THE $H_\infty$ CONTROL PROBLEM FOR NEUTRAL SYSTEMS WITH MULTIPLE DELAYS

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Abstract: This paper presents the $H_\infty$ control problem for linear neutral systems with unknown constant multiple delays, in delay independent case. A sufficient condition for the existence of an $H_\infty$ controller of any order is given in terms of three linear matrix inequalities, when the coefficient $D_{12}$ of the input in the controlled output is zero.

Keywords: Neutral systems, output feedback, $H_\infty$-control.

1. INTRODUCTION

In this paper we consider the $H_\infty$ control problem for linear neutral systems with unknown constant multiple delays in delay independent case. $H_\infty$ control problem is defined as finding a controller such that the $H_\infty$-norm of the closed-loop transfer function is strictly less than an arbitrarily given real $\gamma > 0$. This problem is examined mainly by two approaches: the algebraic Riccati equations (AREs) and the linear matrix inequalities (LMIs). In the literature, various related works for linear systems have been reported, see (e.g. Zhou and Khagonekar (1988); Doyle et. al. (1989), for ARE and Iwasaki and Skelton (1994); Gahinet and Apkarian (1994), for LMI). $H_\infty$ control problem for systems with time-delay has rarely been considered. Recently, the state feedback $H_\infty$-control problem, for linear neutral systems is examined in Mahmoud (2000a,b). The output feedback $H_\infty$ controller design for linear time-delay systems by LMI approach is also achieved in Choi and Chung (1997). But, at the knowledge of the author no paper treats output feedback $H_\infty$-control problem for linear neutral systems.

Consider the $n^{th}$ order linear time-invariant generalized neutral systems $\Sigma$ described by the following equation:

$$\dot{x}(t) - E\dot{x}(t - \tau) = Ax(t) + \sum_{i=1}^{k} A_{di}x(t - d_i) + B_1w(t) + B_2u(t) (1)$$

$$z(t) = C_1x(t) + D_{11}w(t) + D_{12}u(t) (2)$$

$$y(t) = C_2x(t) + D_{21}w(t) + D_{22}u(t) (3)$$

$$x(t_0 + \theta) = \phi(\theta), \ \forall \theta \in [-\max(\tau, d_i), 0], (4)$$

where $i \in \{1, 2, ..., k\}$, $x \in \mathbb{R}^n$ is the plant state, $w \in \mathbb{R}^q$ is any exogenous input, including plant disturbances, measurement noise, etc., $u \in \mathbb{R}^m$ is the control input, $z \in \mathbb{R}^p$ is the regulated output and $y \in \mathbb{R}^k$ is the measured output, $A, A_{di}, B_1, B_2 C_1, C_2$ and $D_{ij}$, for $i, j = 1, 2$ are known real constant matrices of the apropriate dimensions. $\tau > 0$ and all $d_i > 0$'s are unknown constant delays, $\phi \in \mathcal{C}_{\tau,n}$, where $\mathcal{C}_{\tau,n} = \mathcal{C}([-\tau, 0], \mathbb{R}^n)$ be the space of continuous functions taking $[-\tau, 0]$ into $\mathbb{R}^n$. It is assumed that $D_{22} = 0$. It should be noted that this assumption involve no loss of generality, while considerably simplifying algebraic manipu-
where the controller $\Sigma_c$ is given by Assumption 1.1. The triple $(A, B_2, C_2)$ is stabilizable and detectable.

Assumption 1.2. $\lambda |E| < 1$.

We remark that $\Sigma$ is a continuous-time model for which Assumption 1 is quite standard. However, Assumption 2 gives a condition in the discrete-time sense and its role will be clarified in the subsequent analysis.

Consider the $n_c$th order linear time-invariant dynamic $(n_c > 0)$ and static $(n_c = 0)$ controllers

$$\dot{x}_c(t) = K_{21}y(t) + K_{22}x_c(t)$$

$$u(t) = K_{11}y(t) + K_{12}x_c(t)$$

where $x_c \in \mathbb{R}^{n_c}$ is the controller state, $K_{11}$, $K_{12}$, $K_{21}$ and $K_{22}$ have appropriate dimensions. We shall denote the class of controllers by $\Sigma_c$.

Let $x_c(t) = [x^T(t) \ x_c^T(t)]^T$. Then, the closed-loop system, $\Sigma_{cl}$ is the following;

$$\dot{x}_c(t) = EF\dot{x}_c(t - \tau)$$

$$A\dot{x}_c(t) + \sum_{i=1}^k \bar{A}_i F x_c(t - d_i) + Bw(t)$$

$$z(t) = \bar{C} x_c(t) + Dw(t)$$

where

$$\bar{A} = \bar{A} + \bar{B}_2 K \bar{C}_2, \quad \bar{B} = \bar{B}_1 + \bar{B}_2 K \bar{D}_21,$$

$$\bar{C} = \bar{C}_1 + \bar{D}_1 K \bar{C}_2, \quad \bar{D} = \bar{D}_11 + \bar{D}_1 K \bar{D}_21$$

