STABILITY ROBUSTNESS ANALYSIS OF UNCERTAIN DISCRETE-TIME DESCRIPTOR SYSTEMS

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Abstract: The stability robustness problem of uncertain discrete-time descriptor systems is investigated. Uncertainties appear not only in the state matrix but also in the advanced state matrix. Through checking the feasibility of a strict linear matrix inequality problem, the robust stability of the uncertain system can be easily determined. The proposed result also gives a sufficient condition for the existence of a parameter-dependent Lyapunov function for such uncertain systems.

Keywords: Descriptor systems, robust stability, linear matrix inequalities, linear systems, uncertainties.

1. INTRODUCTION

Descriptor systems (also referred to singular systems, or generalized state-space systems, or implicit systems, or semistate systems in the literature) have received a great number of attentions over past 20 years (Dai, 1989; Kucera and Zagala, 1988; Lewis, 1986). It has been demonstrated that, in comparison with the standard state variable model, such system model can capture more essential properties for physical systems, especially in preserving original parameter information (Dai, 1989; Verghese et al., 1981). A lot of applications of descriptor systems have also been found in a variety of area (Clements, 1994; Dai, 1989; Fang, 1992; Hu and Davison, 1994).

The research of robust stability for continuous-time descriptor systems can be referred to (Chou and Lin, 1999; Fang and Chang, 1993; Qiu and Davison, 1992) and the references therein. For discrete-time descriptor systems, the robust stability issue with structured perturbations was firstly investigated in (Fang et al., 1994) and an allowable upper bound of uncertainties was provided there. The robust stabilization problem subject to unstructured perturbations was studied by Xu et al. (Xu et al., 2001). It should be pointed out that all of the work about robust stability mentioned above was developed in the context of uncertainties affecting only the state matrix. When uncertainties appear in both the state matrix and the advanced state matrix of descriptor systems,
the robust stability problem becomes much more complicated. For this problem, continuous-time interval descriptor systems were investigated in (Lin et al., 2001). The main idea is to check rank dropping of certain matrix, in which rank drops while eigenvalues move across the imaginary axis. The technique cannot be applied to discrete-time cases because rank dropping would happen while eigenvalues move across either the origin (stable region) or the unit circle. Up to date, it seems that no result about robust stability of such uncertain system is available in the open literature. This motivates us to study the problem. The paper is concerned with robust stability problem of the following descriptor system

\[ E(\delta)x(k + 1) = A(\delta)x(k) \]  

where the advanced state matrix \( E(\delta) \in \mathbb{R}^{n \times n} \) might be singular. \( E(\delta) \) and \( A(\delta) \) belongs to the following convex polytopic set

\[
\left\{ [E(\delta) A(\delta)] = \sum_{i=1}^{N} \delta_i [E_i, A_i] : E_i \in \mathbb{R}^{n \times n}, \\
A_i \in \mathbb{R}^{n \times n}, \delta_i \geq 0, \sum_{i=1}^{N} \delta_i = 1 \right\},
\]

In the above, \( \delta \) simply represents a vector variable with real elements \( \{\delta_1, \delta_2, \ldots, \delta_N\} \). \( [E_i, A_i] \) are given vertex matrices. For convenience, we define the unit simplex set

\[
\Delta = \{ (\delta_1, \delta_2, \ldots, \delta_N) : \delta_i \geq 0, \sum_{i=1}^{N} \delta_i = 1 \}.
\]

While \( E_i \) and \( A_i \), \( i = 1, 2, \ldots, N \), are given, our problem is to determine if the system (1) is Schur stable for all \( \delta \in \Delta \). Since the variation of the elements in \( E(\delta) \) and \( A(\delta) \) might destroy not only stability but causality and regularity (Fang et al., 1994; Qiu and Davison, 1992), we have to check these three kinds of properties simultaneously while concerning robust stability problem of system (1).

