THE GRÖBNER BASIS METHOD IN CONTROL DESIGN: AN OVERVIEW

D. Nešić I.M.Y. Mareels * T. Glad ** M. Jirstrand ***

* Dept. Electrical and Electronic Engineering, University of Melbourne, Parkville VIC 3010 Australia, d.nesic@ee.mu.oz.au, iven@ee.mu.oz.au
This work was supported by the Australian Research Council under the Large Grants Scheme.
** Division of Automatic Control, Linköping University, S-581-83, Linköping, Sweden
*** MathCore AB, Wallenbergsgata, S-583-35, Linköping, Sweden

Abstract: The paper discusses the usefulness of Gröbner bases methods in a variety of control problems for a class of polynomial systems. Polynomial systems are described by difference or differential equations in which the transition map or vectorfield are polynomials. Gröbner bases methods are useful in both the analysis and the design of polynomial control systems.

Keywords: Gr"obner bases, polynomial systems, control design, symbolic computation

1. INTRODUCTION

The topic of this article is to highlight the effectiveness of symbolic computation software in the analysis and design of control systems. In particular the usefulness of Gröbner bases (D. Cox and O'Shea, 1992) in the context of polynomial control systems is illustrated. A more detailed exposition of this material and a more complete list of references can be found in (Nešić et al., 2001).

We are motivated to consider polynomial systems, as they are the natural generalization of linear systems. Also, because polynomials are universal approximators, polynomial control models can be used as valid models for almost any physical system. This also implies that most likely any possible difficulty that can be encountered in nonlinear systems will also occur in the class of polynomial systems. Finally observe that any nonlinear function (trigonometric, exponential functions) that can be considered as a solution of a polynomial differential or algebraic equation can be treated within the same framework through the introduction of a number of extra states in the model.

The class of polynomial systems considered here is described by:

\[ \sigma x(t) = f(x(t), u(t)) \]
\[ y(t) = h(x(t), u(t)). \] (1)

Here \( x \in \mathbb{R}^n, y \in \mathbb{R}^p, u \in \mathbb{R}^m \) represent respectively the state, the observed output and the manipulated control input of the system. The operator \( \sigma \) is either the derivative \( \frac{dx(t)}{dt} \) for continuous time systems (when \( t \in \mathbb{R} \)) or the forward difference \( \sigma x(t) = x(t + 1) \) for discrete-time systems (when \( t \in \mathbb{N} \)). \( \mathbb{R}, \mathbb{Q}, \mathbb{N} \) and \( \mathbb{C} \) denote respectively the sets of real, rational, natural and complex numbers. The vector functions \( f \) and \( h \) are assumed to have entries that are polynomial
functions in all their variables. The coefficients are assumed to be either rational numbers or integers.

In the context of polynomial control systems, the problems of determination of equilibria and the domain of attraction of a stable closed loop system, as well as testing controllability and observability reduce naturally to the analysis of sets of algebraic polynomial equations and can effectively be addressed using the Gröbner basis method. For an overview of the usefulness of Gröbner basis ideas in the context of linear control systems refer to (Munro, 1999).

2. THE GRÖBNER BASIS METHOD

The central objects in the Gröbner basis theory are polynomial ideals and affine varieties (D. Cox and O’Shea, 1992).

Let \( p_1, \ldots, p_s \) be multivariate polynomials in the variables \( x_1, \ldots, x_n \) whose coefficients are in the field \( k \). The variables \( x_1, \ldots, x_n \) are considered as place markers in the polynomial. The notation \( p_1, \ldots, p_s \in k[x_1, \ldots, x_n] \) is adopted. The affine variety (or variety) defined by the \( s \) polynomials \( p_1, \ldots, p_s \) is the collection of all solutions in \( k^n \) of the system of equations:

\[
\begin{align*}
p_1(x_1, \ldots, x_n) &= 0; \quad \ldots; \quad p_s(x_1, \ldots, x_n) &= 0. 
\end{align*}
\]