$$F^T = \begin{bmatrix} I \\ 0 \end{bmatrix}, E = \begin{bmatrix} E \\ 0 \end{bmatrix}, \bar{A} = \begin{bmatrix} A \\ 0 \end{bmatrix}$$

$$\bar{A}_i = \begin{bmatrix} \bar{A}_i \\ 0 \end{bmatrix}, \bar{B}_i = \begin{bmatrix} \bar{B}_i \\ 0 \end{bmatrix}, \bar{B}_2 = \begin{bmatrix} B_2 \\ 0 \end{bmatrix}$$

$$K = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}, \bar{C}_2 = \begin{bmatrix} C_2 \\ 0 \end{bmatrix}, \bar{D}_21 = \begin{bmatrix} D_{21} \\ 0 \end{bmatrix}$$

$$\bar{C}_1 = \begin{bmatrix} C_1 \\ 0 \end{bmatrix}, \bar{D}_12 = \begin{bmatrix} D_{12} \\ 0 \end{bmatrix}$$

The closed-loop transfer matrix $T_{zw}(s)$ from $w$ to $z$ is given by

$$T_{zw}(s) = \bar{D} + \bar{C} \left( sI - EF e^{-st} \right)^{-1} \bar{A} - \sum_{i=1}^k \bar{A}_i F e^{-sd_i} \right)^{-1} \bar{B}$$

Definition 1.3. Given a scalar $\gamma > 0$. The controller $\Sigma_c$ is said to be an $H_{\infty}$-controller if the following two conditions hold:

(i) $\bar{A}$ is asymptotically stable,

(ii) $\| T_{zw} \|_\infty < \gamma$.

Lemma 1.4. (Schur complement). Given constant matrices $\Omega_1$, $\Omega_2$ and $\Omega_3$ where $0 < \Omega_1 = \Omega_1^T$ and $0 < \Omega_2 = \Omega_2^T$ then $\Omega_1 + \Omega_2 \Omega_3^{-1} \Omega_3 < 0$ if and only if

$$\left[ \begin{array}{cc} \Omega_1 & \Omega_3^T \\ \Omega_3 & -\Omega_2 \end{array} \right] < 0.$$

Lemma 1.5. Given a symmetric matrix $\Omega$ and two matrices $\Gamma$ and $\Sigma$ with appropriate dimensions. The inequality

$$\Omega + \Sigma \Gamma + (\Sigma \Gamma)^T < 0$$

is solvable for $K$ if and only if

$$\bar{\Gamma}^T \Omega \bar{\Gamma} < 0, \quad \Sigma \Omega \Sigma^T < 0$$

where $\bar{\Gamma}$ and $\bar{\Sigma}$ denote the orthogonal complements of $\Gamma$ and $\Sigma$, respectively.


2. THE MAIN RESULTS

Define

$$W = \bar{A}^T P + P \bar{A} + \bar{Q} + \sum_{i=1}^k \bar{S}_i + \bar{C}^T \bar{C}$$

$$+ (P \bar{B} + \bar{B}^T \bar{D}) \Phi^{-1} (P \bar{B} + \bar{C}^T \bar{D})^T$$

$$+ \Psi \Phi^{-1} \Phi^T \Psi + \sum_{i=1}^k \bar{P} \bar{A}_i \bar{S}_i^{-1} \bar{A}_i^T \bar{P}$$

$$\Phi = \gamma^2 I - D^T D$$

$$\bar{R} = Q - E^T (\bar{C}^T \bar{C} + Q + \sum_{i=1}^k \bar{S}_i + \bar{C}^T \bar{C}) E,$$

$$\Psi = P \bar{A} + \bar{Q} + \sum_{i=1}^k \bar{S}_i + \bar{C}^T \bar{C}$$

$$+ (P \bar{B} + \bar{C}^T \bar{D}) \Phi^{-1} \Phi^T \Phi \bar{C}$$

where $\bar{S}_i = F^T S_i F$ and $\bar{Q} = F^T Q F$.

Theorem 2.1. Subject to Assumptions 1 and 2 the closed-loop neutral systems $\Sigma_{cl}$ with multiple delay is asymptotically stable independent of delay and the $H_{\infty}$ performance bound constraint $\| T_{zw} \|_\infty < \gamma$ holds for a given $\gamma > 0$, if there exist matrices $0 < P_T = P_0 < Q_T = Q$ and $0 < S_i^T = S_i$ for $i = 1, 2, ..., k$ satisfying

