A DISCRIMINATION METHOD OF ROOTS AREA FOR POLYTOPIC POLYNOMIALS

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Abstract: The physical parameters of controlled systems (plants) are uncertain and are accompanied by nonlinearity. The state space equation and the characteristic polynomial of the control system should, therefore, be expressed by an interval set of parameters. This paper examines the robust performance evaluation of that type of control system based on the existing area of characteristic roots (i.e., eigenvalues). In particular, in this paper, a sufficient condition for the roots area which is enclosed by a specified circle on a complex variable plane is given by applying the classic Sturm’s theorem (division algorithm) to the four corners of a segment polynomial. The result that is obtained by finite calculations in regard to the coefficients of the segment polynomial, can be extended to general interval polynomials with multiple uncertain parameters. Copyright © 2002 IFAC

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1. INTRODUCTION

The physical parameters of controlled systems (plants) are uncertain and accompanied by nonlinearity. The state space equation and the characteristic polynomial of the control system should, therefore, be expressed by an interval matrix and an interval (polytopic) polynomial, respectively (Kharitonov, 1979; Ackermann, 1993; Barmish, 1994; Bhattacharyya, 1994). The robust stability and the poles location for that type of control system has been elucidated by many researchers (Bartlett et al., 1988; Soh et al., 1997). Nonetheless, in our previous paper, we tried to apply the classic Sturm’s theorem to interval polynomials in regard to the existing area of characteristic roots in a specified contour on a complex variable plane, and derived theorems which correspond to weak-Kharitonov’s theorem based on an assumption for the uncertain systems (Okuyama et al., 1999).

In this paper, by applying Sturm’s theorem (a division algorithm) to the four corners of a segment polynomial, a sufficient condition will be given for the characteristic roots area which is enclosed by a specified circle on the complex plane. The concept of finite calculations based on the division algorithm in regard to the coefficients of the segment polynomial will be extended to general interval polynomials with multiple uncertain parameters. The robust performance of that type of control system will be evaluated by using the discrimination method.

2. INTERVAL SYSTEMS

The state space equation of controlled systems with uncertainty (and nonlinearity) can be expressed by the interval set parameters as follows:

\[
\frac{dx}{dt} = (A + \Delta)x, \quad x \in \mathbb{R}^n, \quad A, \Delta \in \mathbb{R}^{n \times n},
\]

\[
\Delta = \sum_{i=1}^{m} \tilde{q}_i A_i, \quad \tilde{q}_i \in [q_i^-, q_i^+].
\]

If the entries of matrix $A_i$ exist in only the $k$-th row (or column) (i.e., Row$_{j=k} A_i \neq 0$, Row$_{j \neq k} A_i = 0$, $\forall j \in \{1, 2, \ldots, n\}$, and $k \in \{1, 2, \ldots, n\}$), the characteristic polynomial can be written as the following form:
The characteristic polynomial of Eq. (1) can be reduced to the above form. When the entries of matrix $A_i$ exist in two or more rows (columns), interval polynomial (2) can be obtained by applying the interval arithmetic for multiplication, i.e.,

$$[q_i^-, q_i^+] = [q_j^-, q_j^+] \cdot [q_k^-, q_k^+]$$

$$q_i^- = \min\{q_j^-, q_j^+, q_j^+ q_k^-, q_j^+ q_k^+\}$$

$$q_i^+ = \max\{q_j^-, q_j^+, q_j^+ q_k^-, q_j^+ q_k^+\}.$$  

At any rate, the dynamics of the system will be dependent on the form of interval polynomial (2).

In this paper, the relationship between the mathematical model (2) (i.e., (1)) and the location of characteristic roots (in the strict sense, the area of the roots) is examined. The discrimination of the roots area based on Eq. (2), however, becomes a considerably complicated problem. Thus, the problem based on the concept of a set of segment polynomials will be analyzed at first.

