Abstract: This contribution presents a parametric approach to the frequency domain design of state feedback controllers in the general case of multiple closed loop poles. By introducing the set of generalized pole directions as design parameters a compact parametric expression is derived for the closed loop denominator matrix parameterizing the state feedback controller in the frequency domain. No assumptions are imposed on the closed loop poles and the pole directions. Connecting relationships are established between the time and frequency domain approach to parametric state feedback design. A simple example demonstrates the proposed design procedure. Copyright © 2002 IFAC

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1. INTRODUCTION

The parametric approach to the design of state feedback controllers has received considerable attention during the last two decades and it can be carried out using time domain as well as frequency domain techniques. In the time domain a parametric approach was originally proposed independently by Roppenecker (1981) and Fahmy and O’Reilly (1982). Developments of the parametric state feedback solution are reported in O’Reilly and Fahmy (1985) and Roppenecker (1986) while a comprehensive review of the different time domain approaches to the eigenstructure assignment problem is provided by White (1995) and Liu and Patton (1998). Using the set of closed loop eigenvalues and the set of invariant parameter vectors as free design parameters, Roppenecker (1981, 1986) and Fahmy and O’Reilly (1982) derive an explicit expression for the state feedback matrix. Recently a parametric approach to the frequency domain design of state feedback controllers was proposed by Deutscher and Hippe (2000, 2001). Based on the polynomial representation for the plant, a parametric frequency domain parameterization of state feedback controllers was presented. By introducing the set of closed loop poles and the set of pole directions as free design parameters, an explicit parametric expression for the coefficient matrix of the polynomial matrix $\tilde{D}(s)$, parameterizing the state feedback in the frequency domain, was obtained. However, the approach presented in Deutscher and Hippe (2000, 2001) was derived under the assumption, that in the case of multiple closed loop poles all pole directions are linearly independent. The purpose of this contribution is to present a parametric expression for the parametrizing polynomial matrix $\tilde{D}(s)$ without further assumptions on the closed loop poles and their pole directions in order to generalize the results given in Deutscher and Hippe (2000, 2001). This goal is accomplished by applying the concept of generalized pole directions introduced by Antsaklis (1980, 1993). Using these generalized pole directions and the closed loop poles as design parameters a compact parametric expression is derived for the polynomial matrix $\tilde{D}(s)$, that different from Deutscher and Hippe (2000, 2001) directly yields the polynomial matrix $\tilde{D}(s)$. Since the pole directions are closely related with the invariant parameter vectors used in the time domain approach, a connecting relation is presented being valid also in the case of multiple closed loop poles. These connections allow to use time domain results in the frequency domain and vice versa. Furthermore a relationship between the generalized pole directions and the generalized...
eigenvectors in the case of multiple closed loop eigenvalues is established. Since this leads to a result
given in Duan (1993) further insight is provided in the
relationship between the time and the frequency
domain approach to parametric state feedback design.
The next section reviews the state feedback design
in the frequency domain and Section 3 briefly repeats
the parametric approach in the time domain. After
introducing the generalized pole directions in Section
4 the new expression for parametric frequency
domain design of state feedback controllers is derived
as well as connecting relationships to the time
domain approach. A simple example demonstrates the
design procedure.

