CLOSED FORM DIRECT KINEMATICS OF A CLASS OF STEWART PLATFORM

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Abstract: This paper presents the closed form of direct kinematics for a class of Stewart platform. Algebraic technique is explored to establish the model of a class of Stewart platform and the sets of nonlinear polynomial equations are derived. A Modified Dialytic elimination method is applied to solve the sets of nonlinear polynomial equations. Finally a 24th degree polynomial equation was obtained. This new result has been numerically verified by the inverse kinematics. Copyright © 2002 IFAC

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1. INTRODUCTION

Stewart platform was presented for a few decade years. It has high accuracy, sturdiness and capacity of loading comparing with serial manipulators. Generally, each of these mechanisms consists of two platforms, the mobile platform and the base platform. In the parallel manipulator of this paper, the mobile platform is connected to the base via six identical links consisting of a revolute joint attached to the ground followed by an actuated prismatic joint that is connected to the platform by a revolute joint. Only the prismatic joints are actuated (Merlet, 1998).

Griffs and Duffy (1989) investigated a forward displacement analysis for 3-3 Stewart platforms by geometry method. Merlet (1992) used vector technique to obtain a Triangular Symmetric Simplified Manipulator. Innocenti and Parenti-Castelli, (1993) solved 5-5 parallel mechanisms with respect to triangle technique. Sreenivasan and Waldron, (1994) proposed the problem that the mobile platform is similar to the base based on vector and matrix methods. Analytical techniques usually tried to change the sets of original multivariate polynomial equations into a high degree polynomial equation in an unknown by elimination, but elimination process is very difficult. At present, there are several methods to solve sets of multivariate polynomial equations, such as Groebner Bases (Buchberger, 1990) method, Wu (Wu, 1984) elimination method, and Dialytic (McNamee, 1993; Roth, 1993) elimination method. They exist computational complexity problem for complex practical problems in some extent. How to efficiently calculate and how to decrease computational time are crucial problems. Although the classical Dialytic elimination method has been known for a long time, its use has been limited because it is not practical for problems with more than two or three unknowns or equations of high degree (Roth, 1993). Modified Dialytic elimination method attempts to make itself a practical computational tool. In this paper, the establishment of the coordinates and the construction of functions satisfy the decrease of the numbers of variables and the most exponents of variables so that the sets of multivariate polynomial equations can be converted into the type which the modified Dialytic elimination could solve. Then the sets of multivariate polynomial equations in terms of modified Dialytic elimination method were reduced and a 24th degree polynomial equation in one unknown was derived. Moreover, Extraneous solutions were not introduced by taking advantage of the modified Dialytic elimination method.

A brief current state of research and the development of direct kinematics were introduced in section 1. Section 2 established the kinematic equations based on an algebraic method. Then the sets of nonlinear polynomial equations were derived with respect to the modified Dialytic elimination and a 24th degree polynomial equations in one unknown was obtained. In addition, the result was verified by virtue of the inverse kinematics. Finally, the conclusions were drawn.

2. ESTABLISHMENT OF THE DIRECT KINEMATICS
Parallel manipulators that consist of an equilateral triangular mobile platform and an equilateral hexagonal base platform linked by six extensible length links are considered (see Figure 1). A reference frame \( O, x, y, z \) to the base and a mobile frame \( O_1, x_1, y_1, z_1 \) are attached. The origin is \( A_1 \), and the \( x \) axis is the same as the line \( A_1 A_2 \); the origin is \( B_1 \), and the \( x_1 \) axis is parallel to line \( B_2 B_3 \). The side length of mobile platform is \( \sqrt{3}r \), where \( r \) is circumscribe radius; the side length of the base is \( r \). The transform matrix mobile platform relative to the base is \( P \).

\[
P = \begin{bmatrix} u_1 & v_1 & u_2v_3 - u_3v_2 \\ u_2 & v_2 & u_3v_1 - u_1v_3 \\ u_3 & v_3 & u_1v_2 - u_2v_1 \end{bmatrix}
\]