3. SEGMENT POLYNOMIAL

Let us consider the following segment polynomial (i.e., a polynomial with only one interval set coefficient):

$$\tilde{F}(s) = \sum_{i=1}^{m} \tilde{q}_i \left( \sum_{\ell=0}^{n} c_{i,\ell} s^{n-\ell} \right),$$  

where $c_{i,\ell}$ is a real constant coefficient and $s$ is a complex variable which corresponds to the eigenvalue of matrix $A + \Delta$. Of course, if $\tilde{q}_i$ is an element of interval set parameters, the eigenvalue (i.e., the characteristic root) must be an element of bounded sets of complex numbers. In other words, if $\tilde{q}_i$ is represented by a set of interval parameters, the eigenvalues must be represented as a bounded area of complex numbers.

Since interval polynomial (2) is not the generalized form with respect to the state equation (1), the characteristic polynomial of Eq. (1) can be reduced to the above form. When the entries of matrix $A_i$ exist in two or more rows (columns), interval polynomial (2) can be obtained by applying the interval arithmetic for multiplication, i.e.,

$$F(s) = \sum_{i=1}^{m} q_i \left( \sum_{\ell=0}^{n} c_{i,\ell} s^{n-\ell} \right),$$  

where $q_i$ is an $h \times \ell$ matrix.

4. CIRCULAR CONTOUR

In this paper, the following circular contour $\partial \Gamma$ (as shown in Fig. 1) is considered:

$$s = \rho e^{j\theta} + \sigma_0 + j\omega_0. \quad (\theta : -\pi \to \pi).$$  

Here, $\rho$, $(\sigma_0, \omega_0)$ and $\theta$ are the radius, the center and the angle of rotation for the specified circle, respectively.

Any point $s$ on circular contour (7) can also be written by the following rational function of real variable $\alpha$:

$$s = \frac{u + jv + \alpha}{1 - j\alpha},$$  

where $u = \rho + \sigma_0 + j\omega_0$ and $v = \rho - \sigma_0 - j\omega_0$. Here, $\alpha$ can be correspondingly considered as follows:

$$\alpha = \tan(\theta/2).$$  

Circular contours (7) and (8) includes the following extreme case.

(a) In Eqs. (7) and (8), when we consider $R = R, \sigma_0 = -R$ (or $\sigma_0 = R$), $\omega_0 = \epsilon$ and $R \to \infty$, $\epsilon \to 0$, the specified area becomes a left half (or right half) $s$-plane ($\partial \Gamma_3$ as shown in Fig. 1) In this case, a part of the circular contour approaches the imaginary axis on the $s$-plane, provided that variable $\omega$ is considered in a certain limited range $-\omega_1 < \omega < \omega_2$, where

$$\omega_1, \omega_2$$
When applying the above transformation (8) to Eq. (3), the following numerator polynomial for real variable \( \alpha \) can be obtained:

\[
(1-j\alpha)^n \tilde{F}(s) = \sum_{i=1}^{m} \tilde{q}_i \left( \sum_{\ell=0}^{n} c_{i,\ell}(u+j\alpha)^{n-\ell}(1-j\alpha)\right).
\]

(10)

Since Eq. (10) is a polynomial with complex coefficients, it can be written as the following form:

\[
\tilde{\Phi}(j\alpha) = (1-j\alpha)^n \tilde{F}(s) = \tilde{P}(\alpha) + j\tilde{Q}(\alpha),
\]

(11)

where

\[
\tilde{P}(\alpha) = \tilde{a}_{0,0}\alpha^n + \cdots + \tilde{a}_{0,n-1}\alpha + \tilde{a}_{0,n},
\]

(12)

\[
\tilde{Q}(\alpha) = \tilde{b}_{0,0}\alpha^n + \cdots + \tilde{b}_{0,n-1}\alpha + \tilde{b}_{0,n}.
\]

(13)

Coefficients in Eqs. (12) and (13) can be calculated from the expansion of Eq. (10).

