ON THE CONTROL OF LINEAR SYSTEMS HAVING INTERNAL VARIATIONS, PART I—REACHABILITY

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Abstract: Non square implicit descriptions can be used for modelling a set of linear systems, characterized by a certain common structure. Necessary and sufficient conditions, expressed in terms of the overall implicit model, exist for controlling it so that it has a unique behaviour (whatever be the internal structure variations). We enhance from these conditions the parts which are due to the common internal dynamic equation and, respectively, to the algebraic constraints which are “controlled” (in an hidden way) by the degree of freedom. Copyright © 2002 IFAC.

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1. INTRODUCTION

Let us consider the Implicit Description:

\[ \Sigma^i : \quad E \dot{x}(t) = Ax(t) + Bu(t) ; \quad y(t) = Cx(t) \quad (1) \]

where \( E : \mathcal{X} \to \mathcal{X}^i, \quad A : \mathcal{X} \to \mathcal{X}, \quad B : \mathcal{U} \to \mathcal{X} \) and \( C : \mathcal{X} \to \mathcal{Y} \) are linear operators. In (Bonilla and Malabre 1991b), it was shown that when \( \dim \mathcal{X}^i \leq \dim \mathcal{X} \), it is possible to describe linear systems with an internal Variable Structure. Indeed, when \( \dim \mathcal{X}^i < \dim \mathcal{X} \) and if the system is solvable (i.e. possesses at least one solution), solutions are generally non unique. In some sense there is a degree of freedom in (1), which can be used for instance to take into account, in an implicit way, a possible structure variation. For example:

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
\dot{x}(t) \\
y(t)
\end{bmatrix}
= \begin{bmatrix}
0 & 1 & -1 \\
1 & 0 & -1
\end{bmatrix}
\begin{bmatrix}
x(t) \\
u(t)
\end{bmatrix}
+ \begin{bmatrix}
0 \\
1
\end{bmatrix}
\begin{bmatrix}
t
\end{bmatrix}
(2)
\]

with the additional constraint: \[ [ \alpha \quad \beta \quad 1 ] \quad x(t) = 0. \]

If \((\alpha, \beta) = (-1, -1)\), namely \( x_3 = x_1 + x_2 \), then the input-output description is \( \dot{y}(t) + y(t) = u(t) \).

If \((\alpha, \beta) = (-1, 0)\), namely \( x_3 = x_1 \), then the input-output description is \( \dot{y}(t) + y(t) = u(t) \).

**Theorem 1.** (Bonilla et al. 1994) The implicit system (1) is reachable with output dynamics assignment, that is to say, it is reachable and the supremal observable part of the spectrum of \( \lambda(E - BF_d) - (A + BF_p) \) can be chosen arbitrarily using a P.D. feedback, \( u(t) = F_d \dot{x}(t) + F_p x(t) \), if and only if

\[ \mathcal{R}_{\mathcal{X}}^i = \mathcal{X} \quad (3) \]

and

\[ \dim \mathcal{V}_\mathcal{X}^* \cap \mathcal{K}_E - \dim \mathcal{B}/\mathcal{B} \cap \mathcal{E} \leq \dim \mathcal{V}^* \cap \mathcal{E}^{-1} \mathcal{B}(4) \]

where:\[ \mathcal{V}^* = \sup \{ \mathcal{T} \subset \mathcal{K}_C \mid AT \subset ET + \mathcal{I} \} \].

We then realize that using the implicit system framework, we are able to control a system having a variable internal structure, changing inside a

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1 We write \( \mathcal{B} \) and \( \mathcal{E} \) to denote \( \mathcal{I} \mathcal{M} \mathcal{B} \) and \( \mathcal{I} \mathcal{M} \mathcal{E} \), resp.