2. STATE FEEDBACK DESIGN IN THE FREQUENCY DOMAIN

Consider a time invariant, completely controllable and
observable linear system of nth order with p inputs and
m \geq p outputs, described in the time domain
by the state equations
\[ \begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t) \\
y(t) &= Cx(t)
\end{align*} \tag{1} 
\]
with \( B \) having full column rank and \( C \) having full
row rank. In the frequency domain the system is
characterized by its transfer behaviour
\[ \begin{align*}
y(s) &= F(s)u(s) \\
&= C(sI - A)^{-1}Bu(s)
\end{align*} \tag{2} 
\]
The \((m,p)\) system transfer matrix \( F(s) \) in (2) can be
represented in a right coprime matrix fraction
description (MFD)
\[ F(s) = N(s)D^{-1}(s) \tag{3} \]
with the \((m,p)\) polynomial matrix \( N(s) \) and the
column reduced \((p,p)\) polynomial matrix \( D(s) \). The
eigenvalues \( s_i \) of the system (1) (here identical with the
poles of \( F(s) \)) can be determined from the
denominator matrix \( D(s) \) by
\[ \det D(s) = c \det(sI - A) = 0, \ c \neq 0 \tag{4} \]
In the time domain, the state feedback
\[ u(t) = -Kx(t) \tag{5} \]
is parameterized by the constant \((p,n)\) matrix \( K \). In the
frequency domain the state feedback control
is parameterized by the \((p,p)\) polynomial matrix \( \tilde{D}(s) \)
(see e.g. Hippe, 1988). This is the denominator
matrix in the right MFD of the \((m,p)\) reference transfer
matrix (without prefilter)
\[ \begin{align*}
F_{rs}(s) &= C(sI - A + BK)^{-1} \tag{6} \\
&= N(s)\tilde{D}^{-1}(s)
\end{align*} \]
so that the closed loop characteristic polynomial is
\[ \det \tilde{D}(s) = c \det(sI - A + BK), \ c \neq 0 \tag{7} \]
As a consequence of the well known connecting
relation between the time and the frequency domain
representation of state feedback (see e.g. Hippe, 1988)
\[ \tilde{D}(s)D^{-1}(s) = I + K(sI - A)^{-1}B \tag{8} \]
the parameterizing polynomial matrix \( \tilde{D}(s) \) has the
following properties
\[ \Gamma_c[\tilde{D}(s)] = \Gamma_c[D(s)] \tag{9} \]
and
\[ \delta_{i,c}[\tilde{D}(s)] = \delta_{i,c}[D(s)], \ i = 1(l)p \tag{10} \]
Here \( \Gamma_c[\cdot] \) is the highest column degree coefficient
matrix and \( \delta_{i,c}[\cdot] \) is the \( i \)th column degree. Under the
restrictions (9) and (10) the polynomial matrix \( \tilde{D}(s) \),
characterizing the closed loop dynamics of the state
feedback loop in the frequency domain, has exactly
the same number of free parameters as the state feedback
matrix \( K \) (Hippe, 1988).

3. PARAMETRIC STATE FEEDBACK
DESIGN IN THE TIME DOMAIN

This subsection shortly reviews the parametric
approach due to Roppenecker (1982) in the case of
multiple closed loop eigenvalues. To this end con-
der the state feedback controller \( K \) for the system (1)
assigning the set of \( m \leq n \) distinct closed loop eigenvalues \( s_i \), \( \mu = 1(1)m \), with algebraic multiplicity \( n_\mu \)
such that \( n_1 + \ldots + n_\mu = n \). Further denote the
geometric multiplicity of \( s_i \) by \( m_\mu \). Now introduce the
generalized eigenvector \( \tilde{V}_\mu^{(k)} \) of grade \( k \) associated
with the closed loop eigenvalue \( s_i \)
\[ (A - BK - \tilde{s}_i I)^{(k)} \tilde{V}_\mu^{(k)} = 0 \tag{11} \]
\[ (A - BK - \tilde{s}_i I)^{(k-1)} \tilde{V}_\mu^{(k-1)} \neq 0 \]
for \( \mu = 1(1)m, \nu = 1(1)n_\mu, k = 1(1)n_\mu \). As a direct
corollary of (11) one obtains the following relation
between generalized eigenvectors of different grade
\[ (A - BK - \tilde{s}_i I)^{(l)} \tilde{V}_\mu^{(l)} = (A - BK - \tilde{s}_i I)^{(l-1)} \tilde{V}_\mu^{(l-1)} \tag{12} \]
Thus each of the \( m_\mu \) linearly independent eigenvectors \( \tilde{V}_\mu^{(1)} \)
(i.e. generalized eigenvectors of grade 1)\)
associated with the eigenvalue \( s_i \) and satisfying
\[ (A - BK - \tilde{s}_i I)^{(l)} \tilde{V}_\mu^{(1)} = 0 \tag{13} \]
is accompanied by a chain of \( n_\mu \) linearly independent
generalized eigenvectors (11) such that
\( n_{\mu_1} + \ldots + n_{\mu_\nu} m_\mu = n_\mu \). In order to obtain a parametric
expression for the state feedback controller \( K \) rewrite
the characteristic equation (13)
\[ (A - \tilde{s}_i I)^{(l)} \tilde{V}_\mu^{(1)} = BK^{(l)} \tilde{V}_\mu^{(1)} \tag{14} \]
By introducing the set of generalized invariant
parameter vectors of grade \( k \)
\[ \tilde{V}_\mu^{(k)} = K^{(k)} \tilde{V}_\mu^{(1)} \tag{15} \]
for \( \mu = 1(1)m, \nu = 1(1)n_\mu, k = 1(1)n_\mu \) and by assuming
in the sequel that \( \tilde{s}_i \) is different from all eigenvalues
of the system (1) one obtains from (14) and
(15) a parametric expression for the eigenvectors
\[ \tilde{v}_{\mu
u}^{(1)} = (A - \tilde{s}_\mu I)^{-1} B p_{\nu
u}^{(1)} \]
(16)

