Abstract: FIR compensator design for $H_2$-optimal decoupling of measurable or previewed signals in discrete-time linear time-invariant systems is considered. The algorithm for the FIR system weighting matrices computation is based on pseudoinversion techniques aiming to minimize the $l_2$ norm of the overall system impulse responses. The inherent dimensionality constraint of these techniques is overcome by welding problems referring to subsequent time subintervals of the FIR system window. The solution of the $H_2$-optimal state observation problem with unknown inputs straightforwardly comes by duality.

Keywords: $H_2$ optimal decoupling, preview control, FIR systems.

1. INTRODUCTION

It is well known that in the discrete-time case the use of FIR systems is particularly convenient for the solution of the decoupling problem of measurable or previewed signals as well as the perfect tracking problem (that can be considered as a particular case of the former). The dual problem i.e., the possibly delayed unknown-input observation of a linear function of the state as well as left inversion as a particular case is also conveniently solved with FIR systems.

The necessity of using FIR systems is due to the nature of the modes which the optimal finite-time state trajectory arcs consist of. In fact, both stable and anti-stable modes (corresponding to eigenvalues reciprocal to each other) are to be reproduced in the control function.

If a set of geometric-type conditions are met, $H_2$ optimal decoupling or tracking problems and their duals can be solved cost-free (or almost cost-free). This aspect has recently been investigated in (Marro et al., 2000c), where a compensator with a peculiar structure, a parallel of a FIR system and a dynamic unit, has been proposed.

As far as previewed signal decoupling and tracking is concerned, it is well-known that perfect or almost perfect tracking can be achieved also in the nonminimum phase case if the reference signal is known in advance. See, for instance, (Devasia et al., 1996) and (Hunt et al., 1996) for the infinite horizon nonlinear and linear cases, respectively. Instead, refer to (Gross and Tomizuka, 1994) and (Marro and Fantoni, 1996) for two different approaches to the receding horizon SISO case and also to (Marro et al., 2000a) for the introduction of FIR systems in obtaining the noncausal inversion of MIMO discrete-time linear systems.

Regarding the dual problems, FIR filter and smoother design has been extensively investigated and its use is now well established, mainly for the estimation of some state variables of stochastic systems ((Park et al., 2000), (Kwon et al., 1999), (Kwon et al., 1994), (Kwon and Byun, 1989), (Kwon and Kwon, 1987), (Kwon et al., 1983), (Kwon and Pearson, 1978)), although some attempts have also been made to introduce the receding horizon technique for the design of observers in a deterministic environment ((Ling and Lim, 1996)).

The novelty of this contribution consists in presenting a solution of the decoupling problem with preview (hence of its dual) by means of a FIR compensator achieving the minimum $H_2$ norm of the transfer function matrix from the input to be decoupled to the controlled output (or, in the dual case, of the transfer
2. STATEMENT OF THE PROBLEM

Consider the linear discrete time-invariant system \( \Sigma \) described by

\[
\begin{align*}
    x(k + 1) &= Ax(k) + Bu(k) + H h(k), \\
    y(k) &= C x(k) + D u(k) + G h(k),
\end{align*}
\]

with state \( x \in \mathbb{R}^n \), control input \( u \in \mathbb{R}^p \), previewed or measured input \( h \in \mathbb{R}^q \) and controlled output \( y \in \mathbb{R}^m \). Assume that matrix \( A \) is stable, pair \((A, B)\) controllable and matrices \([B' D']\) and \([H' G']\) full column rank. Refer to the block diagram in Fig. 1, where the \( N_p\)-step preview interval of signal \( h(k) \) is accounted for by the \( N_p\)-delay unit, so that the overall system having \( h_p(k) = h(k + N_p) \) as input and \( y(k) \) as output is causal. The particular case \( N_p = 0 \) corresponds to the decoupling of a signal which is measurable but not known in advance. The block \( \Phi \) denotes a FIR system described by

\[
u(k) = \sum_{\ell=0}^{N_p-1} \Phi(\ell) h_p(k - \ell),
\]

with window \( N > N_p\). Referring to Fig. 1, denote by \( W(z) \) the transfer function matrix from \( h_p(k) \) to \( y(k) \). The \( H_2 \) optimal decoupling problem is stated as follows.

