STABILITY ANALYSIS OF INTERCONNECTED IMPLICIT SYSTEMS BASED ON PASSIVITY

Kiyotsugu Takaba

* * * Department of Applied Mathematics and Physics
Graduate School of Informatics
Kyoto University
Kyoto 606–8501, Japan
E-mail: takaba@amp.i.kyoto-u.ac.jp

Abstract: This paper considers the internal stability of an interconnection of two implicit systems based on passivity theory. The passivity for implicit systems is defined as a generalization of the passivity in the traditional input-output framework. We derive a necessary and sufficient condition for the passivity of an implicit system in terms of a constrained linear matrix inequality (CLMI). Based on this CLMI condition, we derive a sufficient condition for the internal stability of an interconnection of two implicit systems.

Keywords: stability, interconnection, implicit system, constrained linear matrix inequality, passivity

1. INTRODUCTION

The implicit system representation has great flexibility in modeling of physical systems because it does not make a priori distinction among input, output and state variables. For this reason, a number of research works on linear implicit systems have been reported in the literature (Aplevich 1991, Geerts and Schmacher 1996, Kuijper 1994, Masubuchi and Shimemura 1997, Masubuchi 2000, Shibasato et al. 1999, Takaba 1999, 2000).

In this paper, we consider the stability of a general interconnection of implicit systems. The interconnection was devised to describe the interactions among systems without using input-output structures in the behavior theory (Willems 1997). When we consider the control of implicit systems, the interaction between the plant and the controller can be described as an interconnection (Masubuchi 2000).

The notion of passivity plays an important role in stability analysis for feedback systems in the traditional state-space or input-output framework (Vidyasagar 1993, van der Schaft 1996). Some attempts to generalize the passivity theorem has been recently made based on the integral quadratic constraints (Megretski and Rantzer 1997) and the quadratic separators (Iwasaki and Shibata 1999). It may also be noted that, in the framework of the behavioral approach, the analysis and control of passive (or dissipative) systems has been studied based on the quadratic differential forms (Willems and Trentelman 1998, 2001).

The purpose of this paper is to derive a new stability condition of a dynamical system described as an interconnection of two implicit systems by generalizing the passivity theorem.

The organization of this paper is as follows. First, we review the basic facts on linear implicit systems in Section 2. In Section 3, we define the passivity of an implicit system, and derive a necessary and sufficient condition for the passivity with respect to a quadratic supply rate. Based on this passivity condition, the internal stability of an interconnection of two implicit systems will be considered in Section 4.
2. LINEAR IMPLICIT SYSTEM

We consider a linear implicit system $\Sigma$ described by

\[ \begin{cases} E \dot{\xi}(t) = F \xi(t), & \xi(0-) = \xi_0 \\ w(t) = G \xi(t) \end{cases} \]  

(1)

where $E, F \in \mathbb{R}^{m \times n}, G \in \mathbb{R}^{l \times n}$ and $\text{rank } E = r \leq \min\{m, n\}$. The system variable vector $\xi(t) \in \mathbb{R}^n$ consists of all variables such as inputs, outputs, state variables and the other redundant variables. The vector $w(t) \in \mathbb{R}^l$ externalizes the external variables. It should be noted that, unlike the conventional state-space or descriptor framework, we do not make a priori distinction among inputs, outputs and states in $\xi(t)$ and $w(t)$.

In the behavioral setting, a dynamical system is characterized in terms of the behavior that is the set of all possible trajectories of the system variables $\xi(t)$ (Willems 1991). For the implicit system $\Sigma$, we can define the behavior as the set of solutions to the differential/algebraic equation (1). It is well-known that an implicit system may admit impulsive response if its initial condition is not consistent with algebraic constraints in (1). It is helpful to introduce the impulsive-smooth distributions $\mathcal{C}_{imp}$ in order to describe the impulsive behavior (Geerts and Schumacher 1996). Hence, we define the behaviors of $\Sigma$ corresponding to the initial value $\xi_0$ as

\[ \mathcal{B}_{imp}(\xi_0) = \left\{ \xi \in \mathcal{C}_{imp} \mid E \dot{\xi} = F \xi \forall t > 0, \xi(0-) = \xi_0 \right\} \]