3. REDUCE THE FUNDAMENTAL EQUATIONS

Let the transform \[
\begin{align*}
u &= u_2y + u_3z \\
w &= y^2 + z^2
\end{align*}
\]

For the direct kinematics of parallel manipulators, articular variables can be expressed in general as a nonlinear algebraic function of links \( l_i \). Then have

\[
V = l_i(l_i) \quad i \in [1,6]
\] (1)

The point coordinates \( A_0, B_0 \) substitute for the above nine equations. Thus, the direct kinematics equations can be derived:

\[
\begin{align*}
x^2 + y^2 + z^2 &= l_1^2 \\
(x - R)^2 + y^2 + z^2 &= l_2^2 \\
(\sqrt{3}/2ru_1 + 3/2ru_2 + x - 3/2R)^2 + (\sqrt{3}/2ru_2 + 3/2rv_2 + y - \sqrt{3}/2R)^2 + (\sqrt{3}/2rv_3 + 3/2rv_3 + z)^2 &= l_3^2 \\
(\sqrt{3}/2ru_1 + 3/2rv_1 + x - R)^2 + (\sqrt{3}/2ru_2 + 3/2rv_2 + y - \sqrt{3}/2R)^2 + (\sqrt{3}/2ru_3 + 3/2rv_3 + z)^2 &= l_4^2 \\
(-\sqrt{3}/2ru_1 + (3/2)rv_1 + x)^2 + (-\sqrt{3}/2ru_2 + (3/2)rv_2 + y - \sqrt{3}/2R)^2 + (-\sqrt{3}/2ru_3 + (3/2)rv_3 + z)^2 &= l_5^2 \\
(-\sqrt{3}/2ru_1 + (3/2)rv_1 + x + (1/2)R)^2 + (-\sqrt{3}/2ru_2 + (3/2)rv_2 + y - \sqrt{3}/2R)^2 + (-\sqrt{3}/2ru_3 + (3/2)rv_3 + z)^2 &= l_6^2 \\
u_1^2 + u_2^2 + u_3^2 &= 1 \\
v_1^2 + v_2^2 + v_3^2 &= 1 \\
u_1v_1 + u_2v_2 + u_3v_3 &= 0
\end{align*}
\]

\[
\begin{align*}
u &= \frac{u_2y + u_3z}{\sqrt{3}r} \\
w &= \frac{y^2 + z^2}{\sqrt{3}r}
\end{align*}
\]

Let the transform \[
\begin{align*}
u &= v_2y + v_3z \\
w &= y^2 + z^2
\end{align*}
\]

Expressed as linear functions with respect to \( y, u_1, v_1 \), the unknowns are the coordinates of position vector \( B_1 \) in base frame \( B_1 = \{x, y, z\} \), and the orientation cosine of mobile platform relative to the base. For convenience, let the set of variables be \( V = \{x, y, z, u_1, u_2, u_3, v_1, v_2, v_3\} \).

The sets of kinematic equations can be obtained from six equations of link lengths and three additional constrains due to the orthogonality of transform matrix \( P \):

\[
\begin{align*}
|A_1B_1|^2 &= l_1^2, & |A_2B_1|^2 &= l_2^2, & |A_3B_2|^2 &= l_3^2, \\
|A_2B_2|^2 &= l_2^2, & |A_3B_3|^2 &= l_3^2, & |A_4B_3|^2 &= l_4^2, \\
\sum_{i=1}^{3} u_i^2 &= 1, & \sum_{i=1}^{3} v_i^2 &= 1, & \sum_{i=1}^{3} u_iv_i &= 0.
\end{align*}
\]