**Problem 1.** \((H_2 \) optimal decoupling problem with preview) Refer to systems (1) and (2) connected as shown in Fig. 1. Given the window \( N \) and the preview time \( N_p \), find the FIR weighting matrices \( \Phi(\ell) \) \((\ell = 1, \ldots, N - 1)\) minimizing \( ||W||_2 \).

It is worth noticing that solution of Problem 1 also applies to the dual problem, as briefly shown in the sequel. Refer to the linear discrete-time system \( \Sigma_d \) described by

\[
\begin{align*}
    x(k + 1) &= A_d x(k) + B_d u(k), \\
    y(k) &= C_d x(k) + D_d u(k), \\
    e(k) &= H_d x(k) + G_d u(k),
\end{align*}
\]

with state \( x \in \mathbb{R}^n \), inaccessible input \( u \in \mathbb{R}^p \), informative output \( y \in \mathbb{R}^m \), output to be estimate \( e \in \mathbb{R}^q \). Matrix \( A_d \) is assumed to be stable, pair \((A_d, C_d)\) observable and matrices \([C_d D_d], [H_d G_d]\) full row rank. Let us consider Fig. 2, where \( \Phi_d \) is a FIR system with weighting matrices \( \Phi_d(\ell) \) \((\ell = 1, \ldots, N - 1)\). Denote by \( W_d(z) \) the transfer function matrix from \( u(k) \) to \( e(k-n_p) \).

**Problem 2.** \((H_2 \) optimal unknown-input estimation of a linear function of the state with delay) Refer to system (3) connected as shown in Fig. 2. Given the window \( N \) and the delay time \( N_p \), find the FIR weighting matrices \( \Phi_d(\ell) \) \((\ell = 1, \ldots, N - 1)\) providing the following correspondences are set: \( A = A_d', B = C_d', C = B_d', D = D_d', H = H_d', G = G_d' \).

2.1 Geometric conditions for perfect decoupling

In this section some geometric conditions guaranteeing perfect or almost perfect decoupling are briefly recalled. They have been proven and applied in (Barbagli et al., 2000) and (Marro et al., 2000). Let us denote by \( \hat{y}^* \) the maximum \((\hat{A}, \text{im}\hat{B})\)-controlled invariant contained in \( \ker \hat{C} \) and \( \hat{y}^* \) the minimum \((\hat{A}, \text{ker}\hat{C})\)-conditioned invariant containing \( \text{im}\hat{B} \), with \( (\hat{A}, \hat{B}, \hat{C}) := (A, B, C) \) if both \( D \) and \( G \) are null matrices and

\[
\hat{A} := \begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix}, \quad \hat{B} := \begin{bmatrix} B \\ D \end{bmatrix}, \quad \hat{C} := \begin{bmatrix} 0 & I_q \end{bmatrix},
\]

if not. Also, define \( \hat{H} := \text{im}H \) if both \( D \) and \( G \) are null matrices, \( \hat{H} := \text{im}[H'G']' \) if not.

If system (1) is minimum phase with respect to \( u \), the condition

\[
\hat{H} \subseteq \hat{y}^*
\]

guarantees that perfect decoupling is achievable with a stable feedforward dynamic unit without any preaction. The condition

\[
\hat{H} \subseteq \hat{y}^* + \hat{j}^*
\]

guarantees that perfect decoupling is achievable with a stable feedforward dynamic unit with only a relative-degree preaction.

On the other hand, if the system (1) is nonminimum phase, condition (6) enables perfect decoupling only
as the preaction time $N_p$ approaches infinity. However, almost perfect decoupling is achievable if $N_p$ is large enough with respect to the time constant of the unstable zero closest to the unit circle. In the above mentioned cases $\|W\|_2$ is zero or can made arbitrarily small. If condition (6) is not satisfied or system (1) is non-minimum phase and the available preaction time is not large enough, the $H_2$ optimal design object of this paper is a convenient resort and the use of FIR compensators instead of dynamic systems unifies and greatly simplifies the synthesis procedures.