Rewriting relation (12) for \( k = 2 \) yields
\[ (A - \tilde{s}_\mu I)\tilde{v}_{\mu
u}^{(2)} = BK\tilde{v}_{\mu
u}^{(1)} + \tilde{v}_{\mu
u}^{(2)} \]
(17)

and hence with (15) the generalized eigenvector \( \tilde{v}_{\mu
u}^{(2)} \) of grade 2 can be expressed as
\[ \tilde{v}_{\mu
u}^{(2)} = (A - \tilde{s}_\mu I)^{-1}(Bp_{\nu
u}^{(2)} + \tilde{v}_{\mu
u}^{(1)}) \]
(18)

Proceeding with this approach one obtains in the general case
\[ \tilde{v}_{\mu
u}^{(n)} = (A - \tilde{s}_\mu I)^{-1}(Bp_{\nu
u}^{(1)} + \tilde{v}_{\mu
u}^{(n-1)}) \]
(19)

Now introduce the \((n,n)\) matrix \( \tilde{V} \) of generalized eigenvectors
\[ \tilde{V} = \begin{bmatrix} \tilde{v}_{11}^{(1)} & \cdots & \tilde{v}_{1n}^{(1)} & \cdots & \tilde{v}_{11}^{(n-1)} & \cdots & \tilde{v}_{11}^{(n)} & \cdots \end{bmatrix} \]
(20)

and the \((n,p)\) matrix \( P \) of generalized invariant parameter vectors
\[ P = \begin{bmatrix} p_{11}^{(1)} & \cdots & p_{1n}^{(1)} & \cdots & p_{11}^{(n-1)} & \cdots & p_{11}^{(n)} & \cdots \end{bmatrix} \]
(21)

Then by writing (15) in matrix form
\[ P = K\tilde{V}^{-1} \]
(22)

and by assuming that \( \tilde{V} \) is nonsingular one has a parametric expression for the state feedback controller \( K \)
\[ K = P\tilde{V}^{-1} \]
(23)

Conversely one can show that (23) assigns the eigenstructure given by (16) and (19) to the closed loop system. However, it should be noted that one cannot assign an arbitrary eigenstructure to the the system (1), as the closed loop eigenstructure is restricted by Rosenbrock’s Control Structure Theorem (see e.g. Kailath, 1980). For instance this theorem implies \( m_p \leq p \), which is an obvious consequence of the definition of the pole directions in the frequency domain approach in the next section.

