PARAMETER IDENTIFICATION FOR SOME LINEAR SYSTEMS WITH FRACTIONAL BROWNIAN MOTION

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Abstract: A parameter identification problem is formulated and solved for a multidimensional parameter of a stochastic system where the parameter appears in the drift term of a stochastic differential equation that has the Brownian motion replaced by a fractional Brownian motion with the Hurst parameter in \((1/2, 1)\). These latter fractional Brownian motions seem to be useful models for many physical phenomena where Brownian motion is not appropriate. A different stochastic calculus is required for these processes because they are not semimartingales. A family of estimates is given that arises from a formal application of a least squares algorithm. The strong consistency of the family of estimates is verified.

Keywords: Parameter identification, Fractional Brownian motion, Linear stochastic systems, Estimation.

1. INTRODUCTION

The identification of parameters for an unknown system is a basic problem in control theory. Since many physical systems that have uncertainty, perturbations, or unmodelled dynamics are modelled as stochastic systems, the problem of the parameter identification of unknown stochastic systems naturally arises. In continuous time, these stochastic systems are typically modelled by stochastic differential equations that contain a Brownian motion. However, from actual measurements of physical systems it is known that Brownian motion is often not an appropriate model for the uncertainty. A family of stochastic processes that have arisen in the modelling of systems in hydrology, economics and telecommunications is fractional Brownian motion. Therefore it is natural to investigate parameter identification of stochastic systems described by stochastic differential equations that have a Brownian motion replaced by a fractional Brownian motion.

Fractional Brownian motion denotes a family of Gaussian processes that are indexed by the Hurst parameter \(H \in (0, 1)\). These processes were defined by Kolmogorov (Kolmogorov, 1940). The first application of such processes was made by Hurst (Hurst, 1951), (Hurst, 1956) who used the properties of these processes to describe the long term storage capacity of reservoirs along the Nile River. Mandelbrot (1995) used these processes to model some economic time series and, most recently, these processes have been used to model telecommunication traffic (e.g., (Leland et al., 1994)). Two of the most important properties of these fractional Brownian motions for modeling

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physical systems are self similarity and, for $H \in (1/2, 1)$, a long range dependence. The self-similarity means that if $a > 0$, then $(\beta^H(at), t \geq 0)$ and $(\beta^H(t), t \geq 0)$ have the same probability law where $(\beta^H(t), t \geq 0)$ is a real-valued (standard) fractional Brownian motion with Hurst parameter $H$. The long range dependence means that if $r(n) = \mathbb{E}[\beta^H(1)(\beta^H(n + 1) - \beta^H(n))]$ then $\sum_{n=1}^{\infty} r(n) = \infty$.

For each $H \in (0, 1)$, a real-valued Gaussian process $(\beta^H(t), t \geq 0)$ is defined on a probability space such that $\mathbb{E}[\beta^H(t)] = 0$ and

$$\mathbb{E}[\beta^H(s)\beta^H(t)] = \frac{1}{2}[t^{2H} + s^{2H} - |t - s|^{2H}]$$

for all $s, t \in \mathbb{R}_+$. This process is called a standard fractional Brownian motion with Hurst parameter $H$. If $H = 1/2$, then the standard fractional Brownian motion is a standard Brownian motion. A standard fractional Brownian motion has a version with continuous sample paths. The 4th variation of a fractional Brownian motion is nonzero and finite for $p = 1/H$. Thus for $H \in (1/2, 1)$ the process $(\beta^H(t), t \geq 0)$ is not a semimartingale and it is also not a Markov process. The former fact implies that the usual stochastic calculus is not applicable for $(\beta^H(t), t \geq 0)$ if $H \in (1/2, 1)$.