Formally, a variety \( V \) is

\[
V(p_1, \ldots, p_s) := \{(a_1, \ldots, a_n) \in k^n : p_i(a_1, \ldots, a_n) = 0, \quad i = 1, \ldots, s \}. 
\]

For instance, a straight line, a parabola, an ellipse, a hyperbola and a single point are all examples of varieties in \( \mathbb{R}^2 \). The polynomial ideal \( I \) that is generated by \( p_1, \ldots, p_s \) is the set of polynomials obtained by combining these polynomials through multiplication and addition with other similar multivariate polynomials, formally:

\[
I = \langle p_1, \ldots, p_s \rangle = \{ f = \sum_{i=1}^{s} g_ip_i : g_i \in k[x_1, \ldots, x_n] \}. 
\]

Polynomials \( p_i, i = 1, \ldots, s \) form a basis for the ideal \( I \). A very useful interpretation of a polynomial ideal \( I \) is in terms of the equations (2).

By multiplying \( p_i \) by arbitrary polynomials \( g_i \in k[x_1, \ldots, x_n] \) and adding them up, the implication of (2) is that \( f = g_1p_1 + \ldots + g_sp_s = 0 \), and of course \( f \in I \). Hence, the ideal \( I = \langle p_1, \ldots, p_s \rangle \) contains all the polynomial consequences of the equations (2).

A notion at the core of the Gröbner basis method is that of monomial ordering (a monomial is a polynomial consisting of a single term). It introduces an appropriate extension of the notion of leading term and leading coefficient familiar from univariate polynomials to multivariate polynomials. Unlike in the univariate case, many different monomial orderings exist: lexicographic, graded lexicographic, graded reverse lexicographic, etc. (D. Cox and O’Shea, 1992). In the sequel we only consider the so called lexicographic or lex ordering. Let \( \alpha, \beta \) be two \( n \)-tuple of integers \( (\alpha = (\alpha_1, \ldots, \alpha_n), \beta = (\beta_1, \ldots, \beta_n) \in \mathbb{N}^n) \). \( \alpha \) is said to succeed \( \beta \) in the lex ordering, denoted as \( \alpha \triangleright \beta \) if in the vector difference \( \alpha - \beta = (\alpha_1 - \beta_1, \ldots, \alpha_n - \beta_n) \), the left-most nonzero entry is positive. One can define \( n! \) lex orderings for polynomials in \( n \) variables. For the polynomial \( f = 2x_1^2x_2^3 + 4x_1^2x_2^2 \), using the lex ordering \( x_1 \triangleright x_2 \triangleright x_3 \) the monomial \( x_1^2x_2^3 \) succeeds the monomial \( x_1^2x_2^2 \) as \( (3, 1, 3) \triangleright (3, 0, 5) \). With this ordering, the leading coefficient and the leading term are respectively \( \text{LC}(f) = 2 \) and \( \text{LT}(f) = 2x_1^2x_2^2 \). On the other hand, the leading term is \( \text{LT}(f) = 4x_1^2x_2^5 \) using lex ordering \( x_3 \triangleright x_2 \triangleright x_1 \).

In general, an ideal \( I \) does not have a unique basis, but given any two different bases \( \langle p_1, \ldots, p_s \rangle \) and \( \langle g_1, \ldots, g_t \rangle \) of \( I \), the varieties \( V(p_1, \ldots, p_s) \) and \( V(g_1, \ldots, g_t) \) are equal. In other words, a variety only depends on the ideal generated by its defining equations. Some bases of an ideal may be simpler or better in some sense than some other bases. Intuitively, if all the polynomials in a given basis of an ideal have a degree that is lower than the degree of any other polynomial in the ideal with respect to a particular monomial ordering, then this basis must be in some sense the simplest basis. In particular, a Gröbner basis of an ideal for a given monomial ordering has such a property and can be thought of as the simplest or canonical basis. Given an ideal \( I \) and monomial ordering, denote the set of leading terms of elements of \( I \) as \( \text{LT}(I) \). The ideal generated by elements of \( \text{LT}(I) \) is denoted \( \langle \text{LT}(I) \rangle \). In general, the ideal generated by the leading terms of a particular ideal \( I \) is not the same as the ideal generated by the leading terms of polynomials in a basis for that particular ideal \( I \). A Gröbner basis is a special basis for which this property does hold and it is formally defined as the set of polynomials \( g_1, \ldots, g_t \) for which \( \langle \text{LT}(I) \rangle = \langle \text{LT}(g_1), \ldots, \text{LT}(g_t) \rangle \). In computation of Gröbner bases, the user specifies the monomial ordering and different monomial orderings produce different Gröbner bases. Given a monomial ordering, the two most important properties of Gröbner bases are:

(1) Every ideal \( I \subset k[x_1, \ldots, x_n] \) other than the trivial ideal \( \{0\} \) has a Gröbner basis.
Furthermore, any Gröbner basis of an ideal $I$ is a basis for $I$.

(2) Given an ideal $I \subseteq k[x_1, \ldots, x_n]$ other than the trivial ideal $\{0\}$, a Gröbner basis of $I$ can be computed in a finite number of algebraic operations.

The first algorithm for computation of Gröbner bases, published in 1960’s, is attributed to B. Buchberger (D. Cox and O’Shea, 1992). Since then a number of improvements have been reported and the algorithm has been implemented in most commercial symbolic software packages. The Buchberger’s algorithm generalizes two well known algorithms: Gauss elimination for sets of multivariate linear algebraic equations and Euclid’s algorithm for computing the greatest common divisor of a set of univariate polynomials.

$$g_1 = x_1 + x_2 - x_3^2,$$
$$g_2 = x_2^3 + x_2^2 + x_3,$$
$$g_3 = 6x_3 - 3x_3^3 - 2x_3^2 - x_3^3 + 4x_3x_2,$$
$$g_4 = 4x_3 - 8x_3^3 + 3x_3^3 + x_3^1. \quad (5)$$

By construction $g_i = 0, i = 1, 2, 3, 4$ has the same solutions as $p_j = 0, j = 1, 2, 3$ but the polynomials $g_i$ have a better structure than the polynomials $p_j$. Indeed, polynomial $g_4$ depends only on $x_3$ and $g_4 = 0$ can be solved numerically. The solutions can be substituted into $g_2 = 0$ and $g_3 = 0$ to obtain polynomials in $x_2$ only that are again solved numerically. This process of back-substitution can be continued until all solutions are found.

Periodic solutions of the polynomial system $x(k+1) = f(x(k))$ can be found in a similar way. It suffices to solve the polynomial $x = f^n(x)$ to determine the $p$-periodic solutions.

To illustrate how to find periodic solutions in the continuous time case, consider the polynomial system $f \left( \frac{dx}{dt}, \ldots, \frac{dx}{dt}, y, t \right) = 0$, where $y$ can be thought of as the output of a closed-loop control system. To approximately compute a periodic solution of this system, the method of harmonic balancing can be used (Mees, 1981). It can be shown under certain conditions that if an approximate periodic solution exists, then the system has a periodic solution that is close to the approximate one. A truncated Fourier series can be considered as a candidate approximate solution $y(t) = \sum_{k=-N}^{+N} c_k e^{-jkd\omega}$, with $c_k \in \mathbb{C}$, and such that $c_k$ and $c_{-k}$ are complex conjugates. Postulating that $y(t)$ is a solution, leads to a set of equations and since $f$ is polynomial, all these equations are polynomial. Using the Gröbner basis method withlex ordering, this set of equations can be solved for $c_{0}, c_1, \ldots, c_N$ and $\omega$ to obtain $y(t)$. However, in practice the actual solution may not be so important and it is often more relevant to know if a solution exists and what is the oscillation frequency $\omega$. This can be obtained by finding the Gröbner basis of the polynomial equations with lex ordering where $\omega$ is the lowest ranking variable.