3. THE MODIFIED FINITE HORIZON LQ PROBLEM

Solution of Problem 1 can easily be obtained by slightly extending an efficient algorithm for solving the finite-horizon linear quadratic optimal control problem, possibly cheap or singular. This extension and the corresponding algorithmic solution are presented below as Problem 3 and Theorem 1.

**Problem 3.** (Finite-horizon LQ problem with a previewed impulse input and constrained final state) Consider system (1) with given initial state $x(0)=x_0$ and final state constrained as

$$\Gamma x(N) = y_f,$$  

where $\Gamma$ and vector $y_f$ are given. The final time $N$ is assumed to be greater than the controllability index of $(A,B)$. Let $h(k) = \hat{h}\delta(k-N_p)$, where both $\hat{h}$ and $N_p < N$ are given. Find a control sequence $u(k)$ ($k=0,\ldots,N-1$) minimizing the cost

$$J := \sum_{k=0}^{N-1} y'(k)y(k) + x'(N)Z'Zx(N).$$

where $Z$, a penalty matrix on the final state, is given.

Let us introduce the following compact notation for the control sequence:

$$u_N := \begin{bmatrix} u(0) \\ u(1) \\ \vdots \\ u(N-1) \end{bmatrix}.\tag{9}$$

A solution of Problem 3 is provided by the following theorem.

**Theorem 1.** A solution of Problem 3 is given as

$$u_N^* = T_N x_0 + V_N y_f + W_N \bar{h},$$

where the matrices $T_N$, $V_N$ and $W_N$ are defined by

$$T_N := -P_{B_N}(\Gamma L_N)^{\#} \Gamma A^N - K (B_N K)^{\#} A_N,$$

$$V_N := P_{B_N}(\Gamma L_N)^{\#},$$

$$W_N := -P_{B_N}(\Gamma L_N)^{\#} \Gamma A^{N-N_p-1} H - K (B_N K)^{\#} H_N,$$

being $P_{B_N} = (I - K (B_N K)^{\#} B_N)$ and

$$A_N := \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{N-1} \end{bmatrix}, B_N := \begin{bmatrix} D \\ CB \\ \vdots \\ CA^{N-2} B \end{bmatrix},$$

$$L_N := \begin{bmatrix} 0 \\ \vdots \\ (CA^{N-1} - H) \\ (ZA^{N-N_p-1} - H) \end{bmatrix},$$

and $K$ is a basis matrix for ker($\Gamma L_N$). The optimal cost is

$$J^* = \begin{bmatrix} x_0 \\ y_f \\ \bar{h} \end{bmatrix}^{\#} \begin{bmatrix} C_N & C_N D_N & C_N E_N \\ D_N C_N & D_N D_N & D_N E_N \\ E_N C_N & E_N D_N & E_N E_N \end{bmatrix} \begin{bmatrix} x_0 \\ y_f \\ \bar{h} \end{bmatrix},$$

where matrices $C_N$, $D_N$ and $E_N$ are defined by

$$C_N = P_{B_N} (A_N - B_N (\Gamma L_N)^{\#} \Gamma A^N),$$

$$D_N = P_{B_N} (\Gamma L_N)^{\#},$$

$$E_N = P_{B_N} (H_N - B_N (\Gamma L_N)^{\#} \Gamma A^{N-N_p-1} H).$$

The proof of Theorem 1 has been omitted. A complete proof of the theorem is given in (Marro et al., 2000b). Theorem 1 provides an algorithm framework to deal with the following finite horizon LQ optimization problems.

1. Standard LQ problem with both the initial and the final state completely assigned. In this case matrices $H$ and $G$ and, consequently, $W_N$, $H_N$ and $E_N$ are not defined. The problem is solved by assuming $\Gamma = I_n$, $y_f = x_f$, and $Z = 0_{1,n}$.