In this paper, some multidimensional stochastic systems are described by linear stochastic differential equations where a Brownian motion is replaced by a fractional Brownian motion with $H \in (1/2, 1)$. A different stochastic calculus is required to study these stochastic differential equations (Duncan et al., 2000). Some other approaches to stochastic calculus for these processes are given in (Decreusefond and Üstünel, 1999), (Lin, 1995) and (Zähle, 1998) Some unknown parameters appear in the drift term of the stochastic differential equation. A family of estimates is given that arise by a formal application of the least squares method with Brownian motion (e.g., (Duncan and Pasik-Duncan, 1990)). It is shown that this family of estimates is strongly consistent. The methods of verification of strong consistency are quite different from those used for Brownian motion that rely heavily on martingale properties. Furthermore, the methods use a stochastic calculus for fractional Brownian motion. In (Duncan and Pasik-Duncan, 2001), a similar parameter identification problem is studied for a scalar parameter. With only a scalar unknown parameter, the methods, that are required, are simpler. In (Kleptsyna and Le Breton, 2001), a family of maximum likelihood estimates is used with a transformation of the processes to verify strong consistency, for a scalar parameter. There seems to be no work on strongly consistent estimators for this kind of parameter identification problem. Furthermore, there seems to be no work on the numerical implementation if identification models.

2. PROBLEM FORMULATION AND MAIN RESULT

Initially, a fractional Brownian motion is more completely described. Let $\Omega = C_0(\mathbb{R}_+ \times \mathbb{R})$ be the Fréchet space of real-valued continuous functions on $\mathbb{R}_+$ with the initial value zero and the topology of local uniform convergence. There is a probability measure, $P^H$ on $(\Omega, \mathcal{F})$ where $\mathcal{F}$ is the Borel $\sigma$-algebra on $\Omega$ such that on the probability space $(\Omega, \mathcal{F}, P^H)$, the coordinate process is a fractional Brownian motion, $(\beta^H(t), t \geq 0)$, that is, $\beta^H(t, \omega) = \omega(t)$ for each $t \in \mathbb{R}_+$ and (almost) all $\omega \in \Omega$. This probability space is used subsequently. Fix $H \in (1/2, 1)$ and let $\phi_H : \mathbb{R} \to \mathbb{R}_+$ be given by $\phi_H(t) = H(2H - 1)|t|^{2H - 2}$. It follows by direct computation that

$$\mathbb{E}[\beta^H(t)\beta^H(s)] = \int_0^t \int_0^s \phi_H(u - v)du dv.$$ 

Let $(X(t), t \geq 0)$ be the $\mathbb{R}^n$-valued Gaussian process that is the solution of

$$dX(t) = A(\alpha_0)X(t)dt + dB^H(t)$$

$X(0) = X_0,$

that is,

$$X(t) = e^{tA(\alpha_0)}X_0 + \int_0^t e^{A(\alpha_0)(t-s)}dB^H(s)$$

where $A(\alpha_0) \in L(\mathbb{R}^n, \mathbb{R}^n)$, $B^H(t) = (\beta^H_1(t), \ldots, \beta^H_n(t))$ and $\beta^H(t), t \geq 0, i = 1, \ldots, n$ are independent real-valued standard fractional Brownian motions with Hurst parameter $H$, $X_0$ is a zero mean Gaussian random vector independent of $(B^H(t), t \geq 0)$,

$$A(\alpha) = A_0 + \sum_{i=1}^p \alpha^i A_i.$$ 

$\alpha = (\alpha^1, \ldots, \alpha^p)^T \in \mathcal{A}$ is an unknown parameter vector, $\mathcal{A}$ is an open set in $\mathbb{R}^p$, $(A_0, A_1, \ldots, A_p)$ are known matrices and the collection of matrices $(A_1, \ldots, A_p)$ is linearly independent.

Let $A(t) \in L(\mathbb{R}^p, \mathbb{R}^p)$ for $t \in \mathbb{R}_+$ be defined by

$$\dot{A}(t) = (A_{ij}(t))$$

where

$$A_{ij}(t) = \int_0^t \langle A_iX(s), A_jX(s) \rangle ds$$

for $i, j \in \{1, \ldots, p\}$ and

$$\dot{A}(t) = (\dot{a}_{ij}(t))$$

(8)
where
\[ \hat{a}_{ij}(t) = \frac{a_{ij}(t)}{a_{ii}(t)} \] (9)
for \( i, j \in \{1, \ldots, p\} \).