Example 2. Consider the van der Pol equation $\dot{y} - a(1 - by^2)y + y = 0$, and postulate the periodic solution $y(t) = \sum_{k=-3}^{+3} c_k e^{-jkd\omega}$, where $c_k = c_{kr} + c_{kr}$, $k = 0, 1, 2, 3$. Substitute $y(t)$ into the van der Pol differential equation and equate the coefficients on the left hand and right hand sides of the equation to obtain four polynomial equations: $p_1 := -c_{1r} \omega^2 - abc_{1r} c_{34} \omega + c_{1r} =$
0; \ p_2 := 2abc_1c_3^2\omega + abc_1^2c_3\omega + 2abc_1c_3\omega + \
abc_1^2\omega - ac_1\omega = 0; \ p_3 := -9c_3\omega^2 - 3abc_3c_3\omega - 
3abc_3\omega - 6abc_1c_3\omega + 3ac_3\omega + c_3 = 0; \ p_4 := 
3abc_3\omega - 9c_3\omega^2 + 3abc_1c_3\omega + 6abc_1c_3\omega - 
3ac_3\omega + abc_1\omega + c_3 = 0, \ c_0 = 0, \ c_2 = 0 \ 
and \ c_1 = 0. \ The \ coefficients \ a \ and \ b \ are \ regarded \ as \ 
variables \ and \ to \ simplify \ computations \ p_1 \ is \ divided \ by \ 
c_1, \ and \ p_2 \ by \ c_1\omega. \ A \ Gröbner \ basis \ of \ ideal \ \( \langle p_1, p_2, p_3, p_4 \rangle \) \ with \ the \ lex \ ordering \ \( c_3 \triangleright c_2 \triangleright c_1 \triangleright \omega \triangleright \) \ contains \ 7 \ polynomials \ which \ are \ large \ (42 \ pages \ of \ Maple \ output) \ but \ it \ contains \ one \ polynomial \ in \ \omega \ and \ a \ only: \ \( g_7 := \frac{32490\omega^{10} + (549a^2 - 6213)\omega^8 + (3745 - 567a^2)\omega^6 + (159a^2 - 842)\omega^4 + (53 - 13a^2)\omega^2 - 1}{625} \). \ Obviously \ for \ a = 0, \ the \ van \ der \ Pol \ oscillator \ is \ a \ simple \ harmonic \ oscillator \ with \ frequency \ \( \omega = 1 \). \ In \ the \ presence \ of \ nonlinear \ damping \ \( a \neq 0 \), \ the \ solution \ of \ \( g_7 = 0 \) \ can \ be \ expanded \ as: \ \( \omega^2 = 1 + a_3a^3 + 
ac0a^2 + ac0a^2 + \ldots \) \ and \ using \ Maple, \ the \ coefficients \ \( a_i \) \ are \ obtained \ \( a_1 = -\frac{1}{8}, \ a_2 = \frac{25}{256}, \ a_3 = -\frac{625}{4065} \), etc.

3.2 Estimating the domain of attraction

Gröbner bases can be used to eliminate some variables of interest from a set of polynomial equations, as illustrated by Example 1, where \( x_1 \) and \( x_2 \) are eliminated from \( g_1 \) in (5). \ The variables that are ranked higher in the lex ordering are eliminated first.

This feature of Gröbner bases can be used to obtain an estimate of the domain of attraction for polynomial systems. \ In general, estimating a domain of attraction is a hard problem. \ One way of obtaining a (possibly conservative) estimate for a domain of attraction is to use Lyapunov functions. \ It is a standard result in Lyapunov theory that if \( x = 0 \) is an equilibrium point for the continuous-time system \( \dot{x} = f(x), \ D \subseteq \mathbb{R}^n \) is a domain containing \( x = 0 \) and \( W : D \rightarrow \mathbb{R} \) is a continuously differentiable function, such that \( W(0) = 0 \), \ and for all \( x \in D - \{0\} \) we have \( W(x) > 0 \) and \( \frac{\partial W}{\partial x} f(x) < 0 \) then \( x = 0 \) is asymptotically stable. \ Given such a Lyapunov function, consider the sets \( \Omega = \{ x \in \mathbb{R}^n : \frac{\partial W}{\partial x} f(x) < 0 \} \) and \( B_d = \{ x \in \mathbb{R}^n : W(x) \leq d \} \). \ If \( B_d \subseteq \Omega \) for some \( d > 0 \), then the set \( B_d \) is an estimate for the domain of attraction.