2 We also write \( \mathcal{K}_\mathcal{X} \) to denote the kernel of a given \( \mathcal{X} \).
Problem 2. Let us consider a set of strictly proper linear systems, with input \( u(t) \) and output \( y(t) \):

\[
F_i(p)y(t) = G_i(p)u(t)
\]  

(5)

If this set of strictly proper linear systems can be embedded in the following set of implicit systems:

\[
\Sigma_i : \begin{bmatrix} E \\ 0 \end{bmatrix} \dot{x} = \begin{bmatrix} A \\ D_i \end{bmatrix} x + \begin{bmatrix} B \\ 0 \end{bmatrix} u ; y = Cx
\]  

(6)

\( i = 1, \ldots, n \), where \( E : \mathcal{X} \rightarrow \mathcal{X} \), \( A : \mathcal{X} \rightarrow \mathcal{X} \), \( B : \mathcal{U} \rightarrow \mathcal{X} \), \( D_i : \mathcal{X} \rightarrow \mathcal{X} \), and \( C : \mathcal{X} \rightarrow \mathcal{Y} \) are linear operators of appropriate dimensions, with: \( \mathcal{X}_i = \mathcal{X} \subseteq \mathcal{X} \) (\( i = 1, \ldots, n \)), and such that the maps \( E \) and \( D_i \) are epic, namely: \( \mathcal{E} = \mathcal{X} \) and \( \text{Im } D_i = \mathcal{X}_i \).

Under which conditions this set of linear systems can be controlled by a fixed P.D. state feedback, \( u(t) = F_p x(t) + F_d \dot{x}(t) \), assigning for all of them the same external closed-loop behaviour, which synthesis is based on the common internal structure, described by the implicit flat system \( E \dot{x}(t) = Ax(t) + Bu(t) \)?

In the framework of implicit flat systems the answer is given by Theorem 1 in a very general way. But, in order to find the control law, we have to go through the generalization of the Morse Canonical Form (Morse 1973), namely the Canonical Form for Descriptor Systems introduced by Lebret and Loiseau (1994); which sometimes could be a hard work, not to say ill-conditioned. So, we want to go deeper in order to find what is the common internal structure of the set of linear systems (6) which enables us to solve Problem 2.

In this way, we will be able to know how to embed the set of linear systems (5) into a flat implicit system (1), having the output dynamics assignment property. And more important, we will get a simpler procedure of synthesis of such a P.D. feedback. For this, in this paper we study the structural consequences of the geometric condition of Theorem 1 when applied to solve Problem 2. Thanks to this study, we give in (Bonilla and Malabrè 2002) a procedure to synthesize P.D. feedbacks for rendering unobservable the variation of structure and assigning at will the closed-loop output dynamics. The paper is organized as follows: In Section 2, we study the structural properties of properness, algebraic reducibility and uniqueness of output trajectories of the global models (6). In Section 3 we study the reachability subspace. And in Section 4 we conclude.

Let us finish this Introductory Section recalling the following subspaces, and their associated algorithms for computing them, for a given \((Y, X, Z)\) implicit system \( \Sigma : Y \dot{x} = Xx + Zu \) (see Verghese 1981, Özçaldiran 1986, Malabrè 1987):

1. The supremal \((X, Y, Z)\) invariant subspace contained in the subspace \( \mathcal{V}_\Sigma \) : 

\[
\mathcal{V}^{\mu+1}_\Sigma = \mathcal{V}^\mu_\Sigma \cap X^{-1} (\mathcal{V}^{\mu+1}_\Sigma + \text{Im } Z)
\]

(7)

2. The infimal \((Y, X, Z)\) invariant subspace containing \( \text{Im } Z \):

\[
\mathcal{S}^{\mu+1}_\Sigma = X^{-1} \text{Im } Z
\]

(8)

3. The supremal \((X, Y, Z)\) reachability subspace contained in the subspace \( \mathcal{V}_\Sigma \) : 

\[
\mathcal{R}^{\mu+1}_\Sigma = \mathcal{V}^{\mu+1}_\Sigma \cap \mathcal{R}^{\mu}_\Sigma
\]

(9)

Özçaldiran (1986) has also shown that:

\[
\mathcal{R}^{\mu}_\Sigma = \mathcal{V}^{\mu}_\Sigma \cap \mathcal{S}^{\mu}_\Sigma
\]

(10)

Related with the supremal \((X, Y, Z)\) invariant subspace contained in the subspace \( \mathcal{V}_\Sigma \), we have the following well known and useful result:

Fact 3. Let \( \mathcal{I}(\Sigma ; \mathcal{T}) \) denote the family of \((X, Y, Z)\) invariant subspaces contained in \( \mathcal{T} \), i.e., \( \mathcal{I}(\Sigma ; \mathcal{T}) := \{ \mathcal{V} \subset \mathcal{T} ; \exists F_p : \mathcal{X} \rightarrow \mathcal{U} \text{ and } F_d : \mathcal{X} \rightarrow \mathcal{U} \text{ such that } (X + ZF_p)\mathcal{V} \subset (Y - ZF_d)\mathcal{V} \} \). Then:

A. \( \mathcal{V} \in \mathcal{I}(\Sigma ; \mathcal{T}) \) iff \( \mathcal{X} \subset \mathcal{Y} + \text{Im } Z \), with \( \mathcal{V} \subset \mathcal{T} \).

B. The set of P.D.-feedbacks which, for a given \( \mathcal{V} \in \mathcal{I}(\Sigma ; \mathcal{T}) \) satisfy \((X + ZF_d)\mathcal{V} \subset (Y - ZF_d)\mathcal{V} \) is called a friend pair of \( \mathcal{V} \), namely:

\[
\mathcal{F}(\mathcal{V}) := \{(F_p, F_d) | (X + ZF_d)\mathcal{V} \subset (Y - ZF_d)\mathcal{V} \text{ and } \mathcal{V} \in \mathcal{I}(\Sigma ; \mathcal{T})\}
\]

(11)

C. \( \mathcal{V}^{\mu+1}_\Sigma = \sup \mathcal{I}(\Sigma ; \mathcal{T}) \)

D. Given a pair \((F_p, F_d)\) of feedbacks and the closed loop implicit system, \( \Sigma_F : (Y - ZF_d)\dot{x} = (X + ZF_p)x + Zu \), then: \( \mathcal{V}^{\mu+1}_\Sigma = \mathcal{V}^{\mu+1}_\Sigma \).

4. The following algorithms are associated with the pencil \( \Psi = [\lambda \mathbf{F} - \mathbf{G}] \) (see for instance (and the references therein) Malabrè 1989):

\[
\mathcal{A}^{\mu+1}_{\mathcal{T}, \Psi} = \{0\}; \mathcal{A}^{\mu+1}_{\mathcal{T}, \Psi} = \mathcal{F}^{-1} \mathcal{G} \mathcal{A}^{\mu}_{\mathcal{T}, \Psi}
\]

(12)

- We write \( X^{-1} \text{Im } X \) in place of the usual domain vector space since we work with different definition domains. If \( \mathcal{V} \supset \mathcal{W} \) then \( X^{-1} \text{Im } X = X^{-1} \text{Im } \mathcal{Y} \).

"
\[ A_{2,2}^0 = G^{-1} \text{Im } G; \quad A_{2,2}^{\mu+1} = G^{-1} F A_{2,2}^{\mu} \]  
\[ B_{1,2}^0 = \text{Im } F + \text{Im } G; \quad B_{1,2}^{\mu+1} = F G^{-1} B_{1,2}^{\mu} \]

The nondecreasing algorithms (12) and (15) converge to \( A_{1,2}^0 \) and \( B_{1,2}^0 \), respectively, and the non increasing algorithms (13) and (14) converge to \( A_{2,2}^0 \) and \( B_{2,2}^0 \), respectively.

2. STRUCTURAL PROPERTIES

In this Section we study the structural properties of properness, algebraic redundancy and uniqueness of output trajectories of the global models (6).

For this, in Proposition 4, the descriptor variable of system (16) is decomposed as the direct sum of \( K_D \) and \( K_E \); the subspace \( K_E \) characterizes the degree of freedom which makes possible the variation of the system, and the subspace \( K_D \) characterizes the set of all possible trajectories solutions of the differential part of (16). In Proposition 8, the operators involved in the externally equivalent reduced state space description (without redundancy) are characterized in terms of insertions and natural projections. And, in Theorem 12, the existence of a P.D. Feedback, guaranteeing the uniqueness of the output trajectory, is geometrically characterized.

