RISK-SENSITIVE CONTROL FOR STOCHASTIC OSCILLATIONS

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Abstract. The problem of controlling a near-Hamiltonian noisy system so as to keep it within a domain of bounded oscillations is considered. An exponential risk-sensitive residence time criterion is introduced as a performance measure. An averaging procedure is developed to obtain the asymptotic solution of the optimal control problem. It is shown that the averaged HJB equation is reduced to a first order PDE with coefficients dependent on the noise intensity in the leading order term, though this intensity tends to zero in the original system. The leading order nearly optimal control is constructed as a nonlinear stationary feedback with parameters dependent on the noise intensity. Copyright © 2002 IFAC.

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1. INTRODUCTION

We propose an interpretation of a problem of control against escape from an admissible domain of variables. For a class of near-Hamiltonian systems, reference operation mode is associated with bounded oscillations (libration) inside the potential well, so that the prescribed operation domain surrounds a fixed stable point. In the phase plane, the reference region is circumscribed by a lobe of the separatrix of a generic (non-perturbed) Hamiltonian system, escape out of this domain is associated with the system failure. Examples are the failure of machine due to loss of “locking” of resonance oscillations and synchronized oscillations (Kovaleva, 1998), loss of engineering integrity of civil structures (Sain, Won and Spencer, 1992), etc.

To have a model for discussion, we restrict our consideration to a system on the plane. Extensions to systems in $\mathbb{R}^n$ are quite obvious.

Consider a process governed by the stochastic differential equations with a small parameter $\varepsilon > 0$

\[
\begin{align*}
q &= \frac{\partial X}{\partial p} \\
p &= -\frac{\partial X}{\partial q} + \varepsilon f(q,p,u) + \varepsilon \sigma(q,p) w(t)
\end{align*}
\]

where $q, p \in \mathbb{R}$, an admissible region $G \in \mathbb{R}^2$ is an open domain with boundary $\Gamma$, $u \in U \subset \mathbb{R}$ is control in a compact control space $U$. The term $f(q,p,u)$ incorporates additional nonconservative and control forces, $\sigma(q,p)$ is the diffusion coefficient, $w(t)$ is the standard Wiener process (Fleming and Rishel, 1975). The function

\[
X(q,p) = \frac{1}{2} p^2 + V(q)
\]
is the Hamiltonian of the unperturbed \((\varepsilon = 0)\) conservative system
\[
\frac{\dot{q}}{q} = \frac{\partial X}{\partial p}, \quad \frac{\dot{p}}{p} = -\frac{\partial X}{\partial q}
\] (3)
with the potential \(V(q)\). Coefficients of (1), (3) are assumed to be sufficiently smooth to justify necessary transformations.

We consider in detail a multistable system for which the reference mode of operation corresponds to “captured” motions within the potential well. The potential \(V(q)\) is assumed to have a stable point \(q_s\), and an unstable hyperbolic point \(q_u\). We let \(q_u = 0\), \(V(q_u) = 0\). In the phase plane the threshold \(X = x_1 = 0\) is associated with the separatrix \(\Gamma\); energy in the domain \(G\) inside the loop of the separatrix is negative and corresponds to the “captured” oscillations (Fig. 1).

Escape from the domain \(G\) through the separatrix \(\Gamma\) is considered as a dangerous event to be avoided. It follows from (2) that an admissible domain is defined by the inequality \(X(p, q) < x_1 = 0\). In practice, the system should not enter a small vicinity of the separatrix, otherwise chaotic oscillations may arise (Guckenheimer and Holmes, 1986). With this regards, the admissible domain \(G\) can be defined as
\[
G: \{X(p, q) < x_1 \leq X^*\}
\] (4)
with a given value \(X^* < 0\) (a dashed curve in Fig.1)

![Phase portrait of system (3) with the separatrix \(\Gamma\). Dashed line corresponds to domain (4)](image)

Let \(\tau = \varepsilon t\) be the natural time scale of the slow variable (Freidlin and Wentzell, 1984, Kovaleva, 1999). Then the mean escape (or residence) time criterion is written as
\[
k(\varepsilon) = E \tau_\varepsilon
\] (5)
where \(E\) is the expectation conditioned on an initial point \((q(0), p(0))\), and
\[
\tau_\varepsilon = \inf (\tau, q, p \notin G)
\] (6)
is the first moment the process \([q(\tau), p(\tau)]\) escapes from the reference domain \(G\).