and

\[ \mathcal{B}_{sm}(\xi_0) = \mathcal{B}_{imp}(\xi_0) \cap \mathcal{C}_{sm} \]

where $\mathcal{C}_{imp}$ and $\mathcal{C}_{sm}$ are defined in Appendix. In general, a trajectory in $\mathcal{B}_{imp}(\xi_0)$ may contain impulses, while $\mathcal{B}_{sm}(\xi_0)$ consists of impulse-free trajectories. We also define the sets of admissible initial conditions as

\[ \mathcal{E}_{imp} = \left\{ \xi_0 \in \mathbb{R}^n \mid \mathcal{B}_{imp}(\xi_0) \neq \emptyset \right\} \]

and

\[ \mathcal{E}_{sm} = \left\{ \xi_0 \in \mathbb{R}^n \mid \mathcal{B}_{sm}(\xi_0) \neq \emptyset \right\} \subseteq \mathcal{E}_{imp} \]

One of the advantages of the implicit system representation is that we can translate properties of a dynamical system into conditions of the matrix pencil $F - sE$. In the remainder of this section, we will summarize the basic results from Geerts and Schumacher (1996a, b), Masubuchi and Shimemura (1997) and Willems (1991).

Definition 1. The pair $(E, F)$ is said to be minimal if

\[ \max_{s \in \mathbb{C}} \text{rank } (F - sE) = m. \]

Since $E \dot{\xi}(t) = F \xi(t)$ represents $m$ differential/algebraic constraints on the system variable $\xi(t)$, the minimality implies that there are no redundant constraints in (1). It may also be noted that $\mathcal{E}_{imp} = \mathbb{R}^n$ holds if $(E, F)$ is minimal (Geerts and Schumacher 1996a, b).

Definition 2. The system $\Sigma$, or the pair $(E, F)$, is said to be autonomous if, for any $\xi_0 \in \mathcal{E}_{imp}$,

\[ \left\{ \xi_1, \xi_2 \in \mathcal{B}_{imp}(\xi_0) \right\} \Rightarrow \left\{ \xi_1(t) = \xi_2(t), \forall t > 0 \right\}. \]

Lemma 1. The system $\Sigma$ is autonomous if and only if

\[ \max_{s \in \mathbb{C}} \text{rank } (F - sE) = n. \]

Definition 3. (i) The system $\Sigma$, or $(E, F)$, is said to be controllable in the sense of behavior if there exist $\xi \in \mathcal{B}$ and $T \geq 0$ such that

\[ \xi(t) = \begin{cases} \xi_1(t) & \text{for } t \leq 0 \\ \xi_2(t - T) & \text{for } t \geq T \end{cases} \]

for any $\xi_1, \xi_2 \in \mathcal{B}$, where $\mathcal{B}$ is the smooth behavior defined over $\mathbb{R}$.

(ii) The system $\Sigma$ is said to be impulse controllable if there exists $\xi \in \mathcal{C}_{sm}$ for all $\xi_0 \in \mathcal{E}_{imp}$, namely $\mathcal{E}_{imp} = \mathcal{C}_{sm}$.

Lemma 2. (i) The system $\Sigma$ is controllable in the behavioral sense if and only if rank $(F - sE)$ is constant for all $s \in \mathbb{C}$.

(ii) Suppose that $(E, F)$ is minimal. Then, the system $\Sigma$ is impulse controllable if and only if

\[ \text{im } E + F \ker E = \mathbb{R}^n. \]

Definition 4. The implicit system $\Sigma$ is said to be internally stable if

(i) The system $\Sigma$ is autonomous.

(ii) The system $\Sigma$ is impulse-free, namely,

\[ \mathcal{B}_{imp}(\xi_0) = \mathcal{B}_{sm}(\xi_0) \forall \xi_0 \in \mathcal{E}_{imp}. \]

(iii) $\lim_{t \to \infty} \xi(t) = 0$ $\forall \xi_0 \in \mathcal{E}_{imp}$.

Lemma 3. The following statements are equivalent.

(i) The implicit system $\Sigma$ is internally stable.