4. PARAMETRIC STATE FEEDBACK DESIGN IN THE FREQUENCY DOMAIN

4.1 Definition of the generalized pole directions

Consider \( m \leq n \) distinct closed loop poles \( \tilde{s}_\mu \) with algebraic multiplicity \( n_\mu \) and geometric multiplicity \( m_\mu \) accompanied by nonzero vectors \( q_{\mu
u}^{(1)} \), \( q_{\mu
u}^{(2)} \), \ldots, \( q_{\mu
u}^{(n_\mu)} \), \( \mu = 1(1)m \), \( \nu = 1(1)m_\mu \), satisfying
\[ \tilde{D}(\tilde{s}_\mu)q_{\mu
u}^{(1)} = 0 \]
\[ \tilde{D}(\tilde{s}_\mu)q_{\mu
u}^{(2)} = -\tilde{D}'(\tilde{s}_\mu)q_{\mu
u}^{(1)} \]
\[ \vdots \]
\[ \tilde{D}(\tilde{s}_\mu)q_{\mu
u}^{(n_\mu)} = -(\tilde{D}'(\tilde{s}_\mu)q_{\mu
u}^{(n_\mu-1)}) + \cdots + \frac{1}{(n_\mu-1)!} \tilde{D}^{n_\mu-1}(\tilde{s}_\mu)q_{\mu
u}^{(1)} \]
where \( P^{(i)}(s) \) denotes the \( i \)th derivative of a polynomial matrix \( P(s) \) evaluated at \( s = s_j \) and the \( m_\mu \leq p \) vectors \( q_{\mu
u}^{(1)}, \mu = 1(1)m, \nu = 1(1)m_\mu \), being linearly independent, then the vectors \( q_{\mu
u}^{(k)} \) are called generalized pole directions of grade \( k \).

Remark 1 The concept of generalized pole directions was originally introduced by Antsaklis (1980, 1993), where it was also shown that if such vectors exist the closed loop pole \( \tilde{s}_\mu \) has algebraic multiplicity \( n_\mu \) and geometric multiplicity \( m_\mu \).

Remark 2 The geometric multiplicity \( m_\mu \leq p \) of the closed loop pole \( \tilde{s}_\mu \) can be computed in the frequency domain by \( m_\mu = p - \text{rank} \tilde{D}(\tilde{s}_\mu) \).

Remark 3 For \( n_\mu \leq n_\mu, \leq \ldots \leq n_\mu, m_\mu \), (24) implies, that \( \tilde{D}(s) \) has the Smith form
\[ \tilde{D}(s) = \text{diag}(\varepsilon_1(s), \ldots, \varepsilon_{n_\mu-1}(s), (s - \tilde{s}_\mu)\nu_1\nu_\mu)(s),\ldots,(s - \tilde{s}_\mu)\nu_1\nu_\mu(e_\nu(s)) \]

with \( e_\nu(\tilde{s}_\mu) \neq 0, i = 1(1)p \), and the \( e_\nu(s) \) obtained by repeating this for each distinct closed loop pole \( \tilde{s}_\mu \) (Antsaklis, 1980).

4.2 Parametric expression for state feedback design

In the following a compact parametric expression is derived for the polynomial matrix \( \tilde{D}(s) \) using the closed loop poles \( \tilde{s}_\mu \) and their generalized pole directions \( q_{\mu
u}^{(k)} \) as free design parameters. To this end one needs the following property.

Property 1 Each self conjugate set of closed loop poles \( \tilde{s}_i, i = 1(n) \), is accompanied by a self conjugate set of generalized pole directions \( q_{\mu
u}^{(1)}, q_{\mu
u}^{(2)}, \ldots, q_{\mu
u}^{(n_\mu)} \), \( \mu = 1(1)m \), \( \nu = 1(1)m_\mu \), such that \( q_{\mu
u}^{(k)}(\tilde{s}_i) = (q_{\mu\nu}^{(k)}(\tilde{s}_i))^* \) whenever \( \tilde{s}_i = \tilde{s}_i^* \) (where * denotes taking the conjugate complex).

Given \( n \) closed loop poles with corresponding generalized pole directions the following Theorem 1 provides an explicit expression for the polynomial matrix \( \tilde{D}(s) \).

