DYNAMIC ANTI-INTEGRATOR-WINDUP CONTROLLER DESIGN FOR LINEAR SYSTEMS

Mitsuru Kanamori* and Masayoshi Tomizuka†

*Department of Control Engineering, Maizuru National College of Technology, 234 Aza Shiroya, Maizuru, Kyoto 625-8511, JAPAN. Email: kanamori@maizuru-ct.ac.jp
†Department of Mechanical Engineering, University of California at Berkeley, 5100B Etcheverry Hall, Berkeley, CA 94720-1740, U.S.A. Email: tomizuka@me.berkeley.edu

Abstract: This paper is concerned with the design of dynamic anti-integrator-windup controllers for critically stable linear systems. The constant gain for anti-windup in the conventional approach is extended to a transfer function. It is also shown that the proposed compensator design is an extension of the Kothare’s linear conditioning for static anti-windup compensator. The effectiveness of the proposed design is demonstrated by numerical simulations of a one-link flexible arm. Copyright © 2002 IFAC

Keywords: Input constraints, Dynamic controller, Anti-windup compensator, Lyapunov design, Flexible robot arm

1. INTRODUCTION

Actuator saturation may adversely affect the performance and stability of a closed loop system if it is not properly taken into consideration at the time of controller design. Kothare, Campo, Morari and Nett (1994) presented a unified structure which includes most of all anti-windup compensators of the static class. Weston and Postlethwaite (2000) have proposed a novel linear conditioning for systems containing saturating actuators based on co-prime factorization, which is an extension of the Kothare’s unified conditioning (Kothare, et al., 1994). On the other hand, Kapoor, Teel and Daoutidis (1996) considered compensators with an integral action, and proposed an approach to address anti-integrator wind up and regulation with constant anti-windup gain for general linear compensator. Niu and Tomizuka (1998; 2000) followed the idea of Kapoor et al. (1996), and developed anti-windup controllers with an internal model to accomplish asymptotic tracking for a class of disturbances. Furthermore, their method has been further extended for asymptotic tracking by output feedback control (Kanamori and Tomizuka, 2001).

In this work, the design method for anti-windup constant gain by Niu, Tomizuka and Kanamori will be extended to the anti-windup transfer function. Some conditions for the output feedback control in the previous work (Niu and Tomizuka, 2000; Kanamori and Tomizuka, 2001) will be eliminated. Though the dynamic anti-windup modification does not satisfy the criteria for anti-windup operators in the unified framework by Kothare, et al. (1994), it will be shown that the proposed compensator is an extension of the Kothare’s linear conditioning for static anti-windup compensators. The proposed approach provides the closed loop system a superior sensitivity property against the dynamics of the actuator saturation. The remainder of this paper is structured as follows. In Section 2, a class of systems of our attention and a few technical preliminaries are presented. In Section 3, the state feedback case is presented. In Section 4, the state feedback result is extended to the output feedback case. In Section 5, the proposed approach is shown to be similar to the one based on co-prime factorization. In Section 6, the error regulation property is quantified. Finally, in Section 7, an illustrative example is provided.

2. PRELIMINARIES

Consider the linear system
\[ \dot{x} = Ax + B\sigma(u), \]
\[ y = Cx, \]  
(1)

where \( x \in \mathbb{R}^n \) represents the \( n \)-dimensional state vector, \( u \in \mathbb{R} \) the control input, and \( y \in \mathbb{R} \) the output. \( A, B \) and \( C \) are constant matrices of appropriate dimensions. The matrix \( A \) has no right half plane eigenvalues and no eigenvalues on the imaginary axis except for a possible single pole at 0, i.e., there exists a positive definite symmetric matrix \( P \) such that \( A^T P + PA \leq 0 \). The function \( \sigma \) represents actuator saturation, and its characteristics are as shown below.

**Assumption A0:** Function \( \sigma : \mathbb{R} \to \mathbb{R} \) satisfies

\[
\sigma(u) = \begin{cases} 
  u_{\min} & \text{for } u < u_{\min} < 0, \\
  u & \text{for } u_{\min} \leq u \leq u_{\max}, \\
  u_{\max} & \text{for } u > u_{\max} > 0.
\end{cases}
\]  
(2)

The function \( \psi(u) \) is defined by

\[
\psi(u) = u - \sigma(u).
\]  
(3)

The idea of \( \sigma \)-stable feedback law is now introduced.