For polynomial systems with a polynomial Lyapunov function \( W \), Gröbner bases can be used to systematically compute \( B_d \). \ Indeed it is feasible to construct the largest such set. \ In general, the quest is to find \( d \) such that \( B_d \) is as large as possible and still inside \( \Omega \). \ For polynomial systems with polynomial Lyapunov functions, \( W(x) - d \) and \( \frac{\partial W}{\partial x} f(x) \) are polynomials, and hence the boundaries of sets \( B_d \) and \( \Omega \) are varieties. \ At the points where the varieties \( V(W - d) \) and \( V(\frac{\partial W}{\partial x} f(x)) \) touch each other, the gradients of \( W \) and \( \frac{\partial W}{\partial x} f(x) \) are parallel. \ Using this information, a system of \( n + 2 \) polynomial equations in \( n + 2 \) variables is obtained \( W - d = 0, \frac{\partial W}{\partial x} f(x) = 0, \frac{\partial W}{\partial x} f(x)) - \lambda \frac{\partial W}{\partial x} f(x) = 0 \).

Computing a Gröbner basis for the above system of equations, where the variable \( d \) has the least rank in the lex ordering, a polynomial equation in \( d \) only is obtained. \ The least positive solution to this equation is the best value of \( d \) for which \( B_d \subseteq \Omega \) and this yields in turn the best estimate for the domain of attraction that can be obtained with the particular Lyapunov function \( W \).

Example 3. \ Consider the system:

\[
\begin{align*}
x_1 &= -x_2 + 2x_2^2x_2 \\
x_2 &= -x_2
\end{align*}
\]  

with the Lyapunov function \( W(x) = 3x_2^2 + 4x_1x_2 + 4x_2^2 \). \ The relevant polynomials are \( p_1 := 3x_2^2 + 4x_2x_2 + 4x_2^2 - d, \ p_2 := -6x_2^2 + 12x_2x_2 - 8x_1x_2 + 8x_1^2x_2 - 8x_2^2, \ p_3 := 6x_1 + 4x_2 - \lambda(36x_2x_2 - 12x_1 - 8x_2 + 16x_2^2x_2), \ p_4 := 4x_1 + 8x_2 - \lambda(12x_2 - 8x_1 + 16x_2^2x_2 - 16x_2), \) and using the lexicographic ordering \( x_2 \triangleright \lambda \triangleright x_1 \triangleright d \), the Gröbner basis of the ideal \( \langle p_1, p_2, p_3, p_4 \rangle \) can be computed. \ It contains the polynomial \( g(d) = 4d^4 - 147d^2 + 786d + 2048d \). \ The approximate solutions of \( g(d) = 0 \) are \( 0, -1.9223, 8.9657, 29, 707 \). \ The smallest positive value of \( d \) for which there is a common solution to the above system of equations is 8.9657. \ The corresponding estimate for the domain of attraction is therefore \( \{ x \in \mathbb{R}^2 : V(x) \leq 8.9657 \} \).