(ii) The pencil $F - sE$ has full column rank for all $s \in \mathbb{C} \cup \{\infty\}$.

(iii) (a) The pencil $F - sE$ has full column rank for all $s \in \mathbb{C}^+$. (b) $\ker E \cap F^{-1} \text{im } E = \{0\}$.

where $\mathbb{C}^+ := \{ s \in \mathbb{C} \mid \text{Re}[s] \geq 0 \}$.

3. PASSIVITY

The notion of passivity, which originally comes from the network theory (Anderson and Vongpanitlerd 1973), is defined as follows:

Suppose that a system contains no stored energy at $t = 0$. The total energy delivered to the system from any external energy source, computed over any time interval $[0, \tau]$, is always non-negative.
We will see how the above passivity is formulated for a linear implicit system $\Sigma$ in (1). Since the interaction between $\Sigma$ and its environment is made through the external variable $w$, the instantaneous power flow into the system can be expressed in terms of $w$. Hence, we can formulate the passivity of the implicit system $\Sigma$ as follows.

**Definition 5.** Let a function $s: \mathbb{R}^l \rightarrow \mathbb{R}$ be given.

(i) The implicit system $\Sigma$ is said to be **passive** with respect to $s$ if

$$\int_{0}^{t} s(w(t)) \, dt \geq 0,$$

$$\forall t \geq 0, \, \forall \xi \in \mathcal{B}_{s} (\xi_0), \, \forall \xi_0 \in \ker E$$

(ii) The implicit system $\Sigma$ is said to be **strongly passive** with respect to $s$ if there exists a constant $\varepsilon > 0$ such that

$$\int_{0}^{t} s(w(t)) \, dt \geq \varepsilon \int_{0}^{t} \xi^T(t) \xi(t) \, dt,$$

$$\forall t \geq 0, \, \forall \xi \in \mathcal{B}_{s} (\xi_0), \, \forall \xi_0 \in \ker E$$

The function $s$ is called a **supply rate** since it represents an instantaneous power flow into the system.

In the following, we consider the passivity with respect to the quadratic supply rate

$$s(w) = w^T \Pi w$$

where $\Pi$ is a symmetric constant matrix.

We make the following assumptions for the system $\Sigma$.

(A3.1) The pair $(E,F)$ is minimal.

(A3.2) The system $\Sigma$ is controllable in the behavioral sense.

(A3.3) The system $\Sigma$ is impulse controllable.

The passivity is closely related to the dissipation theory for a dynamical system. It was shown by Willems (1971) and Trentelman and Willems (1991) that the passivity is equivalent to the existence of a non-negative storage function. Hence, the following theorem can be easily proved in the same way as the dissipation theory for implicit systems (Takaba 1992, 2000).

**Theorem 1.** Suppose that the assumptions (A3.1)–(A3.3) hold, and let a constant symmetric matrix $\Pi$ be given. The implicit system $\Sigma$ is passive with respect to $w^T \Pi w$ if and only if there exists a constant matrix $X$ satisfying the constrained linear matrix inequality (CLMI)

$$E^T X = X^T E \geq 0 \quad (4a)$$

$$F^T X + X^T F \leq G^T \Pi G \quad (4b)$$

Before giving the proof of Theorem 1, we consider the relation between the passivities of the implicit system $\Sigma$ and a state-space system. Under the assumptions (A3.2)–(A3.3), there exist nonsingular matrices $U, V$ such that

$$UEV = \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad U^T V = \begin{bmatrix} A_0 & B_0 & B \\ C_0 & I & 0 \end{bmatrix}. \quad (5)$$

By defining

$$V^{-1} \xi(t) = \begin{bmatrix} x^T(t) & f^T(t) & u^T(t) \end{bmatrix}^T,$$

we obtain

$$\dot{x}(t) = A_0 x(t) + B_0 f(t) + B u(t) \quad (7a)$$

$$f(t) = -C_0 x(t) \quad (7b)$$

Eliminating $f(t)$ from these equations, we see that the implicit system representation (1) is equivalent to the state-space representation

$$\dot{x}(t) = Ax(t) + Bu(t), \quad A := A_0 - B_0 C_0. \quad (8)$$

It is easy to verify that the system (8) is controllable in the sense of state-space systems under the assumptions (A3.1)–(A3.3).