Theorem 1 Consider the right coprime MFD (3) of the \( n \)th order system (1) and the self conjugate set \( S \) of \( m \leq n \) distinct closed loop poles \( \tilde{s}_\mu, \mu = 1(1)m \), with algebraic multiplicity \( n_\mu \) such that \( n_1 + \ldots + n_\mu = n \) and geometric multiplicity \( m_\mu \) with corresponding self conjugate \( n \) generalized pole
directions \( q_{\mu}^{(1)}, q_{\nu}^{(2)}, \ldots, q_{\mu_{n_{\mu}}} \), \( \mu = 1(1)m, \nu = 1(1)n_{\nu} \), such that \( n_{\mu_{1}} + \cdots + n_{\mu_{m_{\mu}}} = n_{\mu} \). The denominator matrix
\[ \tilde{D}(s) = D(s) - FW^{-1}S(s) \quad (25) \]
with the \((n,p)\) polynomial matrix
\[ S(s) = \text{diag}\left[ s_{\mu_{1}(D(s))^{-1}} \ldots s_{1} \right] \quad (26) \]
and the \((p,n)\) constant matrix
\[ F = \begin{bmatrix} f_{11}^{(1)} & \cdots & f_{1n_{\mu}}^{(1)} & f_{12}^{(1)} & \cdots & f_{1n_{\mu}}^{(2)} & \cdots & f_{1n_{\mu}}^{(p)} \end{bmatrix} \quad (27) \]
with constant \((p,1)\) vectors
\[ f_{k1}^{(i)} = D(\tilde{s}_{\mu})q_{\mu}^{(i)} + D^{(i)}(\tilde{s}_{\mu})q_{\mu}^{(i+1)} + \cdots + \frac{1}{(i-1)!}D^{(i-1)}(\tilde{s}_{\mu})q_{\mu}^{(i)} \quad (28) \]
for \( k = 1(1)n_{\mu}, \) and the \((n,n)\) constant matrix
\[ W = \begin{bmatrix} w_{11}^{(1)} & \cdots & w_{1n_{\mu}}^{(1)} & w_{12}^{(1)} & \cdots & w_{1n_{\mu}}^{(2)} & \cdots & w_{1n_{\mu}}^{(n)} \end{bmatrix} \quad (29) \]
with \((n,1)\) vectors
\[ w_{k1}^{(i)} = S(\tilde{s}_{\mu})q_{\mu}^{(i)} + S^{(i)}(\tilde{s}_{\mu})q_{\mu}^{(i+1)} + \cdots + \frac{1}{(i-1)!}S^{(i-1)}(\tilde{s}_{\mu})q_{\mu}^{(i)} \quad (30) \]
for \( k = 1(1)n_{\mu} \), assigns the set \( \tilde{S} \) of closed loop poles and the corresponding generalized pole directions to the closed loop system with real \((n,p)\) matrix \( FW^{-1} \) if and only if Property 1 is satisfied and the matrix \( W \) is nonsingular.