The tool position is located the center of mobile platform, and its coordinates are \( C(rv_1 + x, rv_2 + y, rv_3 + z) \).
The Jacobian matrix $J$ of (7) with respect to $u_3$ was obtained by suppressing $y$: 
\[ \begin{align*}
u_3 & = u - u_2 y \\
v_3 & = v - v_2 y \\
z & = w - y^2 
\end{align*} \]

Three equations $f_1, f_2, f_3$ can be derived (Huang and Wu, 1991; Wu and Huang, 1994):
\[ \begin{align*}
f_1 & = (u_3^2 + u_2^2 - 1)(k_2 - y^2) + (u - u_2 y)^2 \\
f_2 & = (v_3^2 + v_2^2 - 1)(k_2 - y^2) + (v - v_2 y)^2 \\
f_3 & = (u_1 v_3 + u_2 v_2)(k_2 - y^2) + (u - u_2 y)(v - v_2 y) 
\end{align*} \]

Replace $f_1, f_2, f_3$ with formula (3). $f_1, f_2, f_3$ are expressed with respect to $u_1, v_1$ and $y$. Equations $f_1, f_2, f_3$ with one unknown as suppressed are rewritten. The three equations by use of the modified Dialytic elimination method are solved, which homogeneous equations taken derivative have the same zeros as original equations.

For the general quadratics:
\[ \begin{align*}
a_i u_i^2 & + b_i v_i^2 + c_i y^2 + d_i u_i v_i + e_i u_i y \\
+ f_i v_i y + g_i u_i + h_i v_i + i_i y + j_i &= 0 
\end{align*} \] (5)

(5) was obtained by suppressing $y$:
\[ \begin{align*}
a_i u_i^2 & + b_i v_i^2 + d_i u_i v_i + l_i u_i + m_i v_i + n_i = 0 
\end{align*} \] (6)

where $l_i = e_i y + g_i$, $m_i = f_i y + h_i$, 
$n_i = c_i y^2 + i_i y + j_i$

Write homogeneous equations by substituting $u_1 = U_1/W$, $v_1 = V_1/W$, and multiplying $W^2$:
\[ \begin{align*}
a_i U_i^2 & + b_i V_i^2 + d_i U_i V_i + l_i U_i + m_i V_i + n_i W^2 = 0 
\end{align*} \] (7)

The Jacobian matrix $J$ of (7) with respect to homogeneous coordinates is:
\[ \begin{align*}
J &= \begin{bmatrix}
2a_1 U_1 + d_1 V_1 + l_1 W \\
2b_1 V_1 + d_1 U_1 + m_1 W \\
2c_1 U_1 + d_1 V_1 + m_1 V_1 + 2n_1 W \\
2a_2 U_1 + d_2 V_1 + l_2 W \\
2b_2 V_1 + d_2 U_1 + m_2 W \\
2c_2 U_1 + d_2 V_1 + m_2 V_1 + 2n_2 W \\
2a_3 U_1 + d_3 V_1 + l_3 W \\
2b_3 V_1 + d_3 U_1 + m_3 W \\
2c_3 U_1 + d_3 V_1 + m_3 V_1 + 2n_3 W
\end{bmatrix}
\]

The determinant of this matrix was formed. Then a cubic polynomial is as follows:
\[ |J| = A(y)U_1^3 + B(y)U_1^2 V_1 + C(y)U_1 V_1^2 + D(y)U_1 V_1 + E(y)U_1 W^2 + F(y)U_1 V_1 W + G(y)V_1 W + H(y)V_1^2 W + I(y)V_1 W^2 + K(y)W^3 \] (9)

Derivatives of this equation are taken with respect to homogeneous coordinates:
\[ \begin{bmatrix}
a_1(y^2) & b_1(y^2) & l_1(y) & m_1 & n_1(y^2) & U_1^2 & V_1^2 \\
2a_2(y^2) & b_2(y^2) & l_2(y) & m_2 & n_2(y^2) & U_1 V_1 \\
3A(y^6) & 3B(y^6) & 2B(y^6) & 2C(y^6) & F(y^6) & E(y^6) & U_1 W \\
3A(y^6) & 3B(y^6) & 2B(y^6) & 2C(y^6) & F(y^6) & E(y^6) & V_1 W \\
3A(y^6) & 3B(y^6) & 2B(y^6) & 2C(y^6) & F(y^6) & E(y^6) & V_1 W
\end{bmatrix} \]