**Definition 2.1 \( \sigma \)-stable feedback law:** The feedback law \( u_k = Kx \) is \( \sigma \)-stable for the system of Eq. (1) if it guarantees that, for each \( x_0 \in \mathbb{R}^n \), the trajectory of the system \( \dot{x} = Ax + B\sigma(Kx + d) \), \( x(0) = x_0 \) converges to zero asymptotically. The signal \( d \in \mathbb{R} \) is converging to zero and is given by \( z = A_1 z \), \( d = C_1 z \), where \( z \) is the any dimensional state vector and \( A_1, C_1 \) is Hurwitz constant matrix with appropriate dimensions.

Note that there always exists a \( \sigma \)-stable feedback law for a stabilizable, critically stable system of the form of Eq. (1) where \( \sigma \) satisfies assumption A0. The following Lemma establishes the conditions for the \( \sigma \)-stable feedback law:

**Lemma 1:** If the pair \((A, B)\) is stabilizable, \( u_k = Kx \) is a \( \sigma \)-stable feedback law for the system \( \dot{x} = Ax + B\sigma(u) \) where \( \sigma \) satisfies assumption A0 as long as there exists positive definite symmetric matrix \( P \) which satisfies the following inequalities:

\[
A^T P + PA \leq 0, \quad (4)
\]
\[
(A + BK)^T P + P(A + BK) + PBB^T P < 0. \quad (5)
\]

Note that the second inequality (5) implies that the matrix \((A + BK)\) is Hurwitz. In the special case of \( A \) being Hurwitz, \( K = -B^T P \) may be used. Then, Eq. (5) becomes the Riccati inequality and Eq. (4) is eliminated.

Outline of Proof of Lemma 1: This Lemma can be proved by working on the following candidate Lyapunov function:

\[
V = x^T P x + z^T P z, \quad (6)
\]

where noting that \( A_1 \) is Hurwitz, the positive definite symmetric matrix \( P \) is selected to satisfy

\[
A_1^T P + P A_1 + C_1^T C_1 < 0. \]  
(7)

Following steps similar to those in Kapoor, et al (1996) and Niu and Tomizuka (1998), the time derivative of Eq. (6) may be shown to be either negative definite or negative semidefinite for each of three cases in Eq. (2). The negative semidefinite case arises when \( z = 0 \) and

\[
x^T (A^T P + PA)x = 0 \]  
(8)

for nonzero \( x \) under actuator saturation. This situation can be analyzed as shown in Appendix to conclude \( \sigma \)-stability.

The following Lemma establishes global asymptotic stability of the system with disturbance, \( \dot{x} = Ax + B[\sigma(u) - \Gamma] \) where \( \Gamma \) represents the disturbance.

**Lemma 2:** Assume that the pair \((A, B)\) is stabilizable and \( u_k = Kx \) is a \( \sigma \)-stable feedback law for the system \( \dot{x} = Ax + B\sigma(u) \). Then, the following system:

\[
\dot{z} = A_1 z, \quad A_1 : \text{Hurwitz}, \quad \dot{x} = Ax + B[\sigma(Kx + C_1 z + \Gamma) - \Gamma] \]  
(9)

is globally asymptotically stable as long as the condition, \( u_{\min} < \Gamma < u_{\max} \) is satisfied.

Outline of Proof of Lemma 2: Using the following relations for elimination of \( \Gamma \):

\[
u_{\min} \leq u \leq u_{\max} \Rightarrow \\
\sigma(Kx + C_1 z + \Gamma) - \Gamma = Kx + C_1 z \]  
(10)
\[
u_{\max} < u \Rightarrow 0 < u_{\min} - \Gamma < Kx + C_1 z, \]  
(11)
\[
u_{\min} < u \Rightarrow Kx + C_1 z < u_{\min} - \Gamma < 0, \]  
(12)

global asymptotic stability of the system of Eq. (9.9) is assured in the same manner as the proof of Lemma 1.