5. FOUR CORNER POINTS PROBLEM

By using expression (4), Eq. (11) is rewritten as follows:

\[
\Phi(j\alpha) = \Phi(j\alpha, \mu) = (1-j\alpha)^n (\mu P^+(s) + (1-\mu)P^-(s))
\]

\[
= (\mu P^+(\alpha) + (1-\mu)P^-(\alpha)) + j\mu Q^+(\alpha) + (1-\mu)Q^-(\alpha)),
\]

\( \mu \in [0, 1] \)

(14)

The real and imaginary parts of Eq. (14) correspond to those of Eq. (11), i.e.,

\[
\tilde{P}(\alpha) = \mu P^+(\alpha) + (1-\mu)P^-(\alpha),
\]

\[
\tilde{Q}(\alpha) = \nu Q^+(\alpha) + (1-\nu)Q^-(\alpha).
\]

Here, the extreme polynomials are expressed as follows:

\[
P^+(\alpha) = a_{0,0}^+\alpha^n + \cdots + a_{0,n-1}^+\alpha + a_{0,n}^+,
\]

\[
Q^+(\alpha) = b_{0,0}^+\alpha^n + \cdots + b_{0,n-1}^+\alpha + b_{0,n}^+.
\]

(15)

\[
P^-(\alpha) = a_{0,0}^-\alpha^n + \cdots + a_{0,n-1}^-\alpha + a_{0,n}^-,
\]

\[
Q^-(\alpha) = b_{0,0}^-\alpha^n + \cdots + b_{0,n-1}^-\alpha + b_{0,n}^-.
\]

(16)

Thus, the following four corner points (vertices) can be given, and a rectangle together with a line segment (edge) can be drawn in the \( \Phi \)-plane as shown in Fig. 2:

\[
V_1 = (P^+, Q^+), \quad V_2 = (P^-, Q^-),
\]

\[
V_3 = (P^-, Q^+), \quad V_4 = (P^+, Q^-),
\]

where the latter two points are additional ones. (In these expressions, note that polynomials \( P, Q \) and coefficients \( a_{0,\ell}, b_{0,\ell} \) with superscript + do not always denote larger values than those with superscript −). Then, the following four pairs of polynomials can be defined for \( i = 1, 2, 3, 4 \):

\[
P^{(i)}(\alpha) = a_{0,0}^{(i)}\alpha^n + \cdots + a_{0,n-1}^{(i)}\alpha + a_{0,n}^{(i)},
\]

(17)

\[
Q^{(i)}(\alpha) = b_{0,0}^{(i)}\alpha^n + \cdots + b_{0,n-1}^{(i)}\alpha + b_{0,n}^{(i)}.
\]

(18)

As an expression of polynomials with complex coefficients, the following can be given:

\[
\Phi^{(i)}(j\alpha) = P^{(i)}(\alpha) + jQ^{(i)}(\alpha), \quad (i = 1, 2, 3, 4)
\]

(19)

(Note that as for the edges in the \( F \)-plane, two additional polynomials with constant coefficients cannot be determined in general).

As for a polynomial with complex coefficients (19), the following lemma can be shown by using Sturm’s theorem.

[Lemma-1] When coefficient ratios

\[
\frac{b_{0,0}^{(i)}}{a_{1,1}^{(i)}}, \quad \frac{b_{1,1}^{(i)}}{a_{2,2}^{(i)}}, \quad \cdots, \quad \frac{b_{n-1,n-1}^{(i)}}{a_{n,n}^{(i)}}, \quad (i = 1, 2)
\]

are calculated for an extreme polynomial \( \Phi^{(i)}(\alpha) \) (\( i = 1, 2 \)), the number of ratios (20) to be negative \( \rho = R, \quad \sigma_0 = -R, \quad \omega_0 = R \tan \phi_0, \quad R \to \infty \), the specified area becomes a problem of a sectorial area in the left half \( s \)-plane (\( \partial \Gamma_2 \) as shown in Fig. 1).

