GUARANTEED ERROR ROBUST FIR EQUALIZATION FOR UNCERTAIN CHANNELS*

Valery A. Ugrinovskii ∗,∗∗ Ian R. Petersen ∗

School of Electrical Engineering, Australian Defence Force Academy, Canberra ACT 2600, Australia, Email: valu@ee.adfa.edu.au, irp@ee.adfa.edu.au

Abstract: The equalization problem is a problem of estimating a signal transmitted over a telecommunication channel. This problem has a range of applications in telecommunications. In this paper, we consider a version of the problem in which the channel model contains an uncertainty. Our goal is to design a robust finite impulse response (FIR) equalizer which guarantees certain bound on the worst-case estimation error. The design procedure presented in the paper relies on the recent progress in maxdet convex optimization. Another result used in the paper concerns minimax optimal control and filtering of uncertain systems with relative entropy constraints on the uncertainty.

Keywords: Robust filtering, risk-sensitive filtering, equalization, FIR filters

1. INTRODUCTION

The deconvolution (equalization) problem concerns the following. Given a set of measurements $y_k \in \mathbb{R}^q$ taken at the receiving end of a telecommunication line, it is required to estimate the signal $w_k \in \mathbb{R}^r$ transmitted over the channel. The relationship between the transmitted signal $w_k$ and the received signal $y_k$ is determined by the channel and is often modeled using an autoregressive equation

$$y_k = \sum_{i=0}^{k} g_i w_{k-i} + \tilde{v}_k;$$  \hspace{1cm} (1)

where $\{g_i\}$ denotes the impulse response of a discrete time linear time-invariant system (LTI) which models the channel, $\tilde{v}_k \in \mathbb{R}^q$ is a vector measurement noise.

Although deconvolution is a well-studied problem in the field of communications and digital signal processing, research in this area has recently gained a new momentum. One of the most recent developments here relies on advanced robust control and robust filtering methodologies such as $H^\infty$ filtering and risk-sensitive filtering; e.g., see (Einicke and White 1996, Erdogad et al. 2000, Erdogad et al. 2001, Bai and Fu 1999). It turns out that in many cases, robust filters perfectly meet the specifications of the equalization problem. In particular, the robust filtering approach is a quite natural option in the case where the channel model used in the receiver design is obtained using some identification procedure, and hence is not precisely known. Using this approach allows one to address issues such as the robustness of the designed equalizers against channel mismatch.

In this paper, we are concerned with the problem of designing a guaranteed error equalizer. The underlying channel model is assumed to include uncertainty. It is a well known fact that performance of a filter may be significantly compromised due to the presence of uncertainty in the underlying system. Guaranteed error filtering allows one to account for the uncertainty by designing a filter which minimizes an upper bound on the worst-case error caused by the uncertainty and measurement noise. The guaranteed error approach to equalization has previously been considered in the
earlier paper (Ugrinovskii and Petersen 1999a) as an example of the application of the general theory developed in that paper. The equalizer proposed in (Ugrinovskii and Petersen 1999a) was a guaranteed error infinite impulse response (IIR) filter. However, from the practical viewpoint, FIR equalizers have certain advantages over IIR equalizers. Therefore, in this paper we focus on developing a tractable methodology for designing guaranteed error FIR equalizers which are robust against variations in the channel model.

Our approach can be viewed as a further extension of the approach taken in (Ugrinovskii and Petersen 1999a). Fundamental to that approach is the use of the relative entropy functional to describe the size of the uncertainty in the underlying channel model. Using the duality between free energy and relative entropy between two probability measures which occurs in the theory of large deviations (Dupuis and Ellis 1997), it was shown in (Ugrinovskii and Petersen 1999a) that the problem of guaranteed error equalization can be converted into a parameter dependent problem of risk-sensitive filtering; for more details about robust control of stochastic uncertain systems with relative entropy constraint on the uncertainty, see (Petersen et al. 2000a, Ugrinovskii and Petersen 2001, Ugrinovskii and Petersen 1999b, Petersen et al. 2000b).