Furthermore, by substituting (7b) into (6), we get

$$\xi(t) = V \begin{bmatrix} I & 0 \\ -C_0 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix},$$

and

$$w^T \Pi w = \xi^T G^T \Pi G \xi = \begin{bmatrix} x^T \\ u \end{bmatrix} \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} =: s(x,u) \quad (9)$$

where

$$\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} := \begin{bmatrix} I & 0 \\ -C_0 & 0 \\ 0 & I \end{bmatrix} V^T G^T \Pi G V \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}. \quad (10)$$

Thus, the passivity with respect to $w^T \Pi w$ of the implicit system $\Sigma$ is equivalent to the passivity with respect to the quadratic supply rate $s(x,u)$ of the state-space system (8).

**Proof of Theorem 1.** According to Willems (1971), the state-space system (8) is passive with respect to the quadratic supply rate $s(x,u)$ if and only if there exists a non-negative symmetric matrix $P \geq 0$ satisfying the linear matrix inequality (LMI)

$$\begin{bmatrix} A^T P + PA & PB \\ B^T P & 0 \end{bmatrix} \preceq \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix}. \quad (11)$$

Since the passivity of $\Sigma$ is equivalent to prove that of the state-space system (8), it suffices to that the LMI condition (11) is equivalent to the CLMI condition (4). We here define

$$U^{-T} XV = \begin{bmatrix} X_1 & X_2 & X_3 \\ X_4 & X_5 & X_6 \end{bmatrix}. \quad (12)$$
It follows from (4a),(5) and (12) that
\[ X_1 = X_1^T \geq 0, \quad X_2 = 0, \quad X_3 = 0. \]
Moreover, we see from (4b),(5) and (12) that
\[
\begin{bmatrix}
    A_0^T X_1 + X_1 A_0^T + C_0^T X_1 - C_0 X_2 + X_2 B_0 \\
    B_0^T X_1 + X_1 A_0^T & X_1 + X_1^T & X_3 & X_4 \\
    B^T X_1 + X_1 A_0^T & X_1 + X_1^T & X_5 & X_6
\end{bmatrix}
\leq
\begin{bmatrix}
    M_{11} & M_{12} & M_{13} \\
    M_{12} & M_{22} & M_{23} \\
    M_{13} & M_{23} & M_{33}
\end{bmatrix}
\]
(13)
where
\[
\begin{bmatrix}
    M_{11} & M_{12} & M_{13} \\
    M_{12} & M_{22} & M_{23} \\
    M_{13} & M_{23} & M_{33}
\end{bmatrix} := V^T G^T IGV.
\]
By post-multiplying (13) by a nonsingular matrix
\[ L = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \]
and pre-multiplying by its transpose, the CLMI (4) is reduced to the LMI
\[
\begin{bmatrix}
    A^T X_1 + X_1 A & X_1 B_1^T - C_0^T X_1^T + X_1 B_0 \\
    B^T X_1 & 0 & X_3 \\
    X_4 - X_2^T C_0^T + B^T X_1 & X_6 & X_5 + X_5^T
\end{bmatrix}
\leq
\begin{bmatrix}
    Q & S & M_{12} - C_0^T M_{22} \\
    S^T & R & M_{12} - M_{22}^T C_0 \\
    M_{12} - M_{22}^T C_0 & M_{23} & M_{22}
\end{bmatrix}
\]
(14)
where \( X_1 = X_1^T \geq 0, X_4, X_5, X_6 \) are the variables.

We assume that \( X_1, X_4, X_5, X_6 \) satisfy the LMI (14). Then, we see from the upper left 2 \times 2 block of (14) that
\[ A^T X_1 + X_1 A \leq Q S, \]
(15)
Obviously, \( X_1 = X_1^T \geq 0 \) is a solution to the LMI (11).

Conversely, suppose that \( P = P^T \geq 0 \) is a solution of the LMI (11). Then,
\[
\begin{align*}
X_1 &= P \geq 0 \\
X_4 &= X_6 C_0 - B_0^T P + M_{12} - M_{22} C_0 \\
X_5 &= N + \frac{1}{2} M_{22} - N^T \leq 0 \\
X_6 &= M_{23}
\end{align*}
\]
(16)
satisfy the LMI (14).