As mentioned above, since Eqs. (7) and (8) can include most of the problems, the following shows the discrimination of the number of roots in a circular area.

When applying the above transformation (8) to Eq. (3), the following numerator polynomial for real variable \( \alpha \) can be obtained:

\[
(1-j\alpha)^n \tilde{F}(s) = \sum_{i=1}^{m} \tilde{q}_i \left( \sum_{\ell=0}^{n} c_{i,\ell}(u+j\alpha)^{n-\ell}(1-j\alpha)\right).
\]

(10)

Thus, the following four corner points (vertices) can be obtained:

\[
\tilde{P}(\alpha) = \tilde{a}_{0,0}\alpha^n + \cdots + \tilde{a}_{0,n-1}\alpha + \tilde{a}_{0,n},
\]

(12)

\[
\tilde{Q}(\alpha) = \tilde{b}_{0,0}\alpha^n + \cdots + \tilde{b}_{0,n-1}\alpha + \tilde{b}_{0,n}.
\]

(13)

Fig. 2. Four corner points and rectangle.
vertices is 2. The number of polynomials expressed by Eq. (2), the number of drawn on the Φ-plane as shown in Fig 3. As for a view of (hyper)polyhedron (a parallelotope) is general. When complex variable is fixed (frozen), s, ···, n) are calculated by using a division algorithm as shown in APPENDIX.

(Proof) This lemma is a necessary and sufficient condition in regard to the existing area of characteristic roots for the fixed polynomial. In the appendix, the proof will be given systematically by using Sturm’s theorem.

Based on the above premise, the following theorem is obtained as to the above four pairs of polynomials:

[Theorem-1] If the number of ratios (20) to be negative is not changed for the four corner polynomials, the dynamic system that is characterized by segment polynomial Eq. (3) has a robust performance in regard to the invariance of the number of characteristic roots in the specified circle. When considering only one root, for instance, a dominant root, the circle $\partial \Gamma$ (i.e., disc $\Gamma$) gives a sufficient condition for the characteristic root area of the dynamic system with interval set parameters.

(Proof) This theorem is a sufficient condition in regard to the existing area of characteristic roots for the interval polynomial. The proof is obvious from the result in Theorem-1 in which any edge of the rectangles does not pierce the origin. Consequently, as mentioned in the proof of Theorem-1, any edge of the parallelotope (a set of line segments) in the Φ-plane and in the $\Phi$-plane does not pierce the origin.

Although the above results (Theorem-1,2) are only a sufficient condition (i.e., sufficient for edge theorems), the discrimination method proposed in this paper will be useful in robust control systems design (Okuyama et al., 2000).

7. NUMERICAL EXAMPLES

[Example-1] Consider an interval system expressed by Eq. (1) as:

$$\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}, \quad A = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
-1 & -2 & -2
\end{bmatrix}, \quad A_1 = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
-1 & -2 & 0
\end{bmatrix}, \quad \text{and } \tilde{q}_i \in [-0.2, 0.2].$$

The characteristic polynomial can be given by a segment polynomial as follows:

$$\tilde{F}(s) = s^3 + 2s^2 + 2s + 1 + [-0.2, 0.2](2s + 1).$$

When a circle with a center of $(-0.5, 0.8j)$ and a radius of $r = 0.3$ was specified as shown in Fig. 4, the line segments, four corner points and rectangles are drawn in the $\Phi$-plane as shown in Fig. 5. As is obvious from the figure, the series of four corner points (rectangles) well exclude the origin.

The number of the characteristic roots in the specified circle becomes $\mu = 1$ for all the corner polynomials $\Phi^{(i)}(\alpha)$, $i = 1, 2, 3, 4$. With
Fig. 4. Circles and segments of root locus for Example-1.