In recent years, LMI techniques have emerged as a powerful analysis and design tool in solving a number of control problems (Boyd et al., 1994; Doy et al., 1991). Once the problem is transformed to the LMI formulation, it can be solved numerically very efficiently (Gahinet et al., 1995; Ghaoui et al., 1995). In this paper, the robust stability of uncertain system (1) is determined via solving a strict LMI feasibility problem. Using the feasible solution of the derived LMI problem, a parameter-dependent Lyapunov function can also be constructed for system (1). As we know, no method is available up to date for constructing a parameter-dependent Lyapunov function for the uncertain system (1).

Some preliminary results are given in Section 2. Section 3 describes the main results of the paper. A numerical example is given in Section 4 to illustrate the proposed ideas. Conclusion is located in Section 5. The notation used throughout the paper is fairly standard. \( M \geq 0 \) (\( M > 0 \)) stands for \( M \) is positive semidefinite (definite) and \( M < 0 \) for negative definiteness. \( M^t \) means transpose of \( M \). \( \text{rank}(M) \) stands for rank of matrix \( M \). \( |M| \) represents determinant of \( M \). \( I_p \) denotes the identity matrix of dimension \( p \). \( \deg(\cdot) \) means degree of a polynomial.

2. PRELIMINARIES

Before discussing the whole families of system (1), some definitions and properties of single member system of (1) are reviewed and derived.

**Definition 1.** (Dai, 1989; Lewis, 1986) Consider a linear descriptor system \( \dot{x}(k + 1) = Ax(k) \). The system is termed regular if \( |E - \bar{A}| \) is not identically zero. It is termed causal if \( \deg|E - \bar{A}| = \text{rank}(\bar{E}) \) and Schur stable if all roots of \( |E - \bar{A}| = 0 \) lie within the unit disk.

\[ \diamond \]

**Lemma 1.** (Hsiung and Lee, 1999; Xu et al., 2001) The descriptor system \( \dot{x}(k + 1) = \bar{A}x(k) \) is regular, causal, and Schur stable if and only if there exists a matrix \( Y \in \mathbb{R}^{n \times n} \) satisfying

\[ \bar{E}^t Y \bar{E} \geq 0 \]  

\[ \bar{A}^t Y \bar{A} - \bar{E}^t Y \bar{E} < 0. \]

\[ \diamond \]

The inequality forms of (4) and (5) may not suitably be extended to solve the problem raised in this paper. In the following, we introduce a new strict linear matrix inequality condition that is equivalent to Lemma 1 but will be convenient for dealing with our problem. Suppose \( \text{rank}(\bar{E}) = r \leq n \). \( \bar{U} \in \mathbb{R}^{(n-r) \times n} \) is any matrix of full rank and satisfies \( \bar{U} \bar{E} = 0 \).

**Lemma 2.** The system \( \dot{x}(k + 1) = \bar{A}x(k) \) is regular, causal, and Schur stable if and only if the following LMI, with matrix variables \( \bar{P} > 0 \) and \( \bar{Q} > 0 \),

\[ \bar{A}^t \bar{P} \bar{A} - \bar{E}^t \bar{P} \bar{E} - \bar{U}^t \bar{Q} \bar{U} \bar{A} < 0 \]

is feasible, where \( \bar{U} \) is any constant matrix of full rank and satisfies \( \bar{U} \bar{E} = 0 \).

Proof: (\( \Leftarrow \)): Sufficiency can be easily proved by using Lemma 1. Rewrite (6)
\[ \bar{A}'(\bar{P} - \bar{U}'\bar{Q}\bar{U})\bar{A} - \bar{E}'(\bar{P} - \bar{U}'\bar{Q}\bar{U})\bar{E} < 0. \] (7)

Let \( Y = \bar{P} - \bar{U}'\bar{Q}\bar{U} \) which obviously satisfies both (4) and (5). Thus the system \( \bar{E}x(k+1) = \bar{A}x(k) \) is regular, causal, and Schur stable.