Let us consider the following implicit global description, \( \Sigma^g : (E, H, B, C) \),

\[ \Sigma^g : E \dot{x}(t) = H x(t) + B u(t); \quad y = C x(t) \]

\[ E = \begin{bmatrix} E' \\ 0 \end{bmatrix}; \quad H = \begin{bmatrix} A \\ D \end{bmatrix}; \quad B = \begin{bmatrix} B' \\ 0 \end{bmatrix} \]  
(16)

where \( x(t), u(t), \) and \( y(t) \) are, respectively, the descriptor variable, the input, and the output. \( E : \mathcal{X} \rightarrow \mathcal{X}_g, \quad H : \mathcal{X} \rightarrow \mathcal{X}_g, \quad B : \mathcal{U} \rightarrow \mathcal{X}_g, \) and \( C : \mathcal{X} \rightarrow \mathcal{Y} \) are linear operators, such that:

\[ \text{Im } A + B \subset \mathcal{E}; \quad \mathcal{E} \oplus \text{Im } D = \mathcal{X}_g \]  
(17)

2.1 Properness

As indicated in Problem 2, we want to embed the proper systems (5) into an implicit global description (16). So let us study how the properness is reflected in the structural properties of the implicit global description \( \Sigma^g(E, H, B, C) \).

Proposition 4. The pencil \( [\lambda E - H] \) in (16) is internally proper, if it is regular i.e. \( \det(\lambda E - G) \neq 0 \) and it has no infinite zero of order greater than one (there exist no derivators), if and only if:

\[ K_D \oplus K_E = \mathcal{X} \]  
(18)

**Proof of Proposition 4** For the Proof we need the following Lemma, proved in (Bonilla and Malabre 2002):

Lemma 5. If the pencil of the global system, \( \Psi^g = [\lambda E - H] \), is internally proper, then (18) is satisfied.

In view of Lemma 5 we only need to prove that (18) implies the internal properness of the pencil \( [\lambda E - H] \). To prove the properness of the pencil \( [\lambda E - H] \), let us first decompose the subspaces \( \mathcal{X} \) and \( \mathcal{X}_g \) as follows (recall (18) and (17)):

\[ \mathcal{X} = K_D \oplus K_E \quad \mathcal{X}_g = E K_D \oplus D K_E \]

In these bases: \( [\lambda E - H] = \begin{bmatrix} \lambda I - A_1 & -A_2 \\ 0 & -I \end{bmatrix} \),

which is obviously a regular pencil having no infinite zeros of order greater than one. \( \Box \)

Based on the Proof of Proposition 4, let us define:

\[ P : \mathcal{X}_g \rightarrow \mathcal{E}, \quad \text{natural projection along } \text{Im } D \]

\[ V : K_D \rightarrow \mathcal{X} \quad \text{and } Y : \mathcal{E} \rightarrow \mathcal{X}_g \quad \text{insertion maps} \]

Then \( (E, A, B) \) and \( (E, H, B) \) are related by:

\[ P E = E; \quad P H = A; \quad P B = B \]

Moreover, there exist unique maps \( (E, A, B, C) \) such that the lower part of the Diagram of Fig. 1 is commutative, namely:

\[ E V = \overline{E}; \quad H V = \overline{H}; \quad B = \overline{B}; \quad CV = \overline{C} \]

**Fig. 1.** Commutative Diagram of (16).

2.2 Algebraic redundancy

**Definition 6.** (Bonilla and Malabre 1991a) Given an implicit description, \( \Sigma(Y, X, Z, W) \): \( Y = X + Z u, u = W x, \) where \( u, y \) and \( x \) are the input, the output, and the descriptor variable, respectively; and \( Y : \mathcal{X} \rightarrow \mathcal{X}_g, X : \mathcal{X} \rightarrow \mathcal{X}_g, Z : \mathcal{U} \rightarrow \mathcal{X}_g, \) and \( W : \mathcal{X} \rightarrow \mathcal{Y} \) are linear maps of appropriate dimensions. Is called algebraic redundant, any descriptor variable \( \xi(t) \in \mathcal{X} \) which (for all \( t \)) is a constant linear combination of some other descriptor variables and which can be suppressed without
modifying the external behaviour, that is to say, the set of all possible input-output trajectories, \((u, y)\), remains unchanged (see Willems 1983), of system \(\Sigma(Y, X, Z, W)\).