Fig. 5. Rectangles in the Φ-plane for Example-1.

Fig. 6. Circles and root areas for Example-2.

Fig. 7. Rectangles in the Φ-plane for Example-2.

respect to pairs of polynomials \((P^+(α), Q^+(α)), (P^-(α), Q^-(α)), (P^-(α), Q^+(α)), (P^+(α), Q^-(α))\), ratios (20) are calculated, e.g.,

\[
\frac{b_1^{(1)}}{a_1^{(1)}} = 0.544, \quad \frac{b_2^{(1)}}{a_2^{(1)}} = -0.760, \quad \frac{b_3^{(1)}}{a_3^{(1)}} = -1.123.
\]

\[
\frac{b_0^{(3)}}{a_0^{(3)}} = 0.304, \quad \frac{b_1^{(3)}}{a_1^{(3)}} = 0.445, \quad \frac{b_2^{(3)}}{a_2^{(3)}} = -3.26, \quad \frac{b_3^{(3)}}{a_3^{(3)}} = 1.68.
\]

The number of roots in the specified circle did not change. However, when a circle with a center of \((-0.5, 0.85j)\) and a radius of \(r = 0.3\) was specified, \(μ = 0\) appeared in the case of \(i = 1\).

**Example-2** Consider the same system as described in Example-1, but there are three interval set parameters:

\[
x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -2 \end{bmatrix},
\]

\[
A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad \tilde{q}_1 \in [-0.2, 0.2],
\]

\[
A_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -2 & 0 \end{bmatrix}, \quad \tilde{q}_2 \in [-0.1, 0.1],
\]

\[
\tilde{F}(s) = s^3 + 2s^2 + 2s + 1 + [-0.1, 0.1]2s^2 + [-0.1, 0.1]2s + [-0.2, 0.2].
\]

When a circle with a center of \((-0.5, 0.85j)\) and a radius of \(r = 0.3\) is specified as shown in Fig. 6, the paralleloptopes and rectangles with \((m + 1) \cdot 2^m = 32\) corners are drawn in the Φ-plane as shown in Fig. 7. The series of 32 corner points (sets of rectangles) also excludes the origin well. The number of roots in the specified circle did not change. That is, the robust performance was guaranteed in regard to the existing area of dominant roots for the interval system.

**8. CONCLUSIONS**

This paper examined the robust performance evaluation of a control system with interval set parameters. In particular, in this paper, a sufficient condition for the roots area which is enclosed by a specified circle on the \(s\)-plane was given by applying Sturm’s theorem to the four corners of a segment polynomial. The concept of finite calculations based on the division algorithm in regard to the coefficients of the segment polynomial
was extended to general interval polynomials with multiple uncertain parameters.

9. APPENDIX

When notations \( f_0^{(i)}(\alpha) = P^{(i)}(\alpha) \) and \( f_1^{(i)}(\alpha) = Q^{(i)}(\alpha) \) \((i = 1, 2, 3, 4)\) are used for the corner polynomials (17) and (18), the following division algorithm can be executed (Takagi, 1965):

\[
\begin{align*}
  f_{2n-2}^{(i)}(\alpha) &= f_{2n-1}^{(i)}(\alpha)q_{2n-1}(\alpha) - f_{2n}^{(i)}(\alpha), \\
  f_{2n-1}^{(i)}(\alpha) &= f_{2n}^{(i)}(\alpha)q_{2n}(\alpha) - f_{2n+1}^{(i)}(\alpha),
\end{align*}
\]

\(i = 1, 2, \ldots, 10\).