2. Standard LQ problem with the initial state assigned and the final state simply weighted. In this case matrices $H$, $G$, $W_N$, $H_N$ and $E_N$ are not defined like in the previous case. The problem is solved by assuming $\Gamma = 0_n$, $y_f = 0_{n,1}$, while $Z$ is given to define the cost of the final state. The solution does not depend explicitly on the final state (matrices $V_N$ and $D_N$ are null).

3. LQ problem with both the initial and the final state completely assigned and an impulsive input $h(k) = \hat{h}\delta(k-N_p)$. The problem is solved like in case 1 above, but with all the matrices defined.

4. LQ problem with both the initial state assigned, the final state simply weighted, and an impulsive $h(k) = \hat{h}\delta(k-N_p)$. The problem is solved like in case 2 above, with all the matrices defined.

Theorem 1 describes a pseudoinversion procedure to solve the optimal control Problem 3 with a $N_p$-previewed impulse disturbance input and weighted final state. Since the dimensions of the matrices to be pseudo-inverted are proportional to the number of steps $N$ of the control time interval, this technique is subject to a dimensionality constraint depending on the computational capability available. However, this drawback can be overcome by means of the additive procedure described in Section 5.
4. DESIGN OF THE $H_2$ OPTIMAL FIR COMPENSATOR

The solution of the $H_2$ optimal control problem stated in Problem 1 can easily be derived by using the algorithm provided in Theorem 1. The $H_2$ optimal control design of a FIR controller $\Phi$ with the $N_p$-previewed signal $h(k)$ corresponds to solve a finite horizon linear quadratic problem with impulse disturbance of the type stated in Problem 3. In fact, the $H_2$ norm of the transfer function of the overall system from $h_p(k)$ to $y(k)$, see Fig. 1, can be written as

$$\|W\|_2 = \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} W(e^{j\omega}) W^*(e^{j\omega}) d\omega \right)^{\frac{1}{2}}$$

(16)

where $w(k)$ denotes the impulse response matrix corresponding to $W(e^{j\omega})$. By using (16) Problem 1 is easily re-stated in terms of Problem 3, and solved with Theorem 1.

Theorem 2. A solution of Problem 1 is provided by

$$\Phi(j) = \left[ \begin{array}{c} \phi(0) \\ \phi(1) \\ \vdots \\ \phi(N-1) \end{array} \right] W_N,$$

with

$$\phi(i) = 0_n, \quad i \neq j, \quad \phi(j) = I_s,$$

where $W_N$ is defined as in Theorem 1, eq. (15). Matrices $B_N$ and $H_N$ in eq. (11) are computed with $Z = \sqrt{S_o}$, where $S_o$ denotes the solution of the Lur'e equation

$$S - A'SA = C'C.$$

(18)

Proof: Owing to (16), it is immediate to verify that $\|W\|_2^2$ is equal to the cost index (8) in the statement of Problem 3 under the assumptions

$$x(0) = 0_{n,s}, \quad \Gamma = 0_n, \quad y_f = 0_{n,s},$$

$$Z = \sqrt{S_o}, \quad \bar{h} = I_s.$$

Note that $x(N)S_o x(N)$ accounts for the cost from $k=N$ to $k=\infty$ and is evaluated through the Lur'e equation (18), since the control input becomes zero from $k=N$ on.

5. EXTENSION OF THE CONTROL INTERVAL

Referring to Fig. 3, assume that the overall control interval $N_t$ is divided in a finite number of subintervals whose length $N$ satisfies the computational constraint and is greater than the the controllability index of $(A,B)$. Three subarcs can easily be distinguished: it will be shown that the costs on the intervals 1 and 3 are expressed by quadratic forms of $x_1$ and $x_2$, respectively. Once the corresponding cost matrices $S_1$ and $S_2$ have been determined as described in Section 5.2 below, it is possible to derive $x_1$ and $x_2$ as follows.