The following assumptions are made on (5), (8):

(C1) \( A(\alpha) \in L(\mathbb{R}^n, \mathbb{R}^n) \) in (5) is a stable matrix, that is, \( \max \Re(\lambda(A(\alpha))) < c \) for all \( \alpha \in A \) where \( c \) is a constant.

(C2) \[ \liminf_{t \to \infty} |\det \hat{A}(t)| > 0 \quad a.s. \]
where \( \hat{A} \) is given by (8)

Based on a least squares or maximum likelihood method for a linear system with a Brownian motion (Duncan and Pasik-Duncan, 1990), a family of estimates, \( \hat{a}(t), t \geq 0 \) is defined as
\[ \hat{A}(t)\hat{a}(t) = \hat{A}(t)\alpha_0 + B(t) \] (10)
where \( t > 0 \) and \( \hat{a}(0) = 0 \), \( \hat{A}(t) \) is given by (6), \( B(t) = (b_1^H(t), \ldots, b_p^H(t))^T \) and
\[ b_i^H(t) = \int_0^t \langle A_iX(s), dB^H(s) \rangle \] (11)
for \( i \in \{1, \ldots, p\} \).

With the assumption (C2), it suffices to show that
\[ \liminf_{t \to \infty} \frac{|b_i^H(t)|}{a_{ii}(t)} = 0 \quad a.s. \]
for \( i \in \{1, \ldots, p\} \).

Initially an upper bound is given for the \( q \)th absolute moment of \( b_i^H(t) \).

**Lemma 1.** For \( q \geq 1 \)
\[ E[b_i^H(t)]^q \leq M^{qH} \] (12)
for \( i \in \{1, \ldots, p\} \) where \( M \) is a constant that only depends on \( q \) and \( b_i^H(t) \) is given by (11).

**Proof.** Since \( X(t) \) is given by (4), it easily follows that the \( \phi_H \)-derivative (Duncan et al., 2000) of \( X(t) \), \( D_H X(t) \), is
\[ D_H^\alpha X(t) = \int_0^t e^{A(\alpha)(t-s)} \phi_H(s-u)1du \]
where \( 1 = (1, \ldots, 1)^T \).

Applying the rule for the \( \phi_H \)-derivative of a stochastic integral [Theorem 4.2 in Duncan et al. (2000)], it follows that
\[ D_H^\alpha b_i^H(t) = \int_0^t \left( D_H^\alpha X(u), dB^H(u) \right) + \int_0^t \left( X(u), \phi_H(s-u)du \right) \]
\[ = \int_0^t \int_0^u \left\langle e^{A(\alpha)(u-v)} \phi_H(s-v)dv, dB^H(u) \right\rangle + \int_0^t \langle X(s), 1 \rangle \phi_H(s-u)du \]
\[ + \int_0^t \langle X(s), 1 \rangle \phi_H(s-u)du. \]

Now an upper bound is given for the second moment of the Gaussian random variable \( D_H^\alpha b_i^H(t) \).
\[ \begin{align*}
|D_H^\alpha b_i^H(t)|^2 & \leq 2 \left( \int_0^t \left( \int_0^u \langle e^{A(\alpha)(u-v)} \phi_H(s-v)dv, dB^H(u) \right)^2 \right) \\
& + 2 \left( \int_0^s \langle X(u), 1 \rangle \phi_H(s-u)du \right)^2.
\end{align*} \]

**Proof.** It is shown in [Duncan et al. (2000)] that \( X(t), t \geq 0 \) has a limiting Gaussian distribution (see Duncan (2000)) so \( E[X(t)]^2 \) is uniformly bounded, which can also be verified by direct calculation.