**Proposition 7.** (Bonilla and Malabre 1995) Given an implicit description, \(\Sigma(Y, X, Z, W)\), we can restrict the system to \(\mathcal{V}_X^\infty\) in the domain and to \(\mathcal{V}_X^\infty + \text{Im } Z\) in the codomain without modifying the external behaviour. Moreover the induced system, \(\bar{\Sigma}(\bar{\mathcal{Y}}, \bar{\mathcal{X}}): \bar{\mathcal{Y}} = \bar{X} \in\mathcal{V}_X^\infty \rightarrow \mathcal{X}_p/\mathcal{V}_X^\infty + \text{Im } Z\) and \(\bar{\mathcal{X}}: \mathcal{X}/\mathcal{V}_X^\infty \rightarrow \mathcal{X}_p/(\mathcal{V}_X^\infty + \text{Im } Z)\), is the maximal redundant part, i.e. characterizes a certain algebraic redundancy; namely, algebraic relations which can be reduced by simple algebraic manipulations.

**Proposition 8.** The global system (16), \(\Sigma^p(\mathcal{E}, \mathcal{H}, \mathcal{B}, \mathcal{C})\), restricted to \(\mathcal{K}_D\) in the domain and to \(\mathcal{E}\mathcal{K}_D\) in the codomain, contains no algebraic redundant part. Moreover the degree of freedom, characterized by the algebraic redundant descriptor variable, is located in \(\mathcal{K}_E\). Furthermore, the global system (16), \(\Sigma^s: (\mathcal{E}, \mathcal{H}, \mathcal{B}, \mathcal{C})\), is externally equivalent to the reduced state space description (24), \(\Sigma^s: (I, A_0, B_0, \mathcal{C}):\)

\[
\begin{align*}
\Sigma^s: & \quad \mathcal{X}_p = A_0 \mathcal{X}_p + B_0 u; \quad \mathcal{Y} = \mathcal{C} \mathcal{X}_p \quad (24) \\
A_0 & = \mathcal{E}^{-1} \mathcal{A} \quad \text{and} \quad B_0 = \mathcal{E}^{-1} \mathcal{B} \quad (25) \\
\mathcal{C} & = P \mathcal{E} \mathcal{V} = EV; \quad \mathcal{A} = P \mathcal{H} \mathcal{V} = AV \\
\mathcal{B} & = P \mathcal{B} = B \quad ; \quad \mathcal{C} = CV 
\end{align*}
\]

See (Bonilla and Malabre 2002) for the Proof.

### 2.3 Uniqueness of output trajectories

**Proposition 9.** (Lebret 1991, Lewis 1992) The implicit system \(\Sigma(Y, X, Z, W): \mathcal{Y} \mathcal{X} = X \mathcal{X} + Z u, \mathcal{Y} = W \mathcal{X}\), admits at least one solution on the output \(y\) in \(C^\infty\) for all input \(u\) in \(C^\infty\) if and only if

\[
\text{Im } Z \subset B_{1,\Psi} + B_{2,\Psi} \iff \text{Im } Z \subset \text{Im } \Psi 
\]

where \(\Psi\) is the pencil \([\lambda \mathcal{Y} - \mathcal{X}]\). In this case, we will say that the system accepts all inputs. The solution \(y\) is unique if and only if

\[
A_{1,\Psi} \cap A_{2,\Psi} \subset \mathcal{K}_W \iff \mathcal{K}_\Psi \subset \mathcal{K}_W 
\]

\[4\] We write \(X^{-1}\) for the inverse map of \(X\) (when it exists) in order to avoid confusions with the subspace \(X^{-1} \mathcal{T}\).