3.3 Observability and controllability

A special type of Gröbner basis can be used to compare if two ideals are the same or not. \ This is the so called reduced Gröbner basis. \ A reduced Gröbner basis for an ideal \( I \) is a Gröbner basis \( G \) for \( I \) such that: \( LC(p) = 1 \) for all \( p \in G \); and for all \( p \in G \), no monomial of \( p \) lies in \( \langle LT(G - \{ p \}) \rangle \). \ The main property of reduced Gröbner bases is that given an arbitrary ideal \( I \neq \{0\} \) and monomial ordering, \( J \) has a unique reduced Gröbner basis. \ Hence, two ideals \( J_1 \) and \( J_2 \) are the same if and only if their reduced Gröbner bases \( G_1 \) and \( G_2 \) that are computed with the same monomial ordering are the same. \ Most commercial computer algebra systems, such as Maple and Mathematica, have finite algorithms for computation of the reduced Gröbner basis.
The verification of controllability or observability properties can be reduced to the problem of computing maximal control invariant varieties. This is well known for linear systems, but it holds also for polynomial systems of the form $x(t + 1) = f(x(t), u(t))$. For such systems, the variety $V$ is control or input invariant if $f(V, u) \subseteq V$ for all possible $u$. The computation of the maximal invariant subset of a given variety $V$ can be completed in finitely many operations. Consider the defining ideal of the variety $V$, say $J_1 = \langle g_1, \ldots, g_{1,m_1} \rangle$. If the variety corresponding to $J_1$ was control invariant then $g_k \circ f(x, u) = 0$ for all $u$ and all $k = 1, \ldots, m_1$. Polynomials $g_k \circ f(x, u)$ can be viewed as polynomials in $u$ with coefficients in $k[x]$. Denote the collection of all these coefficients as $g_{2,k}$, $k = 1, 2, \ldots, m_2$, with the corresponding ideal $J_2 = \langle g_{2,1}, \ldots, g_{2,m_2} \rangle$. Invariance would imply that $J_1 = J_2$ and if not, then obviously $J_1 \subset J_2$ and the corresponding varieties satisfy $V_1 \supset V_2$. This process can be continued to construct an ascending chain of ideals (or descending chain of varieties) $J_1 \subset J_2 \subset J_3 \subset \ldots$. Any such chain terminates in finitely many steps (D. Cox and O’Shea, 1992). There exists an integer $N$ such that $J_N = J_{N+1}$. The variety $V(J_N)$ is the maximal control invariant subset of $V(J_1)$. Verifying that $J_k = J_{k+1}$ amounts to inspecting the reduced Gröbner bases for $J_k$ and $J_{k+1}$.

Let us illustrate this discussion by considering finite time observability. The discrete-time polynomial system

$$x(t + 1) = f(x(t), u(t))$$
$$y(t) = h(x(t))$$

(7)

is said to be observable if for each pair of initial states $\xi \neq \eta$, there exists an integer $N$ and an input sequence $U_N = \{u(0), \ldots, u(N-1)\}$ such that the solutions starting at $\xi$ and $\eta$ produce different outputs after $N$ steps, that is $h(x(N, \xi, U_N)) \neq h(x(N, \eta, U_N))$. The polynomial $h(x(N, \xi, U_N)) - h(x(N, \eta, U_N))$ can be regarded as a polynomial in elements of $U_N$ with coefficients in $k[\xi, \eta]$. If the ideal generated by these coefficients has a reduced Gröbner basis $\{\xi - \eta_1, \ldots, \xi - \eta_n\}$ then the system is observable. The above discussed algorithm for computing invariant sets streamlines these computations and allows the systematic determination of the integer $N$.

Example 4. Consider the simple Wiener system:

$$x_1(k + 1) = x_2(k)$$
$$x_2(k + 1) = -x_1(k) - 2x_2(k) + u(k)$$
$$y(k) = x_1^2(k)$$

(8)

The system consists of a linear dynamical block and quadratic static nonlinearity, which is at the output of the system. Using the auxiliary linear system

$$\xi_1(k + 1) = \xi_2(k)$$
$$\xi_2(k + 1) = -\xi_1(k) - 2\xi_2(k) + u(k)$$
$$\eta_1(k + 1) = \eta_2(k)$$
$$\eta_2(k + 1) = -\eta_1(k) - 2\eta_2(k) + u(k)$$