If \( f_0^{(i)}(\alpha) \) and \( f_1^{(i)}(\alpha) \) are of the \( n \)-th order for \( \alpha \), \( f_2^{(i)}(\alpha), f_3^{(i)}(\alpha), \ldots, f_{2n}^{(i)} \) are expressed as:

\[
\begin{align*}
  f_2^{(i)}(\alpha) &= a_{1,1}^{(i)}\alpha^{n-1} + \cdots + a_{1,n}^{(i)}, \\
  f_3^{(i)}(\alpha) &= b_{1,1}^{(i)}\alpha^{n-1} + \cdots + b_{1,n}^{(i)}, \\
  &\vdots \\
  f_{2n-2}^{(i)}(\alpha) &= a_{n-1,n-1}^{(i)}\alpha^{n-1} + a_{n-1,1}^{(i)}, \\
  f_{2n-1}^{(i)}(\alpha) &= b_{n-1,n-1}^{(i)}\alpha^{n-1} + b_{n-1,1}^{(i)}, \\
  f_{2n}^{(i)}(\alpha) &= a_{n,n}^{(i)}.
\end{align*}
\]

Here, each coefficient can be given by the following sequential operations:

\[
\begin{align*}
  a_{1,p}^{(i)} &= b_{1,p}^{(i)} \left( a_{0,0}^{(i)} \right) - a_{0,p}^{(i)}, \\
  b_{1,p}^{(i)} &= a_{1,p+1}^{(i)} \left( b_{0,0}^{(i)} \right) - b_{0,p}^{(i)}, \\
  &\vdots \\
  a_{q,p}^{(i)} &= b_{q-1,p}^{(i)} \left( a_{q-1,q-1}^{(i)} \right) - a_{q-1,p}^{(i)}, \\
  b_{q,p}^{(i)} &= a_{q+1,p+1}^{(i)} \left( b_{q-1,q-1}^{(i)} \right) - b_{q-1,p}^{(i)}, \\
  &\vdots \\
  a_{n,n}^{(i)} &= b_{n-1,n}^{(i)} \left( a_{n-1,n-1}^{(i)} \right) - a_{n-1,n}^{(i)}, \\
  (a_{q,n+1}^{(i)} &= 0).
\end{align*}
\]

Thus, argument change \( 2\pi \) for polynomial \( \tilde{F}(s) \) becomes \( (2\pi - n)\pi \) for \( \tilde{\Phi}(\alpha) \) by adding change \(-\pi\) in the argument of \((1-j\alpha)^n\). When \( P^{(i)}/Q^{(i)} \) (or \(Q^{(i)}/P^{(i)}\)) is considered, the number of sign changes which cross zero for \( \alpha : -\infty \to +\infty \) is \( n - 2\mu \).

If the number of sign changes which cross the zero of \( f_1^{(i)}(\alpha)/f_0^{(i)}(\alpha) \) for \( \alpha : \alpha_1 \to \alpha_2 \) is expressed as \( N^{(i)}(\alpha_1, \alpha_2) \) and the number of sign changes of sequence \( f_0^{(i)}(\alpha), f_1^{(i)}(\alpha), \ldots, f_{2n}^{(i)} \) is expressed as \( V^{(i)}(\alpha) \), the following relationship is obtained:

\[
N^{(i)}(\alpha_1, \alpha_2) = V^{(i)}(\alpha_1) - V^{(i)}(\alpha_2).
\]

Since the condition is \( N^{(i)}(-\infty, +\infty) = n - 2\mu \), \( V^{(i)}(-\infty) - V^{(i)}(+\infty) = n - 2\mu \) is obtainable. The condition of Eq. (28) corresponds to observing whether the following ratios are negative or not (The details were described in Okuyama et al., 2001):

\[
\lim_{\alpha \to +\infty} |a_{f_2^{(i)}}(\alpha)|, \ldots, \lim_{\alpha \to -\infty} |a_{f_{2n}^{(i)}}(\alpha)|.
\]

Suppose that the number of negative ratios is \( N \) and the number of positive ratios is \( P \). Thus, \( P + N = n - 2\mu \) can be obtained from Eq. (28). Since \( P + N = n \), \( N = \mu \) is given.

10. REFERENCES