Let the unperturbed uncontrolled system
\[
\begin{align*}
\frac{\dot{q}}{q} &= \frac{\partial X}{\partial p} \\
\frac{\dot{p}}{p} &= -\frac{\partial X}{\partial q} + Ef^0(q,p)
\end{align*}
\] (7)
have an asymptotically stable fixed point with the domain of attraction \(G_u\) such that \(G \subset G_u\). Then for small \(\varepsilon > 0\) most of the orbits of (1) will remain in \(G\) for a very long time – but with the possibility to leave eventually an admissible domain \(G\). The escapes are rare events, and (5) is estimated exponentially when \(\varepsilon\) is small enough
\[
E \tau_\varepsilon \sim \exp(-C_1/\varepsilon), \quad C_1 > 0
\] (8)
see (Dupuis and Ellis, 1997, Freidlin and Wentzell, 1984). Relation (8) implies that the dominant contribution to \(E \tau_\varepsilon\) comes from paths with exponentially long time to escape. Owing to this, criterion (5) may become problematic if escapes over the time interval \(O(1)\) are rare events.

In this connection, Whittle (1990) has proposed an exponential transformation of standard criteria that “biases” these criteria toward paths with very large time to escape. Fleming and Soner (1992) and Dupuis and McEneaney (1997) have given a rigorous mathematical basis for this idea. From this viewpoint, a generalized residence time criterion takes the form
\[
k(\varepsilon) = E \exp[ -\varepsilon^{-1} \int_0^{\tau_\varepsilon} L(q,p,u) \, d\tau]\n\] (9)
with relevant control constraints. We choose a control \(u\) to minimize criterion (9). A proper parameterization of the criterion with small parameter \(\varepsilon\) allows construction of the feedback control sensitive to small noise (“risk-sensitive control”), and (9) can be interpreted as a “risk-sensitive” counterpart of the standard criteria in the small noise perspective.

Dupuis and McEneaney (1997) have discussed in details risk-sensitive control for a small noise diffusion model with time-independent coefficients. This work extends this approach to the problem of escape from the domain of bounded oscillations.
2. MAIN EQUATIONS AND CRITERIA

Reduce (1) to the standard form with slow and fast variables. Following (Guckenheimer and Holmes, 1986), introduce the new variables: the system’s energy $x$ and the phase $\phi$ defined by the equations

$$x(q,p) = \frac{1}{2}p^2 + V(q)$$

so that

$$p(x,q) = \pm[x - V(q)]^{1/2}$$

and

$$\frac{d\phi}{dq} = \frac{\omega(x)}{p(x,q)}, \quad \phi = \omega(x) \int_{0}^{q} p^{-1}(x,q) dq$$

where $\omega(x) = 2\pi/T(x)$, the period $T(x)$ is defined by the formula

$$T(x) = \int_{-\infty}^{\infty} p^{-1}(x,q) dq$$

The integral is calculated along the curve $x = \text{Const}$, $x \in G$. From (10) - (12) one can define the variables $p = p(x,\phi)$, $q = q(x,\phi)$ as $2\pi$-periodic in $\phi$ functions (Guckenheimer and Holmes, 1986).

The change of variables (10), (11), (12) transforms (1) into a system for new variables, the slow energy evolution $x$ and the fast phase $\phi$. By using the Ito formula for the change of variables (Fleming and Rishel, 1975), we obtain (Kovaleva, 1998)

$$x = \varepsilon[F_1(\phi,x,u) + \sigma_1(\phi,x) w(t)] + \frac{1}{2} \varepsilon^2 \sigma_1^2(\phi,x)$$

$$\phi = \omega(x) + \varepsilon[F_2(\phi,x,u,\varepsilon) + \sigma_2(\phi,x) w(t)]$$

with the initial conditions $x(0) = x_0$, $\phi(0) = 0$. In (14) we denote $\sigma(\phi,x) = \sigma[q(\phi,x), p(\phi,x)]$ and

$$F_1(\phi,x,u) = f[q(\phi,x), p(\phi,x)]$$

$$\sigma_1(\phi,x) = \sigma[q(\phi,x), p(\phi,x)] p(\phi,x)$$

$$\sigma_2(\phi,x) = \sigma[q(\phi,x), p(\phi,x)] p(\phi,x)$$

Coefficients $F_2$, $\sigma_2$ require nontrivial derivation. However, as shown later, the resulting approximate equations do not involve these coefficients and their precise form is insubstantial and can be omitted.