By Theorem 5.2 in [Duncan et al. (2000)], there is the inequality
\[ E \left[ \left( \int_0^t \langle X(s), 1 \rangle \phi_H(s-u)du \right)^2 \right] \leq M_3 \left( \int_0^t \phi_H(s-u)du \right)^2 \leq M_4 s^{H-2} \]
because \( (X(t), t \geq 0) \) has a limiting Gaussian distribution (see Duncan (2000)) so \( E[X(t)]^2 \) is uniformly bounded, which can also be verified by direct calculation.

Therefore,
\[ E \left[ \left( \int_0^t \langle X(s), 1 \rangle \phi_H(s-u)du \right)^2 \right] \leq M_3 \left( \int_0^t \phi_H(s-u)du \right)^2 \leq M_4 s^{H-2} \]
and by the Schwarz inequality
\[ E \left[ \left( \int_0^t \langle X(s), 1 \rangle \phi_H(s-u)du \right)^2 \right] \leq (2p)^{2q} \left( \int_0^t E \left[ X(s) D_H^\alpha b_i^H(s) \right]^{1/q} ds \right)^{1/2} \] (13)

where
\[ c = \min \Re(\lambda(A(\alpha))) \] in (5) is a stable matrix, that is, \( \max \Re(\lambda(A(\alpha))) < c \) for all \( \alpha \in A \) where \( c \) is a constant.
for the following equality is satisfied

Lemma 2. The following lemma provides a bound on the variable. It follows by the above that there is an $M_6$ that only depends on $q$ such that

$$\left[ \mathbb{E} \left| D_i^H s^H q^H \right|^2 \right]^{1/2} \leq M_6 s^{2qH - q}.$$  

(14)

Since $D_i^H s^H q^H (s)$ is a zero mean Gaussian random variable, it follows by the above that there is an $M_6$ that only depends on $q$ such that

$$\left[ \mathbb{E} \left| D_i^H s^H q^H (s) \right|^2 \right]^{1/2} \leq M_6 s^{2qH - q}.$$  

Since $(X(t), t \geq 0)$ has a limiting distribution it follows that there is an $M_6$ that only depends on $q$ such that

$$\left[ \mathbb{E} |X(s)|^2 q \right]^{1/2} \leq M_6.$$  

for all $s \geq 0$. Combining the last two inequalities using (13) and (14) it follows that

$$\mathbb{E} |b_i^H(t)|^q \leq M t^H$$

for $i \in \{1, \ldots, p\}$. □

The following lemma provides a bound on the growth of the stochastic integral (11).

**Lemma 3.** For each $\beta > H$ and $\varepsilon > 0$, the following equality is satisfied

$$\limsup_{t \rightarrow \infty} \frac{|b_i^H(t)|}{t^{\beta + \varepsilon}} = 0 \quad \text{a.s.}$$

(15)

for $i \in \{1, \ldots, p\}$ where $b_i^H$ is given by (11).

**Proof.** Let $\beta > H$, $\varepsilon > 0$ and $i \in \{1, \ldots, p\}$ be fixed. Fix $n \in \mathbb{N}$ and consider the sequence of random variables $(b_i^H(k/2^n), k \in \mathbb{N})$. For $k \in \mathbb{N}$, let

$$\Lambda_k = \left\{ \frac{b_i^H \left( k/2^n \right)}{k/2^n} \geq 1 \right\}.$$

Applying Markov’s inequality for $q > 1$ and (12), it follows that

$$P(\Lambda_k) \leq \frac{\mathbb{E} \left| b_i^H \left( k/2^n \right) \right|^q}{k/2^n} \leq M \frac{k^{2qH - q}}{k/2^n}$$

where $M = M \cdot 2^{-nq(H - \beta)}$. Since $H - \beta < 0$, choose $q > 1$ so that

$$\sum_{k=1}^{\infty} \frac{1}{k^{\beta(H - \beta)}} < \infty.$$  

By the Borel-Cantelli Lemma

$$P(\Lambda_k \text{ infinitely often}) = 0.$$  

Thus

$$\limsup_{k \rightarrow \infty} \frac{|b_i^H \left( k/2^n \right)|}{k/2^n} = 0 \quad \text{a.s.}$$