5.1 Solution for interval 2

The cost for the whole interval $[0,N_t]$ can be expressed as

$$c = x_1^T S_1 x_1 + x_2^T S_2 x_2 + \left[ \begin{array}{c} x_1 \\ x_2 \end{array} \right]^T \left[ \begin{array}{cc} C N C_N & C_N D_N C_N' E_N \\ D_N C_N & D_N D_N C_N' E_N \end{array} \right] \left[ \begin{array}{c} x_1 \\ x_2 \end{array} \right]$$

(19)

or, by setting $\xi := [x_1' x_2']$ and consequently defining new matrices,

$$c = \xi^T M \xi + \left[ \begin{array}{c} \xi \\ \bar{h} \end{array} \right]^T \left[ \begin{array}{cc} R_1 & R_2 \\ R_2 & R_3 \end{array} \right] \left[ \begin{array}{c} \xi \\ \bar{h} \end{array} \right].$$

(20)

The optimal value of $\xi$ is derived as

$$\xi_o = -(M + R_1)^{-1} R_2 \bar{h}.$$

(21)

Being $x_1$ and $x_2$ known, the control sequences of intervals 1 and 3 can be derived as $(V_N)_{x_1}$ and $(T_N)_{x_2}$, respectively, where $(V_N)_{x_1}$ and $(T_N)_{x_2}$ denote the global matrices obtained with the iterative procedure described in the subsequent Section 5.2. The control sequence of interval 2 is computed as $T_N x_1 + V_N x_2 + W_N \bar{h}$, according to (10).

5.2 Welding subarcs in intervals 1 and 3

Let us consider Problem 3 with $\bar{h} = 0$. Recall that, if both the initial state $x_0$ and the final state $x_f$ are given, Theorem 1, which refers to interval $[0,N_t]$, defines matrices $T_N$, $V_N$, $C_N$ and $D_N$ that provide an optimal control sequence expressed as

$$u_N^* = T_N x_0 + V_N x_f,$$

(22)

and the corresponding optimal cost as

$$J^* = \left[ \begin{array}{c} x_0 \\ x_f \end{array} \right]^T \left[ \begin{array}{cc} M_1 & M_2 \\ M_2' & M_3 \end{array} \right] \left[ \begin{array}{c} x_0 \\ x_f \end{array} \right],$$

(23)

with $M_1 = C_N C_N$, $M_2 = C_N D_N$, $M_3 = D_N D_N$. We shall show that the control interval can arbitrarily be enlarged by welding optimal subarcs. Let us suppose that the generic time interval $[0,N_t]$ has been divided into two subsequent subintervals, $[0,N_t]$ and $[N_t,N_1 + N_2]$ and that the corresponding control input
and cost matrices \((T_{N_1}, V_{N_1}, M_{1,1}, M_{2,1}, M_{3,1})\) and \((T_{N_2}, V_{N_2}, M_{1,2}, M_{2,2}, M_{3,2})\) have already been computed. Assume \(x(0) = x_0, x(N) = x_1, x(\bar{N}) = x(N_1 + N_2) = x_f\).

The overall cost is
\[
    c = x_1' M_{1,1} x_0 + 2x_1' M_{1,2} x_1 + x_1' M_{3,1} x_1
    + x_1' M_{2,1} x_1 + 2x_1' M_{2,2} x_f + x_f' M_{3,2} x_f.
\]

The value of \(x_1\) minimizing \(c\) is derived by nulling \(\nabla_c x_1\) as
\[
    x_1 = Q_1 x_0 + Q_2 x_f,
\]
where
\[
    Q_1 := -(M_{3,1} + M_{1,2})' M_{2,1},
    Q_2 := -(M_{3,1} + M_{1,2})' M_{2,2}.
\]