Define a control $u^\varepsilon$ minimizing the exponential criterion similar to (9)

$$k^\varepsilon(u) = \mathbb{E}\{\exp[-\varepsilon^{-1} \int_{0}^{\tau_G} L(x,u) d\tau]\}$$

$$u^\varepsilon = \arg \min_{u \in U} k^\varepsilon(u)$$

The Bellman function for criterion (16) is defined as conditional expectation

$$W^\varepsilon(\phi,x) = \min_{u \in U} \{\mathbb{E}_{\phi,x} \exp[-\varepsilon^{-1} \int_{0}^{\tau_G} L(x,u) d\tau]\}$$

where $\mathbb{E}_{\phi,x}$ denotes conditional expectation. It can be shown (Fleming and Soner, 1992) that the HJB equation for $W^\varepsilon(\phi,x)$ is singular and independent of the noise intensity as $\varepsilon \to 0$. To regularize the problem and to make it sensitive to noise, introduce the transformed Bellman function $\hat{V}^\varepsilon(\phi,x)$ (Dupuis and McEneaney, 1997, Fleming and Soner, 1992)

$$\hat{V}^\varepsilon(\phi,x) = -\varepsilon \ln W^\varepsilon(\phi,x)$$

or, from (17), (18)

$$\hat{V}^\varepsilon(\phi,x) = \max_{u \in U} \{-\varepsilon \ln \mathbb{E}_{\phi,x} \exp[-\varepsilon^{-1} \int_{0}^{\tau_G} L(x,u) d\tau]\}$$

A comparison of (17), (18) and (19) proves that $\hat{V}^\varepsilon(\phi,x)$ coincides with the Bellman function for the problem of maximizing the criterion

$$K^\varepsilon(u) = -\varepsilon \ln \mathbb{E}\{\exp[-\varepsilon^{-1} \int_{0}^{\tau_G} L(x,u) d\tau]\}$$

Connection between (16) and (20) yields

$$u^\varepsilon = \arg \min_{u \in U} k^\varepsilon(u) = \arg \max_{u \in U} K^\varepsilon(u)$$

This implies that the requisite control $u^\varepsilon$ can be found as a solution of the HJB equation for function (19). Starting from the equation for $W^\varepsilon(\phi,x)$ and applying logarithm transform (18), we formally obtain the equation for $\hat{V}^\varepsilon(\phi,x)$ in the form

$$\omega(x) \frac{\partial \hat{V}^\varepsilon}{\partial \phi} - \frac{1}{2} \varepsilon^2 \left[\sigma_1^2(\phi,x) + \sigma_2^2(\phi,x) \frac{\partial \hat{V}^\varepsilon}{\partial \phi}\right] + \varepsilon H(\phi,x, \frac{\partial \hat{V}^\varepsilon}{\partial x}, \frac{\partial \hat{V}^\varepsilon}{\partial \phi}) + \frac{1}{2} \varepsilon^2 \left[a_{11}(\phi,x) \frac{\partial^2 \hat{V}^\varepsilon}{\partial x^2} + a_{12}(\phi,x) \frac{\partial^2 \hat{V}^\varepsilon}{\partial x \partial \phi} + a_{22}(\phi,x) \frac{\partial^2 \hat{V}^\varepsilon}{\partial \phi^2}\right] = 0, \quad x < x^\varepsilon$$

$$\hat{V}^\varepsilon(\phi,x) = 0, \quad x = x^\varepsilon$$
where \( a_j(\varphi, x) = \sigma(\varphi, x)\sigma(\varphi, x) \) and

\[
H(\varphi, x, \frac{\partial V^E}{\partial x}, \frac{\partial V^E}{\partial \varphi}) = \max_{u \in U} h(\varphi, x, \frac{\partial V^E}{\partial x}, \frac{\partial V^E}{\partial \varphi}, u)
\]

\[
+ \frac{1}{2} \epsilon \sigma^2(\varphi, x) \frac{\partial^2 V^E}{\partial x^2} + F_1(\varphi, x, u) \frac{\partial V^E}{\partial \varphi} + L(x, u)
\]