Note that optimal filters solving the above parameter dependent risk-sensitive filtering problems are, in general, IIR filters. To ensure the required FIR structure of the guaranteed error filter in this paper we turn to an alternative solution to the FIR risk-sensitive filtering reported in the recent literature (Erdogad et al. 2001). One of the results of the paper (Erdogad et al. 2001) shows that the risk-sensitive FIR equalization problem represents a special case of a general MAX-DET semidefinite programming problem (Vandenberghe et al. 1998). Furthermore, this problem can be efficiently solved using interior point methods (Nesterov and Nemirovsky 1994). The approach of (Erdogad et al. 2001) extends the earlier results of (Wu et al. 1996) on FIR filter design to risk-sensitive FIR filtering.

In this paper, we present a solution to the robust FIR equalization problem which combines the strengths of the approaches mentioned above.

2. STOCHASTIC SYSTEM STATE-SPACE REPRESENTATION FOR THE ROBUST EQUALIZATION PROBLEM

In this section, we present a stochastic state-space model of the equalization system. The block-diagram for the equalization problem with uncertain channel is shown in Figure 1. In this block diagram, $G(z)$ is the transfer function model of the LTI communication channel, $w_k$ denotes the transmitted information sequence and $\tilde{v}_k$ is the channel noise. All the processes are defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We assume that the channel noise $\tilde{v}_k$ is a Gaussian white noise with the covariance matrix $V = V' > 0$. The marginal probability measure of the noise process is denoted $P^\nu$.

$$P^\nu(dv) = (2\pi)^{n/2} \det V^{-1/2} e^{-\frac{1}{2} (v' V^{-1} v)} dv.$$  

The uncertainty in the channel is accounted for by including an uncertainty transfer function $z^{-1} \Delta(z)$ in the channel model; see Figure 1. Here, $\Delta(z) \in H^\infty$. Also, to ensure that $w_k$ and $\phi_k$ are independent, the one-step delay factor $z^{-1}$ is included in the transfer function model of uncertainty. In the sequel, we will use results on robust filtering developed in (Ugrinovskii and Petersen 1999a). The application of these results requires that at time $k$, the output $\phi_k$ of the uncertainty block is independent of the channel noise $\tilde{v}_k$ and the input sequence $w_k$. The one-step delay factor $z^{-1}$ is included to ensure that this requirement is met.

First, we present a state-space representation for the nominal channel; also, see (Ugrinovskii and Petersen 1999a, Erdogad et al. 2001). Consider the system shown on Figure 1 and assume $\Delta(z) = 0$. Let the nominal channel transfer function $G(z)$ have the following form

$$G(z) = C^*(zI - A^c)^{-1} B^c + D^c,$$

with the matrix $A^c$ being stable and the pair $(A^c, B^c)$ being controllable. Then, the nominal noisy channel has the following state-space representation:

$$x_k^{+1} = A^c x_k^{+} + B^c w_k,$$

$$y_k = C^c x_k^{+} + D^c w_k + \tilde{v}_k.$$  

In the original specification of the equalization problem, the input sequence $w_k$ is a random process independent of the initial condition $x_0$ of the system (2) and the channel noise $\tilde{v}_k$. In the sequel, it will be assumed that the random process $w_k$ satisfies the condition

$$\mathbb{P} \left( \frac{1}{K} \sum_{i=0}^{K-1} ||w_i||^2 < \infty \right) = 1$$

for any $k > 0$. This is a more general requirement than that of (Erdogad et al. 2001) where for the sake of consistency with the $H_m$ approach, it was implicitly assumed that $\sum_{i=0}^{\infty} ||w_i||^2 < \infty$. 

![Fig. 1. The equalization problem](image-url)

```plaintext
In the original specification of the equalization problem, the input sequence $w_k$ is a random process independent of the initial condition $x_0$ of the system (2) and the channel noise $\tilde{v}_k$. In the sequel, it will be assumed that the random process $w_k$ satisfies the condition

$$\mathbb{P} \left( \frac{1}{K} \sum_{i=0}^{K-1} ||w_i||^2 < \infty \right) = 1$$

for any $k > 0$. This is a more general requirement than that of (Erdogad et al. 2001) where for the sake of consistency with the $H_m$ approach, it was implicitly assumed that $\sum_{i=0}^{\infty} ||w_i||^2 < \infty$. 
```
Let $Q^{k,w}$ denote a probability measure in $\mathbb{R}^{r(k+1)}$ associated with the random process $w_k$, 

$$Q^{k,w}(dw_0 \times \ldots \times dw_k) := \mathcal{P}(w_0 \in dw_0, \ldots, w_k \in dw_k).$$