Consequently, we have proved that \( \Sigma \) is passive with respect to \( w^T P w \) if and only if the CLMI (4) has a solution.

We also obtain a necessary and sufficient condition for the strong passivity as a corollary of Theorem 1.

\begin{corollary}
Suppose that the assumptions (A3.1)–(A3.3) hold, and let a constant symmetric matrix \( \Pi \) be given. The implicit system \( \Sigma \) is strongly passive with respect to \( w^T P w \) if and only if there exists a constant matrix \( X \) satisfying the CLMI
\[
\begin{align*}
E^T X &= X^T E \geq 0 \\
F^T X + X^T F &< G^T \Pi G.
\end{align*}
\]
(17)
\end{corollary}

\[ w^T \xi(t) \leq \varepsilon \] for some \( \varepsilon > 0 \). In the same way as the proof of Theorem 1, we can show that the above inequality implies the existence of a matrix \( X \) satisfying
\[
\int_0^t \xi^T(t) [G^T \Pi] G \xi(t) dt \geq 0
\]
(18)
Conversely, if the CLMI (16) has a solution \( X \), there exists a \( \varepsilon > 0 \) such that (18) holds. Then, we obtain (17) by simple calculation. Thus, the system \( \Sigma \) is strongly passive with respect to \( w^T P w \).  

\[ \square \]

4. STABILITY OF INTERCONNECTED SYSTEM

We consider the interconnection of two implicit systems \( \Sigma_1 \) and \( \Sigma_2 \):
\[
\begin{align*}
\Sigma_1 : \quad & E_1 \dot{x}(t) = F_1 x(t) \\
& w(t) = G_1 x(t)
\end{align*}
\]
(19)
\[
\begin{align*}
\Sigma_2 : \quad & E_2 \dot{y}(t) = F_2 y(t) \\
& w(t) = G_2 y(t)
\end{align*}
\]
(20)
where \( E_1, F_1 \in \mathbb{R}^{m_1 \times n_1}, E_2, F_2 \in \mathbb{R}^{m_2 \times n_2}, G_1 \in \mathbb{R}^{l \times n_1}, \)
\( G_2 \in \mathbb{R}^{l \times n_2} \). We henceforth denote the interconnected system, or the interconnection, by \( \Sigma_1 \wedge \Sigma_2 \).

Since the interconnection relationship of two systems is characterized by
\[ w(t) = G_1 x(t) = G_2 y(t), \]
we obtain the dynamics of \( \Sigma_1 \wedge \Sigma_2 \) in the implicit form
\[
\Sigma_1 \wedge \Sigma_2 : \quad E \dot{z}(t) = F z(t)
\]
(21)
where
\[
E = \begin{bmatrix} E_1 & 0 \end{bmatrix}, \quad F = \begin{bmatrix} F_1 & F_2 \\
0 & G_1 - G_2 \end{bmatrix}
\]
\[ z(t) = \begin{bmatrix} x(t) \\
y(t) \end{bmatrix} \in \mathbb{R}^n, \quad n = n_1 + n_2, \quad m = m_1 + m_2 + l \]
It is obvious from Lemma 3 that the interconnected system $\Sigma_1 \land \Sigma_2$ is internally stable iff the pencil

$$F - sE = \begin{bmatrix} F_1 - sE_1 & 0 \\ 0 & F_2 - sE_2 \\ G_1 & -G_2 \end{bmatrix}$$

(23)

has full column rank for all $s \in \hat{C}_+ \cup \{\infty\}$. Hence, the following condition is necessary for the internal stability of the interconnected system $\Sigma_1 \land \Sigma_2$.

(A4.1) The pencil $\begin{bmatrix} F_1 - sE_1 \\ G_1 \end{bmatrix}$ has full column rank for all $s \in \hat{C}_+ \cup \{\infty\}$.

Note that (A4.1) is equivalent to the zero-detectability of the subsystem $\Sigma_1$ (Masubuchi 2000).