(23)

Properties of the solution \( V^\epsilon(x, \varphi) \) are beyond the frames of the present study. Coefficients of (21), (22) are assumed to be sufficiently smooth for a smooth solution \( V^\epsilon(x, \varphi) \) to be obtained for all \( \varphi \pmod{2\pi}, x \in \mathbb{G} \). As the function \( V^\epsilon(\varphi, x) \) is found, the solution of the associated optimal control problems can be written in the form (Fleming and Rishel, 1975)

\[
u^\epsilon(\varphi, x) = \arg \max_{u \in U} h(\varphi, x, \frac{\partial V^E}{\partial x}, \frac{\partial V^E}{\partial \varphi}, u) = U(\varphi, x, \frac{\partial V^E}{\partial x}, \frac{\partial V^E}{\partial \varphi})
\]

(24)

To find a limit solution as \( \epsilon \to 0 \), we find the asymptotic solution of a related HJB equation (Bensoussan, 1988; Briand and Hu, 1999). Passage to the limit as \( \epsilon \to 0 \) corresponds to the averaging in the phase \( \varphi \) in the leading order terms. As a result, we obtain the limit equation as a first order PDE with coefficients independent of \( \varphi \). This allows construction of the solution \( V^0(x) \) independent of \( \varphi \) and reduction of the averaged HJB equation to the form

\[
H^0(x, \frac{dV^0}{dx}) - \frac{1}{2} a^0(x)(\frac{dV^0}{dx})^2 = 0, \quad x < \Gamma
\]

\[
V^0(x) = 0, \quad x = \Gamma
\]

(25)

in which

\[
H^0(x, \frac{dV^0}{dx}) = \frac{1}{2\pi a(\varphi)} \int_{-\pi}^{\pi} H(\varphi, x, \frac{dV^0}{dx}, 0) d\varphi
\]

\[
a^0(x) = \frac{1}{2\pi a(\varphi)} \int_{-\pi}^{\pi} a_j(\varphi, x) d\varphi
\]

(26)

From (24), (26) we obtain nearly-optimal control in the form

\[
u^0(\varphi, x) = U^0(\varphi, x, \frac{dV^0}{dx})
\]

(27)

where \( U^0 \) can be found from (24) by setting \( \partial V^E/\partial \varphi = 0 \). As seen from (25) - (27), the feedback control (27) is determined through the coefficients of the first equation in (14).

The averaged Bellman function \( V^\epsilon(\varphi) \) is interpreted as a viscosity solution of (25) (Dupuis and McEneaney, 1997, Fleming and Soner, 1992). Following (Fleming and Soner, 1992), one can prove the uniform convergence

\[
V^\epsilon(\varphi, x) \to V^0(\varphi), \quad \epsilon \to 0, \quad \varphi \pmod{2\pi}, x \in \mathbb{G}
\]

(28)

3. EXPONENTIAL CRITERION IN A SYSTEM WITH ADDITIVE NOISE

The equations of motion are given in the form

\[
\dot{q} = p
\]

\[
p = -V_q(q) + \epsilon u + \epsilon \sigma w(t)
\]

(29)

The change of variables (10), (26) reduces (29) to the standard form similar to (14)

\[
\begin{align*}
\dot{x} &= \epsilon u p + \frac{1}{2} \epsilon^2 \sigma^2 + \epsilon \sigma \omega w(t) \\
\phi &= \omega(x) + \epsilon u k(x, p) + \epsilon \delta(x, p) w(t) + \epsilon^2 \cdots \end{align*}
\]

(30)

where \( p = p(q, x) \), the form of the coefficients \( k, \delta \) is insubstantial, for they are not involved in the resulting equation (25). An uncontrolled noisy system \( (u = 0) \) has the drift coefficient \( (\epsilon \sigma^2)/2 \). This implies that the mean residence time for an uncontrolled system \( E\tau_r \sim 1/\epsilon^2 \), whereas a risk-sensitive control transforms (30) into a dissipative system with the exponentially large residence time.