**Proof.** In order to show that under the stated assumptions each denominator matrix \( \tilde{D}(s) \) can be represented by (25), consider the following expression with \( \Lambda(s) = \text{diag}\left[ s_{\mu_{1}(D(s))^{-1}} \right] \)
\[ \tilde{D}(s) = \Gamma_{s}[D(s)](\Lambda(s) + \tilde{D}_{c}S(s)) \quad (31) \]
Implied by (9) and (10) and the polynomial matrix \( S(s) \) as defined in (26). The constant \((p,n)\) coefficient matrix \( \tilde{D}_{c} \) in (31) contains the free polynomial coefficients of \( \tilde{D}(s) \). Substituting the closed loop poles \( \tilde{s}_{\mu} \) in (31), postmultiplying with their generalized pole directions \( q_{\mu}^{(1)} \) of grade 1 and observing the first line in (24) gives
\[ \Gamma_{s}[D(s)](\Lambda(s) + \tilde{D}_{c}S(s))q_{\mu}^{(1)} = 0 \quad (32) \]
Since \( \Gamma_{s}[D(s)] \) is nonsingular \( (D(s) \) is column reduced), the above bracket must vanish and hence
\[ \tilde{D}_{c}S(\tilde{s}_{\mu})q_{\mu}^{(1)} = -\Lambda(\tilde{s}_{\mu})q_{\mu}^{(1)} \quad (33) \]
for \( \mu = 1(1)m, \nu = 1(1)n_{\nu} \). The nonsingularity of \( \Gamma_{s}[D(s)] \) admits the representation
\[ \tilde{D}_{c} = D_{c} + \Gamma_{s}[D(s)]M \quad (34) \]
for each coefficient matrix \( \tilde{D}_{c} \) in (31) and some \((p,p)\) constant matrix \( M \), where \( D_{c} \) is the \((p,n)\) coefficient matrix of the denominator matrix
\[ D(s) = \Gamma_{s}[D(s)](\Lambda(s) + D_{c}S(s)) \quad (35) \]
in the MFD (3). Now substitute the closed loop poles \( \tilde{s}_{\mu} \) in (31) and its derivative \( \tilde{D}(s) \). Then postmultiply \( \tilde{D}(s) \) with the generalized pole directions \( q_{\mu}^{(1)} \) of grade 1, \( \tilde{D}(s)q_{\mu}^{(1)} \) with the generalized pole directions \( q_{\mu}^{(2)} \) of grade 2 and use the second line of (24) to get
\[ \tilde{D}_{c}(S(\tilde{s}_{\mu})q_{\mu}^{(1)} + S^{(1)}(\tilde{s}_{\mu})q_{\mu}^{(1+1)}) \]
\[ = -\Lambda(\tilde{s}_{\mu})q_{\mu}^{(1)} + \Lambda^{(1)}(\tilde{s}_{\mu})q_{\mu}^{(1+1)} \quad (36) \]
after a simple rearrangement and in view of the nonsingularity of \( \Gamma_{s}[D(s)] \). Proceeding with this approach one obtains in the general case
\[ \tilde{D}_{c}(S(\tilde{s}_{\mu})q_{\mu}^{(1)} + S^{(1)}(\tilde{s}_{\mu})q_{\mu}^{(1+1)}) \]
\[ + \frac{1}{(i-1)!}S^{(i-1)}(\tilde{s}_{\mu})q_{\mu}^{(i)} \quad (37) \]
for \( \mu = 1(1)m, \nu = 1(1)n_{\nu} \). Introducing (34) in (36) and rewriting yields
\[ M(S(\tilde{s}_{\mu})q_{\mu}^{(1)} + S^{(1)}(\tilde{s}_{\mu})q_{\mu}^{(1+1)}) \]
\[ + \frac{1}{(i-1)!}S^{(i-1)}(\tilde{s}_{\mu})q_{\mu}^{(i)} \quad (37) \]
where (35) was used. As the matrix \( W \) is assumed to be nonsingular (see (29)), (37) obtains with (27) the form
\[ M = -FW^{-1} \quad (38) \]
Substituting (34) in (31) and observing (35) yields
\[ \tilde{D}(s) = D(s) + MS(s) \quad (39) \]
such that (25) follows from (38) and (39).
In order to show the sufficient part of Theorem 1 substitute \( \tilde{s}_{\mu} \) in (25) and postmultiply with the vectors \( q_{\mu}^{(1)} \) to give
\[ \tilde{D}(\tilde{s}_{\mu})q_{\mu}^{(1)} = D(\tilde{s}_{\mu})q_{\mu}^{(1)} - FW^{-1}S(\tilde{s}_{\mu})q_{\mu}^{(1)} \quad (40) \]
By noting that \( S(\tilde{s}_{\mu})q_{\mu}^{(1)} = w_{\mu}^{(1)} = We_{\mu} \) and
\[ F_{\mu} = f_{\mu}^{(1)} = D(\tilde{s}_{\mu})q_{\mu}^{(1)} \]
where \( e_{\mu} \) is the \( \mu \)th unit vector, the above bracket must vanish, hence \( \tilde{D}(\tilde{s}_{\mu})q_{\mu}^{(1)} = 0 \), \( \mu = 1(1)m, \nu = 1(1)n_{\nu} \) (41)
Hence the numbers \( \tilde{s}_{\mu} \) and the vectors \( q_{\mu}^{(1)} \) are the closed loop poles and the corresponding generalized pole directions of grade 1 (see (7) and (24)), since for nonzero vectors \( q_{\mu}^{(1)} \) condition (41) only holds if \( \tilde{D}(\tilde{s}_{\mu}) \) is rank deficient. Similarly writing
\[
\begin{align*}
\tilde{D}(\tilde{x}_p)q^{(2)}_{\mu
u} + \tilde{D}^{(1)}(\tilde{x}_p)q^{(1)}_{\mu
u} = D(\tilde{x}_p)q^{(2)}_{\mu
u} - FW^{-1}S(\tilde{x}_p)q^{(2)}_{\mu
u} \\
+ D^{(1)}(\tilde{x}_p)q^{(1)}_{\mu
u} - FW^{-1}S^{(1)}(\tilde{x}_p)q^{(1)}_{\mu
u} = D(\tilde{x}_p)q^{(2)}_{\mu
u} + D^{(1)}(\tilde{x}_p)q^{(1)}_{\mu
u} - FW^{-1}(S(\tilde{x}_p)q^{(2)}_{\mu
u} + S^{(1)}(\tilde{x}_p)q^{(1)}_{\mu
u})
\end{align*}
\]