$$W < 0$$

while

$$\Phi > 0, \quad R > 0$$
Proof 2.2. Let a Lyapunov-Krasovskii functional
\[ V(x_i) = [x_e(t) - \bar{E}F_i x_e(t) - \tau]^T \Phi \]
and denote the term (18) by \( \Omega \). By using the expressions (9), (10) we can rewrite (20) as follows:
\[ \Omega + \Sigma \Gamma + (\Sigma \Gamma)^T < 0 \]
\[
\begin{bmatrix}
\Theta_Y X & B_1 & X C_T X & E \psi X & A_d & X s_q \\
B_1^T & -\gamma I & D_{11}^T & 0 & 0 & 0 \\
C_1 X & D_{11} & -\gamma I & 0 & 0 & 0 \\
E^T \Psi X & 0 & 0 & -R & 0 & 0 \\
A_d & 0 & 0 & 0 & -\Delta_s & 0 \\
X s_q & 0 & 0 & 0 & 0 & \Delta_{s_q}^{-1}
\end{bmatrix}
\]

where \( \Theta_Y := A Y + Y A + Q + \sum_{i=1}^{n} S_i Y + \Theta_X := X A^T + A X, \Psi_Y := Y A + Q + \sum_{i=1}^{n} S_i Y + C_1 Y C_1 + (Y B_1 + C_T D_{11}) \Phi^{-1} D_{11}^T C_1, \Psi_X := A + B_1 \Phi^{-1} D_{11}^T C_1 + X Q + \sum_{i=1}^{n} S_i X + C_1 Y C_1 + C_T D_{11} \Phi^{-1} D_{11}^T C_1), \ A_d := [A_{d_1}, A_{d_2}, \ldots, A_{d_k}], X s_q := [X, X, \ldots X] \text{ and } \Delta_{s_q} := \text{diag}(Q^{-1}, S_1^{-1}, \ldots, S_k^{-1}).

Along similar lines to Gahinet and Apkarian (1994), The inequality (21) is equivalent to
\[
\begin{bmatrix}
X & I \\
I & Y
\end{bmatrix} \geq 0. 
\tag{27}
\]

where \( \tilde{I} := \begin{bmatrix} V_1 & 0 & 0 & 0 & 0 & 0 \\
V_2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & I & 0 & 0 & 0 \\
0 & 0 & 0 & I & 0 & 0 \\
0 & 0 & 0 & 0 & I & 0 \\
0 & 0 & 0 & 0 & 0 & I
\end{bmatrix}, \ \tilde{\Sigma} := \begin{bmatrix} W & 0 \\
0 & 0 & I
\end{bmatrix},
\]

\[
[V_1 \ V_2 \ V_3]^T \text{ and } W \text{ denote any of the basis of the null spaces of }
\begin{bmatrix}
C_2 \ D_{21} (C_1 D_{21} + D_{21} \Phi^{-1} D_{11}^T C_1) E \\
B_2^T
\end{bmatrix}
\]

respectively.

Remark 3.1. In summary, we can say that there exist a positive definite matrix \( P \) and a control gain matrix \( K \), satisfying (20) if and only if there exist symmetric matrices \( X \) and \( Y \) satisfying (26) and (27). So, the solution depends on the existence of \( X \) and \( Y \). Moreover, if rank \( (I - X Y) = k < n \) for solution matrices \( X \) and \( Y \) then there exist a reduced order \( H_\infty \)-controller of order \( k \).

In order to construct an \( H_\infty \)-controller, we first compute some solution \( (X, Y) \) of the LMI’s (26) and (27) by using a convex optimization algorithm for some \( \gamma \) and the positive matrices \( Q, R, S_i \)’s.

As it is noted in Choi and Chung (1997) that if \( k = \text{rank}(I - X Y) = 0 \) then we set \( P = Y \), otherwise, using the matrices \( M \) and \( N \) which are of full column rank such that \( M N^T = I - X Y \), we obtain the unique solution \( P \) to the equation
\[
\begin{bmatrix}
Y & I \\
N^T & 0
\end{bmatrix} = P \begin{bmatrix}
I & X \\
0 & M^T
\end{bmatrix}. 
\tag{28}
\]

An explicit description of all solutions of LMI in (21) can be given as follows in state space:
\[
K = -\rho \bar{\Sigma} \bar{\Xi} \bar{\Xi}^T (\bar{\Xi} \bar{\Xi}^T)^{-1} + U^2 L (\bar{\Xi} \bar{\Xi}^T)^{-1}
\]

where \( \rho \) and \( L \) are free parameters subject to
\[
\Xi := (\Sigma \Sigma^T - I / \rho \Omega)^{-1} > 0, \quad \|L\| \leq \rho
\]

and the matrix \( U \) is defined by
\[
U := I - \Sigma^T \Xi - \Xi \Xi^T (\Xi \Xi^T)^{-1} \Xi \Xi \Omega.
\]

4. CONCLUSIONS

The problem of designing output feedback \( H_\infty \) controllers for linear neutral systems with multiple time-delay has been considered in delay independent case based on the linear matrix inequality (LMI) approach. A necessary and sufficient condition for the existence of \( H_\infty \) controllers of any order is given in terms of three LMIs, when the coefficient \( D_{12} \) of the input in the controlled output is zero. Output feedback \( H_\infty \)-control problem for the same systems in delay dependent case is the subject of further research.

REFERENCES