(⇒): If the system \( \bar{E}x(k+1) = \bar{A}x(k) \) is regular, causal, and Schur stable, there exist two nonsingular matrices \( M \in \mathbb{R}^{n\times n} \) and \( L \in \mathbb{R}^{n\times n} \) such that

\[ M\bar{E}L = \begin{bmatrix} I_r & 0 \\ 0 & I_{n-r} \end{bmatrix}, \quad M\bar{A}L = \begin{bmatrix} A_1 & 0 \\ 0 & I_{n-r} \end{bmatrix} \] (8)

where \( A_1 \in \mathbb{R}^{r\times r} \) whose eigenvalues equal the roots of \( |z\bar{E} - \bar{A}| = 0 \) (Dai, 1989; Verghese et al., 1981). Since \( A_1 \) is Schur stable, there exists an \( r \times r \) matrix \( P_1 > 0 \) satisfying \( A_1^TP_1A_1 - P_1 < 0 \). Then construct \( \bar{P} \) by

\[ \bar{P} = M^t \begin{bmatrix} P_1 & 0 \\ 0 & I_{n-r} \end{bmatrix} M. \]

Let \( \bar{U} \) be any constant matrix of full rank and satisfies \( \bar{U}\bar{E} = 0 \). From the first equality of (8), each \( \bar{U} \) can always be represented as \( \bar{U} = [0 \ W]^t M \), where \( W \in \mathbb{R}^{(n-r)\times (n-r)} \) is a nonsingular matrix. Let \( \bar{Q} = 2W^{-1}W^{-1} \). It is easy to check by direct substitution that any such constructed \( \bar{P} \) and \( \bar{Q} \) satisfy (6) with the corresponding \( \bar{U} \). Q.E.D.

**Remark 1.** Although Lemma 1 and Lemma 2 are equivalent, the latter can directly coincide with the standard Lyapunov inequality when \( E = I_n \) (\( \bar{U}'\bar{Q}\bar{U} \) vanishes in that case). Furthermore, the condition in Lemma 2 is expressed in the standard strict LMI feasibility problem (Boyer et al., 1994). Thus the LMI toolbox of Matlab can be directly applied.

**Definition 2.** The system (1) is quadratically-like admissible if there exist \( P > 0 \) and \( Q > 0 \) satisfying

\[ A_1^tPA_1 - F^tP_1A_1 - PE_1 < 0 \quad \forall \ \delta \in \Delta. \] (9)

Obviously, from Lemma 2, quadratic-like admissibility implies the system (1) is regular, causal, and Schur stable for all \( \delta \in \Delta \). However, quadratic-like admissibility requires common \( P \) and common \( Q \), which generally provides quite conservative results for robust stability. It is natural to ask if there exist parameter-dependent matrices (Feret al., 1996) \( P(\delta) \) and \( Q(\delta) \) satisfying

\[ A_1^tP(\delta)A_1 - F^tP(\delta)E_1 < 0 \quad \forall \ \delta \in \Delta. \] (10)

For checking robust stability of system (1) with Lemma 2, it requires here that \( P(\delta) > 0 \) and \( Q(\delta) > 0 \) for all \( \delta \in \Delta \). In next theorem, an LMI-based approach to the construction of required \( P(\delta) \) and \( Q(\delta) \) is given.

**Theorem 1.** If the following strict LMIs, with matrix variables \( F, X, G, W, P_i > 0, \) and \( Q_i > 0, i = 1, 2, \cdots, N, \)

\[
\begin{bmatrix}
A_i^tF + F^tA_i + E_i^tX + X^tE_i & GA_i - F \\
WE_i - X & 0
\end{bmatrix} < 0,
\]

\[
P_i - U_i^tQ_iU - G - G^t & 0 \\
0 & -P_i - W - W^t
\]

\[
\begin{bmatrix}
A_i^tG_i^t - F^t & E_i^tW_i^t - X^t \\
0 & -P_i - W - W^t
\end{bmatrix} < 0,
\]

\[ i = 1, 2, \cdots, N \] (11)

are feasible, then the parameter-dependent matrices \( P(\delta) \) and \( Q(\delta) \) can be constructed by the way of

\[ P(\delta) = \sum_{i=1}^{N} \delta_i P_i, \quad Q(\delta) = \sum_{i=1}^{N} \delta_i Q_i. \] (12)