We will make use of the following dependence structure of this probability measure (Petersen et al. 2000a):

$$Q^{k,w}(dw_0 \times \ldots \times dw_k) = Q_0^{k}(dw_0)Q_k^{n}(dw_1|w_0) \times \ldots \times Q_k^{e}(dw_k|w_0, \ldots, w_{k-1}).$$

Also, define the probability measure associated with a Gaussian white noise sequence $\hat{w}_k$, $\hat{w}_k \in \mathbb{R}^r$,

$$P^{k,\hat{w}}(dw_0 \times \ldots \times dw_k) = \prod_{i=0}^{k} P^{\hat{w}}(dw_i),$$

$$P^{\hat{w}}(dw) := \mathcal{P}(\hat{w}_k \in dw) = (2\pi)^{-n/2} \det W^{-1/2} e^{-1/2 w^T W^{-1} w} dw.$$ 

Assumption 1. For each $k > 0$, the conditional probability $Q_k^n(dw_0|w_0, \ldots, w_{k-1})$ is absolutely continuous with respect to the reference probability measure $P^{\hat{w}}$. Also, there exists a constant $b > 0$ such that for each $k > 0$,

$$\frac{1}{k+1} h(Q^{k,w}||P^{k,\hat{w}}) \leq b. \tag{3}$$

In equation (3), $h(Q||P)$ denotes the relative entropy functional between a probability measure $Q$ and a given reference probability measure $P$ (Dupuis and Ellis 1997). It is worth mentioning that the relative entropy $h(Q||P)$ can be thought of as characterizing the "distance" between the two probability measures. Hence, equation (3) can be viewed as a constraint on the discrepancy between the probability distributions of the given input process $w_k$ and the Gaussian white noise process $\hat{w}_k$.

Given the state-space description (2) for the nominal channel, we now augment the system (2) in order to include the state-space description of the delay and a FIR filter. This will give us a complete state-space model to which the results of (Ugrinovskii and Petersen 1999a) could be applied in the sequel. The state-space models for the delay and the FIR filter are derived in a similar fashion to those introduced in (Erdogdu et al. 2001).

Let $x_k^d = [w_{k-1}^d \ w_{k-2}^d \ \ldots \ w_{k-d}^d]^T$. Using the vector $x_k^d$ as the state vector the state-space model of the delay operator $\tau^d I$ can be introduced as follows:

$$x_{k+1}^d = A^d x_k^d + B^d w_k, \tag{4}$$

$$y_k^d = C^d x_k^d,$$

Also, the state-space representation of an FIR filter of the form

$$\hat{w}_k = \sum_{i=1}^{\pi} K_i y_{k-i}, \tag{5}$$

is given by the following equations:

$$x_{k+1}^f = A^f x_k^f + A^f c x_k + B^w w_k + B^v \hat{v}_k, \tag{6}$$

$$\hat{w}_k = C^f x_k^f,$$

$$A_f = \begin{bmatrix} 0 & \ldots & 0 \\ I & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & I \end{bmatrix}, \quad A^f c = \begin{bmatrix} C^c \\ \vdots \\ 0 \end{bmatrix},$$

$$B^w = \begin{bmatrix} D^f \\ \vdots \\ 0 \end{bmatrix}, \quad B^v = \begin{bmatrix} I \\ \vdots \\ 0 \end{bmatrix},$$

$$C^f = \begin{bmatrix} K_1 & K_2 & \ldots & K_{\pi} \end{bmatrix}.$$  

Here, $x_k^f = [y_{k-1} \ y_{k-2} \ y_{k-3} \ \ldots \ y_{k-\pi}]^T$. The complete state-space representation of the nominal system is obtained by combining the vectors $x^c$, $x^d$ and $x^f$ into the augmented state vector $x = [x^c, x^d, x^f]^T$:

$$x_{k+1} = Ax_k + B [w_k \ \hat{v}_k], \tag{7}$$

$$\zeta_k = L x_k, \quad \hat{w}_k = K x_k.$$  

Here, the following notation was used

$$A = \begin{bmatrix} A^c & 0 & 0 \\ 0 & A^d & 0 \\ A^f c & 0 & A^f \end{bmatrix}, \quad B = \begin{bmatrix} B^c \ B^d \ B^v \end{bmatrix},$$

$$L = \begin{bmatrix} 0 & C^d \end{bmatrix}, \quad K = \begin{bmatrix} 0 & 0 & C^f \end{bmatrix}.$$  