For the system of Eq. (1), a dynamic compensator in the following form is considered:

\[
\dot{x}_c = -Cx, \quad u = C_c x_c + D_c x, \]  
(13)
where \( x_c \) represents the states of the compensator, and \( C_c \) and \( D_c \) are constant matrices. The dynamics of the compensator is modified by the dynamic anti-windup block \( L(s) \) as shown in Fig.1. This dynamic block is represented in state space as

\[
\begin{bmatrix}
    \dot{x}_d \\
    y_d
\end{bmatrix} =
\begin{bmatrix}
    A_d & B_d \\
    C_d & D_d
\end{bmatrix}
\begin{bmatrix}
    x_d \\
    \psi(u)
\end{bmatrix}.
\] (14)

Note that \( L(s) = C_d(slI - A_d)^{-1}B_d + D_d \). The following assumption is invoked:

**Assumption A1:** The pair \( \begin{bmatrix} 0 & -C \\ 0 & A \end{bmatrix} \) is controllable.

In the following section, a procedure to design \( C_c, D_c \) and \( L(s) \) will be presented such that the origin of the closed-loop system of Eq. (1) under the compensator of Eq. (13), with the dynamic anti-windup modification, is globally asymptotically stable.

### 3. STATE FEEDBACK DESIGN

The system of Eq. (1) under the compensator of Eq. (13) subject to the dynamic anti-windup modification of Eq. (14) may be represented as follows:

\[
\begin{bmatrix}
    \dot{x}_c \\
    \dot{x} \\
    \dot{x}_d \\
    u
\end{bmatrix} =
\begin{bmatrix}
    0 & -C & C_d & D_d \\
    BC_c & A + BD_c & 0 & -B \\
    0 & 0 & A_d & B_d \\
    C_c & D_c & 0 & 0
\end{bmatrix}
\begin{bmatrix}
    x_c \\
    x \\
    x_d \\
    \psi(u)
\end{bmatrix}.
\] (15)

The overall system is depicted in Fig.1.

**Theorem 1:** Consider the system given by Eq. (15). Assume that the following conditions are satisfied:

1. The gain matrix \( \begin{bmatrix} C_c & D_c \end{bmatrix} \) is chosen to make the following matrix \( A_c \) Hurwitz.

   \[
   A_c = \begin{bmatrix} 0 & -C \\ BC_c & A + BD_c \end{bmatrix}.
   \] (16)

2. The anti-windup transfer function is chosen as

   \[
   \dot{\psi}(u) = \begin{bmatrix} 0 & 0 \\ C & D \end{bmatrix}
   \begin{bmatrix}
    x_c \\
    x + x_d
\end{bmatrix} + H(C\hat{x} - Cx),
   \]

   where \( \hat{x} \) refers to the observer states and \( H \) is chosen such that \( (A + HC) \) is Hurwitz, which requires the following assumption.

**Assumption A2:** The pair \( (A, C) \) is observable.

The compensator dynamics will be modified by the

\[
\begin{bmatrix}
    A_d & B_d \\
    C_d & D_d
\end{bmatrix} = \begin{bmatrix} A + BK & B \\ TBF & TB \end{bmatrix},
\] (17)

where, the gain matrices \( K, F \) and \( T \) are

\[
T = (A + BD_c)^{-1}, \quad F = -CT, \quad K = D_c + F.
\] (18)

If \( u_k = Kx \) is a \( \sigma \)-stable feedback law, then the closed-loop system of Eq. (15) achieves global asymptotic stability.

**Proof of theorem 1:** Substituting Eq.(17) to Eq.(15), utilizing the transformation matrix \( J \),

\[
J = \begin{bmatrix} I_c & 0 & -T \\ 0 & I & I \\ 0 & I & 0 \end{bmatrix}, \quad J^{-1} = \begin{bmatrix} I_c & T & -T \\ 0 & 0 & I \\ 0 & I & -I \end{bmatrix},
\]

and noting that \( \psi(u) \) is given by Eq.(3), the closed-loop system is represented as:

\[
\begin{bmatrix}
    \dot{z}_c \\
    \dot{x}
\end{bmatrix} = \begin{bmatrix} \left( A + B\sigma(Kx + C_c z_c) \right) \end{bmatrix} \begin{bmatrix}
    x_c \\
    x + x_d
\end{bmatrix},
\]

where,

\[
\begin{bmatrix}
    z_c \\
    z
\end{bmatrix} = \begin{bmatrix} C_c & -F \end{bmatrix} \begin{bmatrix}
    x_c \\
    x + x_d
\end{bmatrix}.
\] (21)

The system of Eq. (20) is globally asymptotically stable based on Lemma 1 since the matrix \( A_c \) is Hurwitz by the first condition of Eq. (16) in the theorem. This in turn implies that the system of Eq. (15) is globally asymptotically stable.

