete cross-sections in the $z$-direction. The process is dominated by the time delay associated with the flow from the injectors to the measuring point, so the response of the controller is poor. The solid line shows the response of the process when controlled using a P+I controller with a (hypothetical) injector placed at the centre of the pipe. The dashed line shows the response to the more realistic case of an injector at the edge of the pipe. Because the injector at the edge is injecting the substance into the slower moving fluid near the edge of the pipe, it takes time for the substance to diffuse to the swifter flowing fluid in the centre of the pipe, so compared to the injector at the centre, the time delay is longer. As a result, the response of the controller using the injector at the edge is slower, as shown in the figure.

6. CONCLUSIONS

This paper has described a state space version of the partial differential equation that governs the diffusion of a substance in a fluid flow along a cylindrical pipe. The state space model can be used as the basis of a system for regulating the concentration of the substance in the pipe and a system for controlling the average concentration at a particular point along a pipe has been simulated. The control system requires knowledge of the states of the system and process tomography is an important tool for providing estimates of the states. The combination of a tomographic measuring system, together with a feedback controller provide a mechanism for controlling concentrations in fluid flows.

7. REFERENCES