**Theorem 2.** Under the assumption (A4.1), the interconnected system $\Sigma_1 \land \Sigma_2$ is internally stable if there exist constant matrices $X, Y$ and a constant symmetric matrix $\Pi$ which satisfy the CLMI

$$
E_1^TX = X^TE_1 \geq 0 \tag{24a}
$$

$$
F_1^TX + X^TF_1 \leq -G_1^T\Pi G_1 \tag{24b}
$$

$$
E_1^TY = Y^TE_2 \geq 0 \tag{25a}
$$

$$
F_1^TY + Y^TF_2 < G_2^T\Pi G_2 \tag{25b}
$$

**Proof:** We first show that the pencil $F - sE$ in (23) has full column rank for all $s \in \hat{C}_+$. We assume on the contrary that there exist a complex number $\lambda \in \hat{C}_+$ and a nonzero vector $v = [v_1^T \ v_2^T]^T$ satisfying

$$
\begin{bmatrix} F_1 - \lambda E_1 \\ 0 \\ G_1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0
$$

(26)

Then, pre-multiplying (24b) by $v_1^T$ and post-multiplying by $v_1$ yield

$$
v_1^T (F_1^TX + X^TF_1 + G_1^T\Pi G_1) v_1 = 2(\text{Re}\lambda) v_1^T E_1^T X v_1 + v_1^T G_1^T\Pi G_1 v_1 \leq 0
$$

Since $\text{Re}\lambda \geq 0, E_1^TX \geq 0$ and $G_1 v_1 = G_2 v_2$, we obtain

$$
v_1^T G_1^T\Pi G_2 v_2 \leq 0.
$$

(27)

Similarly, pre-multiplying (25b) by $v_2^T$ and post-multiplying by $v_2$ yield

$$
v_2^T (F_2^TY + Y^TF_2 - G_2^T\Pi G_2) v_2 = 2(\text{Re}\lambda) v_2^T E_2^T Y v_2 - v_2^T G_2^T\Pi G_2 v_2 \leq 0
$$

where the equality holds only when $v_2 = 0$. Since $\text{Re}\lambda \geq 0$ and $E_2^TY \geq 0$, we have

$$
v_2^T G_2^T\Pi G_2 v_2 \geq 2(\text{Re}\lambda) v_2^T E_2^T Y v_2 \geq 0
$$

(28)

It follows from (27) and (28) that

$$
v_1^T G_1^T\Pi G_2 v_2 = 2(\text{Re}\lambda) v_2^T E_2^T Y v_2 = 0
$$

Hence, we get

$$v_2^T (F_2^TY + Y^TF_2 - G_2^T\Pi G_2) v_2 = 0
$$

This implies that $v_2 = 0$ and $G_2 v_2 = 0$. Since we have $(F_1 - \lambda E_1) v_1 = 0$ and $G_1 v_1 = 0$, we obtain $v_1 = 0$ from the assumption (A4.1). This contradicts the assumption of $v = [v_1^T \ v_2^T]^T \neq 0$. Hence, the pencil $F - sE$ has full column rank for all $s \in \hat{C}_+$.

We next show that $F - sE$ has full column rank at $s = \infty$, namely $\ker E_1 \cap F^{-1}\ker F_1 = \{0\}$. We assume that there exists a nonzero vector $u \in \ker E_1 \cap F^{-1}\ker F_1$. Then, there exist constant vectors $w_1$ and $w_2$ such that

$$
\begin{bmatrix} E_1 & 0 \\ 0 & E_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} F_1 & 0 \\ 0 & F_2 \\ G_1 & -G_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}
$$

with $u = [u_1^T \ u_2^T]^T$. Under the conditions of (29), pre-multiplying (24b) by $u_1^T$ and post-multiplying by $u_1$ yield

$$
0 = u_1^T (-F_1^TX - X^TF_1) u_1 \geq u_1^T G_1^T\Pi G_1 u_1 = u_2^T G_2^T\Pi G_2 u_2
$$

(30)

Similarly, pre-multiplying (25b) by $u_2^T$ and post-multiplying by $u_2$ yield

$$
0 = u_2^T (F_2^TY + Y^TF_2) u_2 \leq u_2^T G_2^T\Pi G_2 u_2
$$

(31)

where the equality holds iff $u_2 = 0$. It follows from (30) and (31) that $u_2 = 0$ and hence $G_1 u_1 = 0$. Together with (29), this implies $u_1 \in \ker E_1 \cap F^{-1}\ker F_1$.