(9)

and the ideal $J_1 = \langle \eta_1^2 - \xi_1^2 \rangle$, the following ideals are iteratively constructed: $J_2 = \langle \eta_1^2 - \xi_1^2, \eta_2^2 - \xi_2^2 \rangle$, $J_3 = \langle \eta_1^2 - \xi_1^2, \eta_2^2 - \xi_2^2, (\eta_1 + 2\eta_2)^2 - (\xi_1 + 2\xi_2)^2 \rangle$, $J_4 = \langle \eta_1^2 - \xi_1^2, \eta_2^2 - \xi_2^2, (\eta_1 + 2\eta_2)^2 - (\xi_1 + 2\xi_2)^2, (\eta_1 + 2\eta_2) - (\xi_1 + 2\xi_2), (2\eta_1 + 2\eta_2)^2 - 2(\xi_1 + \xi_2)^2, 2\eta_1 - \xi_1 - \xi_2, -2\eta_1 - 3\eta_2 + 2\xi_1 + 3\xi_2 \rangle$. Using the lexicographic order $\xi_1 > \xi_2 > \eta_1 > \eta_2$, the reduced Gröbner basis for $J_4$ is $G_4 = \{\eta_1 - \xi_1, \eta_2 - \xi_2\}$ and therefore the system (8) is observable with $N = 4$.

4. DIFFERENTIAL ALGEBRA IN CONTROL

A set of tools that is related to Gröbner bases comes from the differential algebra. In the 1980s differential algebra was introduced into control problems, see (Fliess and Glad, 1993). Differential algebra can be used to transform a polynomial system from one representation to another. These concepts were introduced in the 1930s by the American mathematician Ritt, (Ritt, 1950). In particular, the Ritt’s algorithm can be regarded as a natural generalisation of the Buchberger’s algorithm used to compute Gröbner bases. Later a more systematic algebraic treatment was given by Kolchin, (Kolchin, 1973). The theory was also extended to partial differential equations (Kolchin, 1973) and difference equations (Cohn, 1979). Differential algebra plays an important role in realization theory of nonlinear control systems. In the context of control, the problems that lend themselves to differential algebra are varied such as, the determination of observability, identifiability, the calculation of zero dynamics, regulator computations, and tracking control.

5. CONCLUDING REMARKS

Gröbner bases are useful for a range of other control problems, such as the inverse kinematic problem and motion planning in robotics (D. Cox and O’Shea, 1992), the computation of the switching surfaces in the solution of the time optimal control problem (U. Walther and Tannenbaum, 1999),
identifiability, input-output equivalence of different state space realizations, normal forms and zero dynamics (Fortell, 1995), analysis of hybrid control systems, computation of limit cycles for continuous-time and discrete-time polynomial systems, observability of continuous-time polynomial systems, forward accessibility of discrete-time polynomial systems, etc. The Gröbner basis method has also been used to solve the so called cover problem, which has been found to be equivalent to several problems for linear control systems, such as the linear functional observer problem, the model matching problem, the deterministic identification problem and the disturbance decoupling problem (Munro, 1999). Our presentation was focused on the use of Gröbner bases for commutative rings over infinite fields. More precisely, we concentrated on applications of Gröbner bases for polynomials with rational coefficients. Two important related areas are Gröbner bases for commutative rings over finite fields and Gröbner bases for non-commutative rings. The former was used for analysis and synthesis of discrete event dynamic systems in (Germundsson, 1995; Gunnarsson, 1997) and the latter was shown to be useful in solving a range of control theoretic problems that involve manipulations of polynomial matrices in (Kronewitter III, 2000). In particular, in (Kronewitter III, 2000) a new extra term in the expansion of the classical state feedback optimal control problem in the singular perturbation form was obtained using non-commutative Gröbner bases.

An important drawback of the Buchberger’s algorithm is that even with the best known versions of the algorithm, it is still easy to generate examples for which the computation of Gröbner basis takes tremendously long time and/or consumes a huge amount of storage space. The main reasons for this are that the total degree of the intermediate polynomials can be extremely large and that the coefficients of the Gröbner basis demand extremely large integers. Even if the original ideal generators are polynomials of small degrees with simple coefficients, this may be the case. In general, the intermediate polynomials encountered whilst generating a Gröbner basis can have total degrees of the order of $2^d$, where $d$ is the total degree of the ideal generators. Although this appears to be a rather negative result, this illustrates worst case behaviour. On the average the running time and the required storage space seem to be much more manageable. In this context it is worth mentioning that different monomial orderings may demand very different computational requirements. Some experimentation with the lex ordering is recommended.

6. REFERENCES