### 4. OUTPUT FEEDBACK DESIGN USING OBSERVER

When all the process state variables are not available for measurement, the dynamics of the compensator of Eq. (13) is augmented with observer. Let us consider the following dynamic output feedback compensator:

\[
\dot{\hat{x}} = A\hat{x} + B\sigma(\hat{u}) + H(C\hat{x} - Cx),
\]

\[
\begin{bmatrix} x_c \\
    x + x_d
\end{bmatrix} = \begin{bmatrix} C_c & -F \end{bmatrix} \begin{bmatrix}
    x_c \\
    x + x_d
\end{bmatrix}.
\] (22)

Fig.1. State feedback design for anti-windup.
Theorem 2: Consider the system given by Eqs. (1), (14) and (22). Assume that the every condition in Theorem 1 is satisfied. If \( u_s = Kx \) is \( \sigma \)-stable, then the closed loop system achieves global asymptotic stability.

Proof of theorem 2: Utilizing the following transformation matrix \( J_o \) to the state vector \( \begin{bmatrix} x_e^T & \hat{x}^T & x^T & x_d^T \end{bmatrix}^T \):

\[
J_o = \begin{bmatrix} I_e & 0 & 0 & -T \\ 0 & I & 0 & 1 \\ 0 & 0 & I & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad J_o^{-1} = \begin{bmatrix} I_e & 0 & T & -T \\ 0 & I & -1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\]

(23)

The system may be represented in the following form by a procedure similar to the one in the proof of theorem 1:

\[
\begin{align*}
\dot{z}_o &= A_o z_o, \\
\dot{x} &= Ax + B \sigma(Kx + C_o z_o), \\
z_o &= \begin{bmatrix} x_e - Tx_d \\ \hat{x} + x_d \\ x + x_d \end{bmatrix}, \quad C_o = \begin{bmatrix} C & D & -K \end{bmatrix}
\end{align*}
\]

(24)

(25)

The system of Eq. (24) is globally asymptotically stable based on Lemma 1 because \( A_o \) is Hurwitz. This in turn implies that the system given by Eqs. (1), (14) and (22) is globally asymptotically stable.

Remark 1: In comparison with the previous methods (Niu and Tomizuka, 2000; Kanamori and Tomizuka, 2001), the conditions for global asymptotic stability have been relaxed. In particular, the present approach does not involve a matrix inequality (Eq. (12) in Kanamori and Tomizuka, 2001), which is not easy to satisfy.

5. OUTPUT FEEDBACK DESIGN BASED ON CO-PRIME FACTORIZATION

It will be shown below that the proposed dynamic anti-windup compensator is similar to the one based on co-prime factorization by Weston et al. (2000), which is an extension of the Kothare’s linear conditioning. Let us consider the output feedback with anti-windup compensator as shown in Fig.2. Matrices \( A_k, B_k, \) and \( C_k \) of the compensator are chosen such that the closed-loop system is identical to the system using observer as follows:

\[
A_k = \begin{bmatrix} 0 & 0 \\ B C_e & A + B D_e + H C \end{bmatrix}, \quad B_k = \begin{bmatrix} -I_e \\ -H \end{bmatrix}
\]

(27)

\[
C_k = [C_e D_e].
\]

The transfer function \( N(s) \) and \( M(s) \) are also chosen by utilizing the state feedback gain matrix \( K \) such that the matrix \( (A + BK) \) is Hurwitz:

\[
N(s) = C(sI - (A + BK))^{-1}B,
\]

(28)

\[
M(s) = I - K(sI - (A + BK))^{-1}B.
\]

Theorem 3: Consider the system given by Eqs. (1), (27) and (28). If \( u_s = Kx \) is \( \sigma \)-stable, then the closed loop system achieves global asymptotic stability.