Then, we obtain $u_1 = 0$ from the assumption (A4.1). This contradicts $u = [u_1^T \ u_2^T]^T \neq 0$. Hence, the pencil $F - sE$ has full column rank at infinity.

Consequently, we conclude by Lemma 3 that the interconnected system $\Sigma_1 \land \Sigma_2$ is internally stable since $F - sE$ has full column rank for all $s \in \hat{C}_+ \cup \{\infty\}$. ■

**Remark:** As pointed out in Takaba (1999,2000), the CLMI in Theorems 1.2 and Corollary 1 can be equivalently converted to linear matrix inequalities (LMIs) that do not have any equality constraints. Therefore, we can easily check the stability of the interconnected systems by using existing convex optimization softwares such as LMI Control Toolbox.

In view of Theorem 1 and Corollary 1, we can easily notice that Theorem 2 is a generalization of the passivity theorem for linear time-invariant feedback systems.

**Corollary 2.** In addition to (A4.1), we assume

(A4.2) The pair $(E_1, F_1)$ is minimal, controllable in the behavioral sense, and impulse controllable.
(A4.3) The pair \((E_2, F_2)\) is minimal, controllable in the behavioral sense, and impulse controllable.

Then, the interconnected system \(\Sigma_1 \land \Sigma_2\) is internally stable if there exists a constant symmetric matrix \(\Pi\) such that

(i) The system \(\Sigma_1\) is passive with respect to \(-w^T \Pi w\).

(ii) The system \(\Sigma_2\) is strongly passive with respect to \(w^T \Pi w\).

5. CONCLUSION

In this paper, we have derived a necessary and sufficient condition for the passivity of an implicit system in terms of a CLMI. Based on this CLMI condition, we have also derived a sufficient condition for the internal stability of an interconnection of two implicit systems as a generalization of the passivity theorem. The result in this paper will be a basic tool for robust stability analysis and robust stabilization of implicit systems.

It also remains as a future research topic to extend the present results to the stability analysis of an interconnection of time-varying or nonlinear systems, since the traditional passivity theorem is applicable to such a kind of feedback systems. (Vidyasagar 1993, van der Schaft 1996, Megretski and Rantzer 1997).

References


APPENDIX

We here introduce the definition of impulsive-smooth distributions (Geerts and Schumacher 1996a, 1996b).

We denote the set of distributions defined on \(\mathbb{R}\) with support on \([0, \infty)\) by \(\mathcal{K}\). The set \(\mathcal{K}\) is closed under convolution. It should be noted that typical examples of the elements of \(\mathcal{K}\) is the Dirac delta distributions \(\delta\) and its derivatives.

Linear combinations of the delta distribution and its derivatives are called impulsive distributions, and the set of these distributions is denoted by \(\mathcal{E}_{p-imp}\). A general form of an impulsive distribution is given by \(\sum_{i=0}^{l} c_i \delta^{(i)}\), where \(c_i (i = 1, 2, \ldots, l)\) is a real-valued constant vector with appropriate dimension and where \(\delta^{(i)}\) denotes the \(i\)-th derivative of \(\delta\).

Another subset of \(\mathcal{K}\) is the set of smooth distributions \(\mathcal{E}_{sm}\), which can be identified with the functions continuously differentiable on the interval \((0, \infty)\).

We here define the set of impulsive-smooth distributions \(\mathcal{E}_{imp}\) which is the set of distributions with support on \([0, \infty)\) as the direct sum of \(\mathcal{E}_{p-imp}\) and \(\mathcal{E}_{sm}\):

\[\mathcal{E}_{imp} = \mathcal{E}_{p-imp} + \mathcal{E}_{sm}\]

The element \(x \in \mathcal{E}_{imp}\) is expressed as

\[x = x_{p-imp} + x_{sm} \quad x_{p-imp} \in \mathcal{E}_{p-imp}, \quad x_{sm} \in \mathcal{E}_{sm}\]