By substitution in (24) we obtain the cost matrices referring to the overall interval \([0, \bar{N}]\) as
\[
    M_1 := M_{1,1} + 2V_{N_1} Q_1 + Q_1' (M_{3,1} + M_{1,2}) Q_1,
    M_2 := M_{2,1} Q_2 + Q_1' (M_{3,1} + M_{1,2}) Q_2 + Q_2' M_{2,2},
    M_3 := Q_2' (M_{3,1} + M_{1,2}) Q_2 + 2Q_2' M_{2,2} + M_{3,2}.
\]

and the corresponding control input matrices as
\[
    T_N = \begin{bmatrix} T_{N_1} + V_{N_1} Q_1 \\
    T_{N_2} Q_1 \end{bmatrix},
    V_N = \begin{bmatrix} V_{N_1} Q_2 \\
    T_{N_2} Q_2 + V_{N_2} \end{bmatrix}.
\]

The above described procedure can be iterated to achieve the solution of the problem in an arbitrarily large control interval, starting from two intervals for which direct computation as provided in Theorem 1 is feasible. At the last iteration, if the final state is not sharply assigned but just weighted (like in the case of interval 3), matrices \(V_{N_2}, M_{2,2}\) and \(M_{3,2}\) should be omitted in eqs. (26), (27) and (28) since they are not defined.

6. AN EXAMPLE

Let
\[
    A = \begin{bmatrix} 0.5 & -0.4 & 0 \\
    0.1 & 0.7 & 0 & -0.5 \\
    0 & 0 & 0.4 & 0 \\
    0.2 & 0 & 0 & 0.6 \end{bmatrix},
    B = \begin{bmatrix} 1 \\
    0 \\
    1 \\
    0 \end{bmatrix},
    H = \begin{bmatrix} 0 \\
    1 \\
    1 \\
    4 \end{bmatrix},
    C = \begin{bmatrix} 1 & 0 & 0 & 0 \\
    0 & 1 & 0 & 0 \end{bmatrix},
    D = \begin{bmatrix} 0 \\
    0 \\
    0 \\
    0 \end{bmatrix},
    G = \begin{bmatrix} 0 \\
    0 \end{bmatrix}.
\]

The system \((A, B, C, D)\) is both left and right invertible. The plant is nonminimum phase (with invariant zeros 0.8 and 1.1). Condition (6) is satisfied because of right invertibility, so that perfect decoupling could be achieved at the limit as both \(N_p\) and \(N - N_p\) approach infinity. Suppose that, due to restricted preview time of the signal to be decoupled, a preaction \(N_p = 20\) is only possible and assume \(N = 40\) for the FIR compensator window. Fig. 4 shows the FIR gains that optimally decouple a previewed unit impulse \(h(k) = \delta(k - N_p)\) occurring at \(k = N_p\) and the corresponding optimal responses. The square of the \(H_2\) optimal norm computed in this case is \(\|W\|_2^2 = 0.246\).

Fig. 4. FIR gains (left) and optimally decoupled outputs (right) for \(N = 40, N_p = 20\).

Fig. 5. FIR gains (left) and optimally decoupled outputs (right) for \(N = 40, N_p = 0\).
Suppose now that no preview of signal $h(k)$ is available. A FIR compensator with $N_p = 0$ and $N = 40$ is derived. The FIR gains and corresponding optimal responses referring to this case are shown in Fig. 5. The square of the $H_2$ optimal norm is $\|W\|^2_2 = 11.333$ in this case, hence significantly greater than the preview case.

7. CONCLUDING REMARKS

A design procedure for a FIR system with given window $N$, providing $H_2$-optimal decoupling of a $N_p$-step previewed signal has been described. The use of a FIR compensator instead of a dynamic unit, although not extensively treated in the literature, is advisable since preaction reduces the $H_2$ norm of the transfer function matrix from the input to be decoupled to the controlled output also in the minimum phase case if the geometric conditions (5) and (6) recalled in Section 2.1 are not satisfied. Since feedthrough matrices $D$ and $G$ are present in the system equations, the disturbance decoupling problem also includes as a particular case $H_2$ optimal right inversion (or tracking). The results obtained are directly applicable to the dual problem, $H_2$ optimal unknown-input observation of a linear function of the state with $N_p$-step postknowledge and, as a particular case, left inversion (estimation of an unknown input).

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9. REFERENCES