Proof of theorem 3: Utilizing the transformation matrix \( J_e \) to the state vector \( \begin{bmatrix} x_e^T & x^T & x_d^T \end{bmatrix}^T \):

\[
J_e = \begin{bmatrix} I_e & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad J_e^{-1} = \begin{bmatrix} I_e & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & -1 \end{bmatrix}
\]

(29)

The transformed closed-loop system may be represented in the following form:

\[
\begin{align*}
\dot{z}_w &= A_w z_w, \\
\dot{x} &= Ax + B \sigma(Kx + C_o z_w), \\
z_w &= \begin{bmatrix} x_k \\ x + x_d \end{bmatrix}
\end{align*}
\]

(30)

(31)

where, \( x_d \) is the states of the anti-windup compensator given by \( N(s) \) and \( M(s) \). The system of Eq. (30) is globally asymptotically stable based on Lemma 1 because \( A_w \) is Hurwitz. This in turn implies that the system given by Eqs. (1), (27) and (28) is
Remark 2: the system of Eq. (30) is in the same form as the system of Eq.(24). Thus, the proposed compensator is similar to the one based on co-prime factorization. This implies that the proposed method an extension of the Kothare’s linear conditioning for static anti-windup.

Remark 3: Note that the selection of the linear compensator (27) is independent from the selection of $K$ in Eq.(28) for σ-stability. On the other hand, in the proposed method the selection of $K$ in Eq. (17) and the selection of the linear compensator (13) are coupled and they must be performed simultaneously for σ-stability.

Remark 4: $z_\omega$, $z_{wo}$ and $z_{wc}$ in Eqs. (20), (24) or (30) converges to zero because $A_\omega$, or $A_{wo}$ is asymptotically stable. Note that the state of the anti-windup transfer function $x_\omega$ appears in all components of $z_\omega$ in Eq.(20) and $z_{wo}$ in Eq.(24) while it appears only in the second component of $z_{wc}$ in Eq.(30). This implies that in the proposed approach all parts of the over all system work in a synchronous manner to deal with actuator saturation while in the method based on co-prime factorization, the input to the compensator $[A_k, B_k, C_k]$ is the second component of $z_{wo}$, $x + x_d$, and the state of the compensator $x_k$ converges to zero as a consequence of the convergence of $x + x_d$. This implies that handling of actuator saturation in the proposed approach is more direct and effective than that in the co-prime factorization approach.

### 6. ERROR REGULATION PERFORMANCE

The regulation performance of the compensator based on state feedback is presented in this section. The output feedback case is omitted because of the limitation of space. The system of Eq.(1) is cast in the framework of the linear regulator theory and the following system is considered,

$$
\dot{x} = Ax + B\sigma(u) + B_w w,
$$

$$
e = r - Cx,
$$

where $w \in \mathbb{R}^n$ and $r \in \mathbb{R}$ represent constant disturbance and constant reference, respectively. The symbol $e \in \mathbb{R}$ represents the error that needs to be regulated, and $B_w$ is a constant matrix of appropriate dimension. Define the new variables,

$$
\bar{x}_e = x_e - \bar{x}_e,
$$

$$
\bar{x} = x - \bar{x},
$$

where $\bar{x}_e$ and $\bar{x}$ represent the converged values of the states $x_e$ and $x$, respectively. Then, they satisfy

$$
\begin{bmatrix}
0 \\
0
\end{bmatrix} =
\begin{bmatrix}
0 & -C \\
BC_e & A + BD_e
\end{bmatrix}
\begin{bmatrix}
\bar{x}_e \\
\bar{x}
\end{bmatrix} +
\begin{bmatrix}
r \\
B_w w
\end{bmatrix}.
$$

The closed-loop system with disturbance $w$ and reference $r$ in Fig.1 is represented by

$$
\begin{bmatrix}
\bar{x}_e \\
\bar{x}
\end{bmatrix} =
\begin{bmatrix}
0 & -C & C_d & D_d \\
BC_e & A + BD_e & 0 & -B
\end{bmatrix}
\begin{bmatrix}
\bar{x}_e \\
\bar{x}
\end{bmatrix} +
\begin{bmatrix}
r \\
B_w w
\end{bmatrix}.
$$

$\Gamma$ represents the asymptotic output of the actuator (i.e., $u(t) \to \Gamma$ as $t \to \infty$) that is determined by

$$
\Gamma = C_r \bar{x}_e + D_r \bar{x}.
$$

Note that the error regulation is identical to the regulation of $\bar{x}$ because of the relation $e = -C\bar{x}$.