In accordance with the problem, we introduce the cost criterion in the form

\[
K^\epsilon(u) = -\epsilon \ln \mathbb{E}[\exp(-\epsilon^{-1} \int_0^{\tau_G} [\omega(x) - u^2/2r] d\tau)]
\]

(31)

From (20), (31), we have \( L(x, u) = \omega(x) - u^2/2r \). Choice of the performance criterion in form (31) implies that, by analogy with deterministic systems (Kovaleva, 1999), criterion (31) maximizes not the moment of escape, but the phase of escape from the reference region \( \mathbb{G} \): \( x_0 \leq x < \Gamma \). The problems are similar, but the HJB equation for the phase control problem is regular as \( \epsilon \to 0 \). The HJB equation for the problem (29), (31) is given by
\[ \alpha(x) \frac{\partial U^\varepsilon}{\partial \phi} + \varepsilon H(\phi, x, p, \frac{\partial U^\varepsilon}{\partial x}, \frac{\partial U^\varepsilon}{\partial \phi}) \]

\[ - \frac{1}{2} \varepsilon \left( \sigma_p \frac{\partial U^\varepsilon}{\partial x} \right)^2 + R(\phi, x, \frac{\partial U^\varepsilon}{\partial \phi}) = 0, \quad x < x_\Gamma \]

\[ V^\varepsilon(\phi, x) = 0, \quad x = x_\Gamma \quad (32) \]

where the term \( R \) includes coefficients vanishing as \( \partial V/\partial \phi = 0 \), and

\[ H(\phi, x, \frac{\partial U^\varepsilon}{\partial x}, \frac{\partial U^\varepsilon}{\partial \phi}) = \max \left\{ \frac{\partial U^\varepsilon}{\partial x} + k(x, p) \frac{\partial U^\varepsilon}{\partial \phi} - u^2/2r \right\} + \omega(\chi) \quad (33) \]

This yields

\[ u^\varepsilon = r[p \frac{\partial U^\varepsilon}{\partial x} + k(x, p) \frac{\partial U^\varepsilon}{\partial \phi}] \quad (34) \]

and

\[ H^\varepsilon = \frac{1}{2} r[p \frac{\partial U^\varepsilon}{\partial x} + k(x, p) \frac{\partial U^\varepsilon}{\partial \phi}]^2 + \omega(\chi) \quad (35) \]

Substitute (35) in (32) and average in the phase \( \phi \).

As a result, we obtain the reduced averaged equation (25) in the form

\[ \frac{1}{2} (r - \sigma^2) \left( \frac{\partial V^0}{\partial x} \right)^2 I(x) = -1, \quad x < x_\Gamma \]

\[ V^0(x) = 0, \quad x = x_\Gamma \quad (36) \]

where \( I(x) \) is the action integral (Guckenheimer and Holmes, 1986)

\[ I(x) = \int p^{-1}(x, q) dq \quad (37) \]

From (36) we find

\[ \frac{\partial V^0}{\partial x} = -\lambda(x) = -[2\phi(x)]^{1/2} \quad (38) \]

This yields a nearly-optimal control in the form

\[ u^0 = -r\lambda(x)p \quad (39) \]

Control (39) is organized as time-invariant feedback equivalent to additional nonlinear viscous damping with the coefficient dependent on the system parameters.

The solution of (36) exists if the condition

\[ \gamma = \sigma^2 - r > 0, \quad \sigma^2 > r \quad (40) \]

holds. An interpretation of this condition is given in Section 4.

4. RISK-SENSITIVE CONTROL AND DIFFERENTIAL GAME IN SYSTEMS WITH AVERAGING

Dupuis and McEneaney (1997) have proved a direct connection between risk-sensitive control and deterministic game. It has been shown that the limit solution of the risk-sensitive HJB equation can be interpreted as the value function of an associated deterministic game in which the minimizing player substitutes for noise. We extend the results to stochastic oscillatory systems.