where $\varepsilon > 0$. There is a set $\Gamma$ with $P(\Gamma) = 0$ such that if $\omega \in \Gamma^c$ then

$$\limsup_{k \rightarrow \infty} \frac{|b_i^H \left( k/2^n, \omega \right)|}{k/2^n} = 0$$

for all $n \in \mathbb{N}$. Since $\{k/2^n, k \in \mathbb{N}\} \subset \{k/2^{n+1}, k \in \mathbb{N}\}$ for all $n$ and assumption (C1) is satisfied, it follows that $\limsup \frac{|b_i^H(t)|}{t^{\beta + \varepsilon}}$ converges to 0 as $t \rightarrow \infty$, where $D = \{k/2^n : k, n \in \mathbb{N}\}$. Since $\frac{|b_i^H(t)|}{t^{\beta + \varepsilon}}, t \geq 0$ has continuous sample paths, it follows that

$$\limsup_{t \rightarrow \infty} \frac{|b_i^H(t)|}{t^{\beta + \varepsilon}} = 0 \quad \text{a.s.}$$

□

**Lemma 3.** Let $(X(t) = (X_1(t), \ldots, X_n(t))^T, t \geq 0$ be the solution of (3). The following inequality is satisfied

$$\liminf_{t \rightarrow \infty} \frac{1}{t} \int_0^t X_i^2(s)ds > 0 \quad \text{a.s.}$$

(16)

for $i \in \{1, \ldots, n\}$.

**Proof.** Let $A = A(\alpha_0)$ where $\alpha_0$ is the true parameter. Since $A$ is a stable matrix, it is well known that the Lyapunov equation

$$A^T Y + Y A = -I$$

has a (unique) symmetric, positive definite solution $Y$. Apply a small generalization of the Itô formula [Theorem 4.3 in Duncan et al. (2000)] to the real-valued process $(\langle YX(t), X(t) \rangle, t \geq 0)$. Then

$$\langle YX(t), X(t) \rangle - \langle YX(0), X(0) \rangle$$

$$= \int_0^t \langle YX(s), AX(s) \rangle ds$$

$$+ \int_0^t \langle YX(s), dB^H(s) \rangle$$

$$+ \int_0^t \langle YAX(s), X(s) \rangle ds$$

$$+ \int_0^t \langle YdB^H(s), X(s) \rangle$$

$$+ \int_0^t \langle Ye^{A(\alpha_0)(s-t)} \times \phi_H(s-u) \rangle ds$$

$$- \int_0^t \langle X(s), X(s) \rangle ds$$

$$+ \int_0^t \langle YX(s), dB^H(s) \rangle$$

$$+ 2 \int_0^t \langle Ye^{A(\alpha_0)(s-t)} \times \phi_H(s-u) \rangle ds.$$  

(17)

By the stability of $A(\alpha_0)$, it follows by direct computation that
\[
\lim_\mathcal{L} \sup_{t \to \infty} \frac{\langle YX(t), X(t) \rangle}{t} = 0 \quad \text{a.s.} \quad (18)
\]

By the proof of Lemma 2, it follows that
\[
\lim_\mathcal{L} \sup_{t \to \infty} \frac{1}{t} \int_0^t |\langle YX(s), dB^H(s) \rangle| = 0 \quad \text{a.s.} \quad (19)
\]

and by direct computation
\[
\lim_\mathcal{L} \frac{1}{t} \int_0^t \int_0^s \text{tr} \left( Y e^{A(s-u)} \right) \phi_H(s-u) duds = \int_0^\infty \text{tr} \left( Ye^{A(v)} \right) \phi_H(v) dv. \quad (20)
\]

It follows from (18)–(20) that
\[
\lim_\mathcal{L} \inf \frac{1}{t} \int_0^t \langle X(s), X(s) \rangle ds > 0 \quad \text{a.s.}
\]

To obtain the corresponding result for each component of \( X \), in (16), it can be assumed by coordinate change that \( Y \) is diagonal and (3) has the same form. Then the inequality (16) is satisfied for each component of the new orthonormal basis from which (16) follows for each component of the original basis. \( \square \)

The strong consistency of \( (\hat{\alpha}(t), t \geq 0) \) is verified using the previous lemmas.