The state-space representation of the uncertain channel is derived using a similar augmenting procedure and is as follows:
\( x_{k+1} = Ax_k + B \left( \begin{bmatrix} 0 \\ \phi_k \\ \tilde{v}_k \end{bmatrix} + \begin{bmatrix} w_k \\ v_k \end{bmatrix} \right) \), \hspace{1cm} (8)

\( \xi_k = Lx_k, \)

\( \tilde{w}_k = Kx_k, \)

\( \xi_k = Cx_k, \)

\( \phi_k = \Delta(z) \xi_k, \)

\( C = \begin{bmatrix} 0 & C^T_1 & 0 \\ \end{bmatrix}, \quad C^T_1 = \begin{bmatrix} 1 & 0 & \ldots & 0 \end{bmatrix}. \)

Note the additional output \( \xi_k \) is defined so that \( \xi_k = w_{k-1} \). The introduction of this output referred to as an uncertainty output converts the system shown in Figure 1 into an uncertain system given in LFT form to which the robust filtering ideas developed in (Ugrinovskii and Petersen 1999a) could now be applied.

The approach of reference (Ugrinovskii and Petersen 1999a) requires that the uncertainty is characterized in terms of a certain relative entropy constraint similar to the constraint (3). Therefore our next step will be to introduce a relative entropy uncertainty description for the uncertainty in the transmission channel. First, we make the following assumption regarding the size of the uncertainty present in the channel.

**Assumption 2.**

\[ \|V^{-1/2} \Delta(z)\|_{\infty} < 1. \] \hspace{1cm} (9)

The required relative entropy uncertainty description is obtained by converting the constraint (9) into a relative entropy constraint. To this end, define the ‘perturbed measurement noise’ process

\( v_k := \tilde{v}_k + \phi_k \) \hspace{1cm} (10)

which represents the channel noise corrupted by the uncertainty. The conditional probability measure \( Q^\xi_k(\cdot|w_0, v_0, \ldots, w_{k-1}, v_{k-1}) \) associated with \( v_k \) can be computed via a probability measure transformation using the fact that \( \tilde{v}_k \) is a Gaussian i.i.d. sequence. Indeed, since the marginal distribution of \( \tilde{v}_k \) is given by the probability measure \( P^\xi_k \), and also since \( \tilde{v}_k \) is independent of \( \phi_k \) and \( w_{k-1} \), then it follows from (10) that

\[ Q^\xi_k(\cdot|w_0, v_0, \ldots, w_{k-1}, v_{k-1}) = e^{-(v_k - \frac{1}{2} \phi_k)V^{-1} \phi_k} P^\xi_k(\cdot|dv). \]

Finally, since \( v_k \) depends on \( \tilde{v}_k \) and \( \phi_k \), then \( v_k \) is independent of \( w_k \). Therefore, the joint conditional probability of \( w_k, v_k \) is given by the product of the corresponding marginal probabilities \( Q^w_k \times Q^v_k \).

Recall the definition of admissible uncertainty given in (Ugrinovskii and Petersen 1999a). Let \( \mathcal{F}_k \) be the complete filtration generated by the process \( x_k \). Also, let

\[ Q_k = \left\{ Q_k(\cdot|w_k \times dv_k|w_0, v_0, \ldots, w_{k-1}, v_{k-1}) \right\} \]

be a given collection of conditional probabilities. Define the probability measures \( Q^k, P^k \) as follows

\[ Q^k(dw_0 \times dv_0 \times \ldots \times dw_k \times dv_k) \]

\[ = \prod_{i=0}^k Q_k(dw_i \times dv_i|w_{i-1}, v_{i-1}), \]

\[ P^k(dw_0 \times dv_0 \times \ldots \times dw_k \times dv_k) \]

\[ = \prod_{i=0}^k P_k(dw_i \times dv_i). \]

For an \( \mathcal{F}_k \)-measurable random variable \( \eta \), let \( E^Q_{x_0} \eta \) and \( E^P \eta \) denote the expectations with respect to \( Q^k \) and \( P^k \), given an initial condition \( x_0 \) of the system (8).