(42)

where (25) was used, one can verify that
\[
\tilde{D}(\tilde{x}_p)q^{(2)}_{\mu
u} + \tilde{D}^{(1)}(\tilde{x}_p)q^{(1)}_{\mu
u} = 0
\]

(43)
such that \(\tilde{D}(s)\) has the generalized pole directions \(q^{(2)}_{\mu
u}\). Proceeding with this approach it is straightforward to show that by Remark 1 (25) assigns the closed loop poles and their generalized pole directions as given in Theorem 1. From Property 1 it follows that the matrix \(FW^{-1}\) is real.

\[\square\]

Remark 4 If each closed loop pole of multiplicity \(n_q\) \(\leq p\) is accompanied by a set of \(n_q\) linearly independent pole directions, the constant matrices \(F\) and \(W\) in the parametric expression (25) obtain the simple forms
\[
F = [D(\tilde{x}_p)q_1 \ldots D(\tilde{x}_p)q_m]
\]
and
\[
W = [S(\tilde{x}_p)q_1 \ldots S(\tilde{x}_p)q_m]
\]
(44)
(45)
where \(q_1\) denotes the generalized pole directions of grade 1. Compared to Deutscher and Hippe (2000, 2001) this is a more compact representation of the polynomial matrix \(\tilde{D}(s)\) under the stated assumptions.