The condition that equation (10) has a solution is the determinant of the coefficient matrix equals to zero. Set the determinant of coefficient matrix equal to zero. Expand and derive a 24th degree polynomial equation in unknown $y$. Where $a_i(y^2)$ express $a_i$ is function with respect to unknown $y$, and the most exponent is 2, so the determinant of the coefficient matrix is function in terms of $y$.

\[ g(y) = 0 \] (11)

Set $W=1$, and obtain
\[ \begin{bmatrix}
a_1 & b_1 & c_1 & d_1 & l_1 & m_1 & n_1 & U_1^2 & V_1^2 & n_2 \\
a_3 & b_3 & c_3 & c_1 & d_3 & l_3 & m_3 & U_1 V_1 & V_1^2 & 1 \\
3A & B & 3G & 2D & 2C & F & E & U_1 & V_1 \end{bmatrix} \]

(12)

$u_1, v_1$ can be solved from formula (12), $x, v_2, u_2$ as well as mid-variables $u, v, w$ can be solved from formula (3), and $u_3, v_3, z$ can be solved from following formula:
\[ \begin{align*}
z &= \sqrt{w - y^2} \\
u_3 &= (u - u_2 y)/z \\
v_3 &= (v - v_2 y)/z
\end{align*} \] (13)

In a word, the degree of equation $g(y) = 0$ is 24th, which can be solved and 24 solutions can be obtained. Once given the value of unknown $y$, other unknowns are uniquely determinate.

4. NUMERICAL EXAMPLE AND ANALYSIS

4.1 Numerical Example

Let the side length of equilateral triangle of the mobile platform be 10, that is $r = 10/\sqrt{3}$; the side
length of equilateral hexagon of the base platform be $R = 7$; the six link lengths be $l_1 = 12.1$, $l_2 = 12.3$, $l_3 = 12.3$, $l_4 = 12.5$, $l_5 = 12.3$, $l_6 = 12.2$ respectively. The minimum length of link is 10.2, and the maximum length of link is 13. In computer algebra system Mathematica the three solutions that satisfy formula (13) as well as constraint conditions (14) can be calculated and obtained. They are as follows:

<table>
<thead>
<tr>
<th>$u_1$</th>
<th>$v_1$</th>
<th>$x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1117244</td>
<td>-0.543086</td>
<td>0.839595</td>
</tr>
<tr>
<td>0.482097</td>
<td>0.73255</td>
<td>0.480576</td>
</tr>
<tr>
<td>3.15143</td>
<td>-0.549797</td>
<td>11.6797</td>
</tr>
</tbody>
</table>

From formula (11), the sets of multivariate equations have 24 complex solutions can be known, and 24 complex solutions by formula (11), (12), (3), and (13) can be easily solved. Because the sets of solutions are extended, redundant solutions can be reduced by constraint condition (14). Finally, three real solutions to satisfy practical problem are obtained.

5. CONCLUSION

In this paper, an algebraic method was explored to establish the kinematic model of a class of parallel manipulators. The establishment of the coordinates and the construction of functions made the decrease of the numbers of variables and the most exponents of variables. In the mean while, the kinematic equations could be reduced in terms of a modified Dialytic elimination method and a 24th degree polynomial equation in an unknown would be obtained. Then a close form of direct kinematics was performed. The solutions satisfying practical problems could be easily obtained. Also, this algorithm is very simple comparing with previous methods. Moreover, Extraneous solutions were not introduced in terms of the modified Dialytic elimination method.

**REFERENCES**