**Definition 1.** (cf. (Ugrinovskii and Petersen 1999a))

Let \( b \) be a given positive constant. A collection of conditional probabilities \( Q \) is said to define an admissible uncertainty if for all \( k = 0, 1, \ldots, h(Q^k p^k) < \infty \) and also the following stochastic uncertainty constraint is satisfied:

\[ \frac{1}{2(k+1)} \sum_{i=0}^k \|\xi_i\|^2 + b \geq \frac{1}{k+1} h(Q^k p^k) \]

(11)

for all \( k = 0, 1, \ldots \).

We denote the set of collections of conditional probabilities \( Q \) defining the admissible uncertainty by \( \Xi \). The corresponding probability measures \( Q^k \) are also called admissible probability measures.

We now note that given the uncertain system (8) which satisfies Assumptions 1, 2, the corresponding collection of conditional probabilities

\[ Q^{w,v} = \left\{ Q^w_0 \times Q^v_0 \times Q^w_1 \times Q^v_1, \ldots \right\} \]

is admissible. Indeed, using the chain rule (Dupuis and Ellis 1997) and the conditions (9), (3) one can obtain

\[ \frac{1}{k+1} h(Q^k p^k) \]

\[ = \frac{1}{2(k+1)} \sum_{i=0}^k \phi_i V^{-1} \phi_i \]

\[ + \frac{1}{k+1} h(Q^{w,v} p^{w,v}) \]

\[ \leq \frac{1}{2(k+1)} \sum_{i=0}^k \|\xi_i\|^2 + b. \]

### 3. Guaranteed COST FIR Equalization

Using the notation introduced in the previous section, the FIR equalization problem is to design an FIR filter \( K(z) \) which minimizes the error between the equalizer output \( \tilde{w}_k \) and \( w_{k-d} \), the delayed transmitted sequence. In (Erdogad et al. 2001), the problem is attacked using the \( H^\infty \) and \( H_\infty \) methodologies. It is demonstrated in (Erdogad et al. 2001) that an FIR equalizer can be
chosen to be a suboptimal filter which achieves a given bound on the $H_\infty$ norm of a transfer function of the system (7) from the disturbance input $[w_k', \xi_k']$ to the equalization error output
\[ e_k := \hat{w}_k - w_{k-d} = \hat{w}_k - \xi_k. \]
This transfer function corresponds to the nominal channel model and does not account for the uncertainty in the channel. In contrast to (Erdogad et al. 2001), we seek to establish a bound on a worst case of the long run error
\[ J(K, Q) = \lim_{k \to \infty} \frac{1}{2(k+1)} E_{\tau_0} \sum_{i=0}^{k} \| e_i \|^2. \]  
(12)

Specifically, the guaranteed error FIR equalization problem is to find an FIR filter $K^*(\cdot)$ and a constant $\gamma > 0$ such that
\[ \sup_{Q \in \Xi} J(K^*, Q) \leq \gamma. \]  
(13)

The derivation of a solution to the guaranteed error FIR equalization problem relies on the duality relationship between free energy and relative entropy (Dupuis and Ellis 1997). Associated with the system (7) driven by the Gaussian white noise input $\hat{w}_k$, consider a risk sensitive cost functional
\[ \mathcal{J}_\tau(K) = \tau \log E_{\tau_0} \sum_{i=0}^{\infty} \| (\hat{w}_i - \xi_i)^2 + \tau \| \xi_i \|^2 \]  
(14)

Also, let
\[ \gamma_\tau = \inf_K \mathcal{J}_\tau(K), \]  
(15)

where the infimum is taken over the set of FIR filters (5). If $\gamma_\tau < \infty$, the corresponding optimal risk sensitive FIR filter is denoted by $K^\tau$.  