**Lemma 10.** The following statements hold true:

1. If the geometric condition (17.a) is satisfied then the implicit system (1), \(\Sigma^s(E, A, B, C)\), admits at least one solution on the output \(y\) in \(C^\infty\) for all input \(u\) in \(C^\infty\).
2. If the geometric condition (17.b) is satisfied then the implicit global description (16), \(\Sigma^p(\mathcal{E}, \mathcal{H}, \mathcal{B}, \mathcal{C})\), admits at least one solution on the output \(y\) in \(C^\infty\) for all input \(u\) in \(C^\infty\).
3. If in addition to the geometric condition (17.a), we take a P.D. feedback, \(u = F_d \mathcal{X} + F_p x + v\), such that the closed-loop system, \(\Sigma^p_f(E - BF_d, A + BF_p, B, C)\), satisfies:

\[
\text{Im } (E - BF_d) = \varepsilon 
\]

then the closed-loop system, \(\Sigma^p_f(E - BF_d, A + BF_p, B, C)\), admits at least one solution on the output \(y\) in \(C^\infty\) for all input \(u\) in \(C^\infty\).

**Theorem 11.** Given the implicit system (1) satisfying the geometric condition (17.a), let us take a P.D. feedback, \(u = F_d \mathcal{X} + F_p x + v\), such that the closed-loop system, \(\Sigma^p_f(E - BF_d, A + BF_p, B, C)\), satisfies the geometric condition (29). Then

\[
\text{Ker } (E - BF_d) \subset \mathcal{V}^* 
\]

\[
\dim (\text{Ker } E) \leq \dim (\mathcal{V}^* \cap E^{-1} \mathcal{B}) 
\]

only if the solution of the output, \(y\), of the closed-loop system, \(\Sigma^p_f(E - BF_d, A + BF_p, B, C)\), is unique.

**Theorem 12.** Given the implicit system (1) satisfying the geometric condition (17.a), let us take a P.D. feedback, \(u = F_d \mathcal{X} + F_p x + v\), with \((F_p, F_d)\) \(\in \mathcal{F}(\mathcal{V}_E^{\infty})\) and such that the closed-loop system, \(\Sigma^p_f(E - BF_d, A + BF_p, B, C)\), satisfies the geometric condition (29). Then, the output, \(y\), of the closed-loop system is unique if and only if the geometric conditions (30) and (31) are satisfied.

### 3. Reachability

**Theorem 13.** The reachability subspace \(\mathcal{R}_{X, \Sigma^s}\) of the implicit flat system, \(\Sigma^s: (E, A, B, C)\), and the reachability subspace \(\mathcal{R}_{X, \Sigma^p}\) of the reduced state space system, \(\Sigma^s: (I, A_0, B_0, \mathcal{C})\), are related as follows:

\[
\mathcal{R}_{X, \Sigma^s} = \mathcal{R}_{X, \Sigma^p} + \mathcal{V} \mathcal{R}_{\mathcal{K}_D, \Sigma^s} 
\]

where \(\Sigma^p_0: E \mathcal{X} = A \mathcal{X}\). Moreover, \(\mathcal{R}_{X, \Sigma^p_0}\) is the subspace reachable by the degree of freedom, \(\mathcal{X}\), of the global system,

\[
\Sigma^p_0: E \mathcal{X} = \mathcal{H} \mathcal{X} + \mathcal{W} \mathcal{D} \mathcal{X}; \quad y(t) = C \mathcal{X}(t)
\]
namely:
\[ \mathcal{R}^*_{X,\Sigma_0} = \mathcal{R}^*_{X,\Sigma_0} \]  \hfill (34)

where: \( W: \text{Im} \ D \rightarrow \mathcal{X}_0 \) is the natural insertion map of \( \text{Im} \ D \) in \( \mathcal{X}_0 \).

**Proof of Theorem 13** For the Proof we need the following Lemma proved in (Bonilla and Malabre 2002):

**Lemma 14.** If \( \text{Im} \ A + B \subset \mathcal{E} \) then:
\[ \mathcal{V}_{K,D,\Sigma}^* = K_{D} \quad ; \quad \mathcal{V}_{X,\Sigma_0}^* = \mathcal{X} = \mathcal{V}_{X,\Sigma_0}^* \]  \hfill (35)

1. Let us first show that:
\[ S_{K,D,\Sigma}^{\mu+1} = \sum_{j=0}^\mu A_0^j \text{Im} \ B_0 \]  \hfill (36)

Indeed from (8) we get: \( S_{K,D,\Sigma}^{1} = \text{Im} \ B_0 \). Note that (25) implies: \( \text{Im} \ B_0 \subset K_{D} \) and \( \text{Im} \ A_0 \subset K_{D} \), then: \( S_{K,D,\Sigma}^{\mu+1} = A_0S_{K,D,\Sigma}^{\mu} + \text{Im} \ B_0 \). And we conclude by induction.