**Theorem 1.** Let \( (\hat{\alpha}(t), t \geq 0) \) be the family of estimates of \( \alpha_0 \) given by (10). This family of estimates is strongly consistent, that is,
\[
\lim_{t \to \infty} \hat{\alpha}(t) = \alpha_0 \quad \text{a.s.} \quad (21)
\]

**Proof.** It follows from (10) that this family of linear equations can be rewritten as
\[
\hat{\alpha}(t) = \hat{A}(t) \alpha_0 + \tilde{b}^H(t), \quad (22)
\]

where
\[
\tilde{b}^H(t) = \left( \tilde{b}_1^H(t), \ldots, \tilde{b}_p^H(t) \right)^T
\]

and
\[
\tilde{b}_i^H(t) = \frac{\tilde{b}_i^H(t)}{a_{ii}(t)} \quad \text{for } i = 1, \ldots, p.
\]

By (C2) it follows that there is a random time \( T \) such that \( \hat{A}^{-1}(S) \) is bounded above by a random variable for \( S \geq T \).

By (22), it follows that
\[
|\hat{\alpha}(t) - \alpha_0| \leq \left| \hat{A}^{-1}(t) \right| \left| \tilde{b}^H(t) \right|.
\]

By (16) and diagonalizing \( A^T A_i \), it follows that
\[
\lim_\mathcal{L} \sup_{t \to \infty} \frac{1}{t} \int_0^t \langle A_i X(s), A_i X(s) \rangle ds > 0 \quad \text{a.s.}
\]

Since
\[
\frac{1}{t} \int_0^t \langle A_i X(s), dB^H(s) \rangle = \frac{1}{t} \int_0^t \langle A_i X(s), A_i X(s) \rangle ds
\]

it follows by (15) and (23) that (21) is satisfied. \( \square \)

The family of estimates, \( (\hat{\alpha}(t), t \geq 0) \) can be expressed recursively as the solution of a stochastic differential equation. Let
\[
\langle \hat{\alpha}X, Y \rangle = \langle (A_1X, Y), \ldots, (A_pX, Y) \rangle^T.
\]

Then
\[
\hat{\alpha}(t) = \hat{A}^{-1}(t) \int_0^t \langle \hat{A}X, dX(s) - A_0 X(s) ds \rangle.
\]

Since
\[
d\hat{A}^{-1}(t) = -\hat{A}^{-1}(t) d\hat{A}(t) \hat{A}^{-1}(t)
\]

it follows that
\[
d\hat{\alpha}(t) = \hat{A}^{-1}(t) \langle \hat{A}X, dX(t) - A(\hat{\alpha}(t)) X(t) dt \rangle.
\]

**Remark 1.** It is elementary to construct examples that satisfy the assumptions (C1) and (C2). For example, let \( n = 2, p = 2 \)

\[
A_0 = \begin{bmatrix} -5 & 0 \\ 0 & -5 \end{bmatrix} \quad A_1 = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad A_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}
\]

and \( A = (0,1) \times (0,1) \).

3. CONCLUSIONS

In this paper, a family of estimates is given that arise from the formal application of a least squares algorithm. It seems interesting that this least squares algorithm for Brownian motion can be used for a fractional Brownian motion to obtain a family of strongly consistent estimates. Thus the least squares algorithm is robust in \( H \in [1/2, 1) \). The same approach given here can be used for a model (3) that has the vector of scalar fractional Brownian motions with different Hurst parameters in \( (1/2, 1) \) and a constant nonsingular diffusion coefficient other than the identity. For future work, it is desirable to investigate a weighted least squares algorithm for identification of the parameter vector as well as the numerical implementation of these various algorithms. In addition to the well known applications of fractional Brownian motion to hydrology, economic data and telecommunications, some applications to modelling of epileptic seizures are planned.

REFERENCES