**Theorem 1.** Suppose there exists $\tau^* > 0$ which attains the minimum in
\[ \inf_{\tau} (\gamma_\tau + \tau d); \]  
(16)

Here, the infimum is taken over the set of $\tau > 0$ such that $\gamma_\tau < \infty$. Then the corresponding FIR filter $K^\tau = K_{\tau^*}$ guarantees the following bound on the error functional:
\[ \sup_{Q \in \Xi} J(K^\tau, Q) \leq \inf_{\tau} (\gamma_\tau + \tau d). \]  
(17)

**Proof:** The above claim follows from the results of (Ugrinovskii and Petersen 1999a).

4. **DESIGN OF A GUARANTEED COST FIR EQUALIZER**

Note that the condition $\gamma_\tau < \infty$ implies that
\[ \| T_{\tau, K^\tau}(\cdot) \|_\infty \leq \tau, \]  
(18)

\[ T_{\tau, K^\tau}(z) := \left[ \frac{K^\tau - L}{\sqrt{\tau} C} \right] (J - A)^{-1} B; \]

e.g., see (Glover and Doyle 1988). Therefore the FIR equalizer which guarantees the upper bound (17) on the worst case error can be found as one of the equalizers which satisfy the $H_\infty$ norm bound condition:
\[ \| T_{\tau, K}(\cdot) \|_\infty \leq \tau. \]

Also, from Theorem 1, it follows that the equalizer $K^\tau$ solves the FIR equalization problem with the risk-sensitive error functional (14) in which $\tau = \tau^*$. Given $\tau > 0$ and an FIR equalizer $K$, the risk-sensitive cost (14) associated with the equalizer $K$ can be evaluated in closed form (Jacobson 1973, Whittle 1990):
\[ \mathcal{J}_{K, \tau}(K) = \frac{\tau}{2} \log \det \left[ \left( I - \frac{1}{\tau} B' \Pi_\tau B \right)^{-1} \right], \]  
(19)

where
\[ \Pi_\tau = \Pi_\tau(K) \text{ denotes the minimal positive semi-definite solution of the algebraic Riccati equation} \]
\[ \Pi_\tau = (K - L)'(K - L) + \tau C' C + A' \Pi_\tau A \]
\[ + A' \Pi_\tau B (\tau I - A' \Pi_\tau B)^{-1} B' A, \]  
(20)

Thus, to find the equalizer $K_\tau$ which attains the infimum in (15), the minimum of the expression on the right-hand side of (19) must be found for each $\tau$. In contrast to the IIR equalization problem where the solution to corresponding the risk-sensitive optimization problem is available in a closed form (Ugrinovskii and Petersen 1999a), the minimization problem (15), (19) involves solving the Riccati equation (20) for an arbitrary filter matrix $K$. This can be avoided by using a semidefinite programming approach proposed in (Erdogad et al. 2001).

We now outline the procedure of solving the guaranteed error FIR equalization problem. Suppose that there exists $\tau > 0$ such that $\gamma_\tau < \infty$. Note that the existence of such $\tau > 0$ is a necessary condition for the problem of guaranteed equalization to have a solution. For this $\tau > 0$, the optimal risk-sensitive FIR filter $K_\tau$ can be found by solving the following convex optimization problem (Erdogad et al. 2001)
\[ \gamma_\tau = \min_{K, Z} \left[ -\frac{\tau}{2} \log \det \left( I - \frac{1}{\tau} B' Z B \right) \right] \]  
(21)

with the LMI constraint
\[ \begin{bmatrix} Z - A' Z A S(\tau, K) - A' Z B S(\tau, K) & I \\ S(\tau, K) & 0 \\ -B' Z A & \tau \Sigma - B' B \end{bmatrix} \succeq 0, \]
Here, \( S(\tau, K) := [K' - L' \sqrt{\tau} C'] \).

The above convex optimization problem is a special case of the MAX-DET convex optimization problem (Vandenberghe et al. 1998) and can be efficiently solved using interior point methods (Nesterov and Nemirovsky 1994).

Once we are able to calculate \( \gamma' \) for any \( \tau > 0 \), the infimum on the right-hand side of equation (16) can now be found using the line search. The corresponding optimal \( \tau^* \) gives rise to the guaranteed error FIR equalizer as described in Theorem 1.

5. CONCLUSION

In this paper, a numerically tractable algorithm for designing a guaranteed error FIR equalizer for an uncertain transmission channel has been proposed. The procedure outlined above involves the minimization over solutions to a certain convex optimization problem. This convex optimization problem includes a scalar parameter so that the minimization can be performed using the line search.

6. REFERENCES