2. Let us next show that:
\[ S_{X,\Sigma_0}^{\mu+1} = (E^{-1}A)^{\mu+1} K_E + \sum_{j=0}^\mu (E^{-1}A)^j E^{-1}B \]  \hfill (37)

Indeed, since \( B + \text{Im} \ A \subset \mathcal{E} \), then \( E^{-1}(AS + BT) = E^{-1}AS + E^{-1}BT \), for any \( S, T \subset \mathcal{X} \). We then realize from (8) that:
\[ S_{X,\Sigma_0}^{\mu+1} = (E^{-1}A)^{\mu+1} K_E + \sum_{j=0}^\mu (E^{-1}A)^j E^{-1}B \]  \hfill (38)

From (26) and (25), we get: \( (E^{-1}A)^j E^{-1}B = (E^{-1}A)^{j-1} E^{-1}A E^{-1}B = (E^{-1}A)^{j-1} E^{-1}(A+B) \) \( E^{-1}E^{-1} \text{Im} \ B_0 = (E^{-1}A)^{j-1} E^{-1}(A+B) \) \( \text{VIm} \ B_0 + K_E \) = \( (E^{-1}A)^{j-1}(E^{-1}A \text{Im} \ B_0 + E^{-1}AK_E) = (E^{-1}A)^{j-1}(E^{-1}A \text{Im} \ B_0 + E^{-1}AK_E) \) = \( (E^{-1}A)^{j-1}(E^{-1}A \text{Im} \ B_0 + E^{-1}AK_E) = \cdots = (E^{-1}A)(V \text{Im} \ B_0 + E^{-1}AK_E) = E^{-1}E^{-1} \text{Im} \ B_0 + E^{-1}AK_E. \)

3. Let us now show that:
\[ S_{X,\Sigma_0}^{\mu+1} = (E^{-1}A)^{\mu+1} K_E \]  \hfill (40)

Indeed, (40) directly follows from (8).

And then, from (36), (37), (40) we get:
\[ S_{K,D,\Sigma}^{\mu+1} = S_{K,D,\Sigma}^{\mu+1} + V S_{X,\Sigma_0}^{\mu+1} \]  \hfill (41)

Finally, (41), Lemma 14, and (10) imply (32).

In order to prove (34) let us compute algorithm (8) for system (33). For \( i = 0 \), we get from (20), (22) and (40): \( S_{X,\Sigma_0}^0 = K E = K_E = S_{X,\Sigma_0}^0 \).

Assuming that, for \( i \geq 0 \): \( S_{X,\Sigma_0}^i = S_{X,\Sigma_0}^i \), we get:
\[ S_{X,\Sigma_0}^{\mu+1} = E^{-1} \left( H S_{X,\Sigma_0}^\mu + W \text{Im} \ D \right) \]
\[ = E^{-1} \left( H S_{X,\Sigma_0}^\mu + K_E \right) = E^{-1}P^{-1} \rho \text{H} S_{X,\Sigma_0}^\mu \]  \hfill (42)

On the other hand from (7) we get (recall that \( \text{Im} E = \mathcal{E} \) and (17,b)): \( \mathcal{V}_{X,\Sigma_0}^0 = \mathcal{X} \) and
\[ \mathcal{V}_{X,\Sigma_0}^\mu = H^{-1} \left( H S_{X,\Sigma_0}^\mu + W \text{Im} \ D \right) = H^{-1} \left( \text{Im} E + W \text{Im} \ D \right) = H^{-1} \mathcal{X} = \mathcal{X}, \]
The which implies: \( \mathcal{V}_{X,\Sigma_0}^\mu = \mathcal{X} \). And then, from (42), (40) and (10) we get (34).