Consider a deterministic controlled oscillatory system reduced to the standard form similar to (14)

\[ x = \varepsilon \left[ F_1(\phi, x, u) + \sigma_1(\phi, x)v(t) \right] \]

\[ \phi = \omega(\chi) + \varepsilon \left[ F_2(\phi, x, u) + \sigma_2(\phi, x)v(t) \right] \quad (41) \]

where the scalars \( u \in U \) and \( v \in L^2 \) are strategies for the maximizing and minimizing players, respectively.

Write the payoff of the game in the form

\[ k^u(u, v) = \int_0^{\tau_G} \left[ L(x, u) + \frac{1}{2} v^2 \right] d\tau \quad (42) \]

As in previous sections, \( \tau = \varepsilon r \) is the time scale corresponding to the slow variable \( x \), \( \tau_G \) is the first moment the system escapes from a given domain \( G \).

We choose a control \( u^0 \) maximizing (42) and a control \( v^\varepsilon \) minimizing (42). The game value is defined as

\[ V^\varepsilon(\phi, x) = \max_{u \in U} \min_{v \in L^2} \left[ E_{\phi, x} \int_0^{\tau_G} \left[ L(x, u) + \frac{1}{2} v^2 \right] d\tau \right] \quad (43) \]

where \( V^\varepsilon(\phi, x) \) is the solution of the Isaacs equation (Basar and Bernhard, 1991)

\[ \omega(\chi) \frac{\partial V^\varepsilon}{\partial \phi} + \varepsilon \max_{u \in U} \min_{v \in L^2} \left[ h(\phi, x, \frac{\partial V^\varepsilon}{\partial x}, \frac{\partial V^\varepsilon}{\partial \phi}) + (\sigma_1 \frac{\partial V^\varepsilon}{\partial x} + \sigma_2 \frac{\partial V^\varepsilon}{\partial \phi}) v + \frac{1}{2} v^2 \right] = 0, \quad x < x_\Gamma \]

\[ V^\varepsilon(\phi, x) = 0, \quad x = x_\Gamma \quad (44) \]

The function \( h^\varepsilon \) in (44) coincides with (23). Obvious transformations yield the optimal strategy for the maximizing and minimizing players in the form
\[
\begin{align*}
  u^* &= \arg \max_{u \in U} h(\varphi, x, \frac{\partial V^E}{\partial x}, \frac{\partial V^E}{\partial \varphi}, u) \\
  v^E &= -\left(\sigma_1 \frac{\partial V^E}{\partial x} + \sigma_2 \frac{\partial V^E}{\partial \varphi}\right)
\end{align*}
\]

The form of the maximizing control \(u^*\) is similar to (24). Substitution of (45) into (44) and averaging in \(\varphi\) yield the averaged equation coinciding with (25). The solution \(V^E(x)\) is thus both the approximate solution of the risk-sensitive stochastic control problem (1), (20) and the approximate solution of the associated deterministic differential game (42), (47). In both cases, the nearly-optimal control \(u^E(\varphi, x)\) takes the form (24).

Condition (40) agrees with the game interpretation of the control problem. As it follows from the previous consideration, equation (36) can be interpreted as a counterpart of the averaged Isaacs-Bellman equation for a differential game in a system similar to (1). In this game noise stands for a minimizing player aimed at achieving a boundary of the admissible domain, while control stands for a maximizing player countering the noise player’s goal. With regards to this interpretation, the goal (a boundary of the region) can be achieved if inequality (40) holds. Otherwise, noise intensity is insufficient to make an orbit achieve the boundary of the region. In this case the averaging procedure cannot be employed, and risk sensitive control should be found from relationships (32), (33), (34).

5. CONCLUSIONS

We propose an interpretation of the problem of control against escape for a class of oscillatory systems. To design a control sensitive to weak perturbations in the leading order terms, we use the risk sensitive control method. The method proposes an exponential transformation of the mean escape time criteria. This transformation “biases” the criterion towards the paths with rather long time to escape and thus makes it sensitive to weak perturbations.

The work develops an asymptotic method for risk-sensitive control problems associated with control against escape from the potential well. It is shown that the approximate solution can be found as a solution of the averaged HJB equation. The solution depends on the system structure and on the noise intensity in the leading order term, even though the noise intensity is small. Nearly-optimal control is found as time-invariant nonlinear feedback.

It is demonstrated that the small noise limit of the risk sensitive control problem can be associated with a solution of a deterministic differential game.

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