4.3 Relation between generalized pole directions and generalized closed loop eigenvectors

Considering system (1) a relation between the new design parameters in the frequency domain and the closed loop eigenvectors can be obtained by introducing the right coprime MFD
\[
(sI - A)^{-1}B = N_x(s)D^{-1}(s)
\]
(46)
and using it to rewrite (8) as
\[
\tilde{D}(s) = D(s) + KN_x(s)
\]
(47)
Substituting \(s\) by the closed loop poles \(\tilde{x}_p\) in (47) and multiplying the result from the right by the generalized pole directions \(q^{(1)}_{\mu
u}\) of grade 1 gives
\[
\tilde{D}(\tilde{x}_p)q^{(1)}_{\mu
u} = D(\tilde{x}_p)q^{(1)}_{\mu
u} + KN_x(\tilde{x}_p)q^{(1)}_{\mu
u}
\]
which implies
\[
KN_x(\tilde{x}_p)q^{(1)}_{\mu
u} = -D(\tilde{x}_p)q^{(1)}_{\mu
u}
\]
(48)
in view of the first line in (24). Now consider the expression
\[
(A - BK - \tilde{x}_p I)N_x(\tilde{x}_p)q^{(1)}_{\mu
u} = (A - BK - \tilde{x}_p I)N_x(\tilde{x}_p)q^{(1)}_{\mu
u} - BKN_x(\tilde{x}_p)q^{(1)}_{\mu
u}
\]
(49)
Substituting (48) in (49), and observing that (46) implies
\[
(sI - A)N_x(s)x = BD(s)x
\]
(50)
expression (49) can be rewritten as
\[
(A - BK - \tilde{x}_p I)N_x(\tilde{x}_p)q^{(1)}_{\mu
u} = 0
\]
(51)
Comparing this with the eigenvector equation (13) the relation between the generalized pole directions \(q^{(1)}_{\mu
u}\) of grade 1 and the closed loop eigenvectors \(\tilde{v}^{(1)}_{\mu
u}\) is obviously given by
\[
\tilde{v}^{(1)}_{\mu
u} = N_x(\tilde{x}_p)q^{(1)}_{\mu
u}, \quad \mu = 1(1)m, \quad \nu = 1(1)m
\]
(52)
Next it is verified that the generalized closed loop eigenvectors \(\tilde{v}^{(2)}_{\mu
u}\) of grade 2 can be expressed as
\[
\tilde{v}^{(2)}_{\mu
u} = N_x(\tilde{x}_p)q^{(2)}_{\mu\nu} + N_x(\tilde{x}_p)q^{(1)}_{\mu
u}
\]
(53)
To this end substitute (52) and (53) in (12) for \(k = 1\) yielding
\[
-(\tilde{x}_p I - A)N_x(\tilde{x}_p)q^{(2)}_{\mu\nu} = -(\tilde{x}_p I - A)N_x(\tilde{x}_p)q^{(1)}_{\mu\nu} - BKN_x(\tilde{x}_p)q^{(2)}_{\mu\nu} - BKN_x(\tilde{x}_p)q^{(1)}_{\mu\nu} = N_x(\tilde{x}_p)q^{(1)}_{\mu\nu}
\]
(54)
which gives with (50) and
\[
(sI - A)N_x(s)x = -N_x(s)x + BD^{(1)}(s)x
\]
(55)
obtained from (50) by differentiating with respect to \(s\) the relation
\[
-BD(\tilde{x}_p)q^{(2)}_{\mu\nu} + N_x(\tilde{x}_p)q^{(1)}_{\mu\nu} = -BD^{(1)}(\tilde{x}_p)q^{(1)}_{\mu\nu} - BKN_x(\tilde{x}_p)q^{(2)}_{\mu\nu} - BKN_x(\tilde{x}_p)q^{(1)}_{\mu\nu} = N_x(\tilde{x}_p)q^{(1)}_{\mu\nu}
\]
(56)
Equation (56) and (47) imply that
\[
-B(D(\tilde{x}_p)q^{(2)}_{\mu\nu} + KN_x(\tilde{x}_p)q^{(2)}_{\mu\nu}) + D^{(1)}(\tilde{x}_p)q^{(1)}_{\mu\nu} + KN_x^{(1)}(\tilde{x}_p)q^{(1)}_{\mu\nu} = -B(D(\tilde{x}_p)q^{(2)}_{\mu\nu} + \tilde{D}(\tilde{x}_p)q^{(1)}_{\mu\nu})
\]
(57)
must vanish. This directly follows from the second line in (24) such that relation (53) is verified. Repeating this approach for all generalized eigenvectors one finally has the relationship between the generalized eigenvectors and the pole directions
\[
\tilde{v}^{(1)}_{\mu\nu} = N_x(\tilde{x}_p)q^{(1)}_{\mu\nu}
\]
\[
\tilde{v}^{(2)}_{\mu\nu} = N_x(\tilde{x}_p)q^{(2)}_{\mu\nu} + N^{(1)}(\tilde{x}_p)q^{(1)}_{\mu\nu}
\]
(58)
\[
\tilde{v}^{(\mu\nu)}_{\mu\nu} = N_x(\tilde{x}_p)q^{(\mu\nu)}_{\mu\nu} + \ldots + \frac{1}{1(1)m}N^{(\mu\nu-1)}(\tilde{x}_p)q^{(1)}_{\mu\nu}
\]
for \(\mu = 1(1)m, \nu = 1(1)m\).

Remark 5 The same result was also derived in Duan (1993) using a time domain approach to eigenstructure assignment. However, different from Duan (1993) this contribution shows how the vectors \(q^{(k)}_{\mu\nu}\) are related with the frequency domain design of state feedback controllers. This provides new insights in the relationship between the time and frequency domain approach to eigenstructure assignment in linear systems.

4.4 Relation between generalized pole directions and generalized invariant parameter vectors

For given closed loop pole locations \(\tilde{x}_p\) one can immediately substitute (52) and (48) in (15) to get
such that the closed loop denominator matrix \( \tilde{D}(s) \) in (25) is given by

\[
\tilde{D}(s) = D(s) - FW^{-1}S(s) = \begin{bmatrix} 0 & 3 + s \\ -3 - 4s - s^2 & -2 \end{bmatrix}
\]

6. CONCLUSIONS

In this contribution a parametric approach was presented to the frequency domain design of state feedback controllers in the general case. This was accomplished by introducing the generalized pole direction as new design parameters. Since the proposed approach does not require any assumptions on the closed loop poles and their generalized pole directions it extends the recent approach of Deutcher and Hippe (2000, 2001) to the case of multiple closed loop poles. As an additional result it was shown how the generalized pole directions are related with the eigenstructure assignment approach in linear systems. A simple example demonstrated the design procedure.

REFERENCES


Deutcher, J. and P. Hippe (2001). Parametric compensator design in the frequency domain. Accepted for publication in Int. J. Control.