Let us come back to the illustrative example (2), which related global system (16) is:
\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
x \end{bmatrix}
+ \begin{bmatrix}
0 & 1 & -1 \\
1 & 0 & -1 \\
\alpha & \beta & 1
\end{bmatrix}
\begin{bmatrix}
x \\
u \\
y
\end{bmatrix}
\]
\hfill (43)

Doing \( x = T_R \xi \) and premultiplying the differential equation of (43) by \( T_L \), where \( T_R = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ -\beta & -\alpha & 1 \end{bmatrix} \), \( T_L = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \), we get:
\[
\begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\xi \\
\xi + \rho_1 \\
\xi \rho_3
\end{bmatrix}
\]
\hfill (44)

In the global system \( \Sigma^g(E, H, \mathcal{B}, C) \) shown in (44), we have enclosed the implicit flat system \( \Sigma^g(E, A, B, C) \) by the dash line boxes and the reduced state space system \( \Sigma^g(I, A_0, D_0, \mathcal{C}) \) by the solid line boxes; the matrix \( D \) is the row below the solid line of \( H \).

Let us note for (44) that: \( 1) K_E \subset K_D = \mathcal{X} ; 2) \) from (36), we get: \( V \mathcal{R}_{K,D,\Sigma}^* = \text{Span} \{ [0 \ 1 \ 0]^T \} \()); \( 3) \) from (40), we get: \( \mathcal{R}_{X,\Sigma_0}^* = \text{Span} \{ [0 \ 0 \ 1]^T \} \(); \( 4) \) from (37), we get: \( \mathcal{R}_{X,\Sigma_0}^* = \mathcal{X} \), i.e. the implicit flat system \( \Sigma^g(E, A, B, C) \) is reachable, whatever be the values of \( \alpha, \beta \). Of course this reachability is achieved by both, the input action, through \( V \text{Im} \ B_0 \), and the degree of freedom, through \( \text{Im} \ D \); \( 5) \) the observability matrix, \( \begin{bmatrix} \mathcal{C} \\
\mathcal{C} \mathcal{A}_0 \end{bmatrix} \), is
\[
\begin{bmatrix}
-\beta \\
-\alpha + \beta \\
\beta - (1 + \beta) (\alpha + \beta) \\
-(1 + \alpha + \beta) (\alpha + \beta)
\end{bmatrix}
\]
geometries of the reduced state space descriptions of the global regular descriptor system (6) and the conditions of Theorem
The aim of our research is to express the geometric and derivative feedback. We have found the common internal structure of linear systems (6) which enables us to solve Problem 2.

In this paper we have considered the problem of controlling the set of linear systems (5) through the use of a non square implicit model $\Sigma'(I, A_0, B_0, \mathcal{C})$. As we have pointed out in the introductory section, there are two geometric conditions, recalled in Theorem 1, which have to hold in order to solve such a problem. These conditions are expressed in terms of the geometry of the flat implicit system (1) and then, it is not easy to point out which common geometry makes this set of linear systems have the same closed-loop behaviour when controlled by a unique proportional and derivative feedback.

We are now able to embed the set of linear systems (5) into a flat implicit system (1), having the output dynamics assignment property. Thanks to the study of this paper, in (Bonilla and Malabre 2002) we give a procedure to synthetize the fixed P.D. control law for rendering unobservable the variation of structure and assigning at will the closed-loop output dynamics, through the use of a non square implicit model (5).

4. CONCLUDING REMARKS

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In this paper we have considered the problem of controlling the set of linear systems (5) through the use of a non square implicit model $\Sigma'(E, A, B, C)$. As we have pointed out in the introductory section, there are two geometric conditions, recalled in Theorem 1, which have to hold in order to solve such a problem. These conditions are expressed in terms of the geometry of the flat implicit system (1) and then, it is not easy to point out which common geometry makes this set of linear systems have the same closed-loop behaviour when controlled by a unique proportional and derivative feedback.

The aim of our research is to express the geometric conditions of Theorem 1 in terms of the geometry of the global regular descriptor system (6) and the geometries of the reduced state space descriptions (24), for each algebraic restriction: $0 = D_x x(t)$. We have found the common internal structure of the set of linear systems (6) which enables us to solve Problem 2.

5. REFERENCES