### 7. DESIGN EXAMPLE

A simulation study for a one link flexible arm is presented below. The dynamics of the arm is described by

$$
A =
\begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & -10 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & -223.1 & -0.7528
\end{bmatrix},
B =
\begin{bmatrix}
1.958 \\
0 \\
0 \\
-2.167
\end{bmatrix},
C =
\begin{bmatrix}
1 & 0 & -5.326 & 0
\end{bmatrix},
B_w =
\begin{bmatrix}
0 & 2 & 0 & 0
\end{bmatrix}.
$$

Note that the system is critically stable with one eigenvalue at 0. The disturbance $w$ is a unit step
that the matrix $D_c$ function entering the system at time $t=0$. The reference $r$ is a pulse signal with a period of 4 sec and a magnitude of 1. The design procedure is as follows: (1) The gain matrices $C_r$ and $D_u$ are chosen to satisfy the desired response or performance of the system; (2) The matrix $K$ is checked whether $K$ is a $\sigma$-stable feedback gain by using LMI analysis tool; (3) The function $L(s)$ for anti-windup modification is determined; and (4) Estimating the worst disturbance, $u_{\text{min}} < \Gamma < u_{\text{max}}$ is checked. The gain matrices $C_r$ and $D_u$ are chosen to satisfy the optimal regulator theory such that the matrix $K$ is a $\sigma$-stable feedback gain. The gain matrices and the dynamic anti-windup block are

\begin{align}
C_r &= 387, \\
D_u &= \begin{bmatrix} -50.3 & -3.27 & 163 & 128 \end{bmatrix}, \\
K &= \begin{bmatrix} 25.6 & 333 & 62.2 & 91.6 \end{bmatrix}, \\
L(s) &= -0.02s^4 + 0.89s^3 + 19.2s^2 + 145s + 437 \\
&\quad s^4 + 37.1s^3 + 619s^2 + 5070s + 11200.
\end{align}

Figure 3 represents the performance by the proposed dynamic anti-windup design. Figure 3(a) shows that the desired performance is obtained if the actuator is free from saturation. Figure 3(b) shows the windup effect with saturation limits, $\pm 5V$. Figure 3(c) shows the performance attained by the dynamic anti-windup modification with the same saturation limits. If anti-windup is not taken into account, performance is devastating as shown in Fig. 3(b). By the proposed anti-windup design, the deterioration of performance is kept minimal as shown in Fig. 3(c).

8. CONCLUSIONS

A dynamic anti-windup approach for critically stable systems has been presented. In the output feedback case, an observer must be introduced. Otherwise, the anti-windup design can be performed in a manner similar to the state feedback case design without any additional conditions. Furthermore, it was shown that the dynamic anti-windup compensator is an extension of the Kothare’s static linear conditioning.

REFERENCES


APPENDIX

For asymptotic stability, it is sufficient to show that $V$ will not remain zero along any finite segment of the state trajectory. The relation represented by Eq. (8) for nonzero $\mathbf{x}$ happens when $\sigma(u)$ takes either $u_{\text{min}}$ or $u_{\text{max}}$ and $\mathbf{x}$ is aligned with the eigenvector of $A$ associate with its 0 eigenvalue. Let $v_0$ denote such an eigenvector: i.e. $A v_0 = 0$. Since $(A, B)$ is stabilizable, either $B u_{\text{min}}$ or $B u_{\text{max}}$ is not aligned with $v_0$. Therefore,

\[ A v_0 + B \sigma(u) \neq 0, \quad \sigma(u) = u_{\text{min}} \text{ or } u_{\text{max}}. \]  

(A1)

(A1) is sufficient to conclude that the relation represented by Eq. (8) may be satisfied only instantaneously, and $\dot{V}$ will not remain positive semi-definite over any finite time interval.