Proof: Let \( F, X, G, W, P_i > 0, \) and \( Q_i > 0, i = 1, 2, \cdots, N \) be the solutions to (11). Define \( \Omega_i \) as

\[ \Omega_i = \begin{bmatrix} A_i^tF + F^tA_i + E_i^tX + X^tE_i & GA_i - F \\
WE_i - X & 0 \\
P_i - U_i^tQ_iU - G - G^t & 0 \\
0 & -P_i - W - W^t
\end{bmatrix}. \] (13)

3. MAIN RESULT

Once the vertices \([E_i, A_i]\) are given, assume \( \text{rank}(E(\delta)) = r \) for all \( \delta \in \Delta \). Let \( U \in \mathbb{R}^{(n-r)\times n} \) be of full rank and satisfy \( UE_i = 0 \) for all \( i \). For extending the result of Lemma 2 to system (1), we firstly introduce the following definition.
Denote $\Omega(\delta) \equiv \sum_{i=1}^{N} \delta_i \Omega_i$ and obviously $\Omega(\delta) < 0$ for all $\delta \in \Delta$. With the notations of (2) and (12), $\Omega(\delta)$ can be also explicitly expressed as

$$
\Omega(\delta) = \begin{bmatrix}
A(\delta)^{t}F + F^{t}A(\delta) + E(\delta)^{t}X + X^{t}E(\delta) \\
GA(\delta) - E \\
WE(\delta) - X \\
A(\delta)^{t}G^{t} - F^{t} \\
P(\delta) - U^{t}Q(\delta)U - G - G^{t} \\
0 \\
-P(\delta) - W - W^{t}
\end{bmatrix}.
$$

Since $\Omega(\delta) < 0$ for all $\delta \in \Delta$, it quickly follows that for each $\delta \in \Delta$

$$
[I_n \ A(\delta) \ E(\delta) \ A(\delta) \ E(\delta)]\Omega(\delta) = A(\delta)^{t}P(\delta)A(\delta) - E(\delta)^{t}P(\delta)E(\delta) - A(\delta)^{t}U^{t}Q(\delta)U A(\delta) < 0. \tag{14}
$$

The fact that $P(\delta) > 0$ and $Q(\delta) > 0$ for all $\delta \in \Delta$ is obtained from (12). Q.E.D.

**Remark 3.** If there exist $P(\delta) > 0$ and $Q(\delta) > 0$, for all $\delta \in \Delta$, satisfying (10), the system (1) is termed robustly admissible. Note that the system (1) is robustly admissible can still imply, based on Lemma 2, that every member of system (1) is regular, causal, and Schur stable. From the viewpoint of conservativeness, robust admissibility is better than quadratic-like admissibility.

We are now in a position to state the main result of this paper.

**Theorem 2.** The system (1) is regular, causal, and Schur stable for all $\delta \in \Delta$ if the strict LMIs in Theorem 1 are feasible. Q.E.D.

**Remark 4.** $P(\delta)$ can be used to form a parameter-dependent Lyapunov function $V(\delta) = x^{t}(k)E(\delta)P(\delta)E(\delta)x(k) \geq 0$ for system (1). Along the state trajectory $x(k)$ of system (1), for any $\delta \in \Delta$,

$$
V(k+1, \delta) - V(k, \delta) = x^{t}(k+1)E(\delta)(P(\delta) - U^{t}Q(\delta)U)E(\delta)x(k+1) - x^{t}(k)E(\delta)P(\delta)E(\delta)x(k) \\
= x^{t}(k)(A(\delta)E(\delta)(P(\delta)A(\delta) - E(\delta)P(\delta)E(\delta) \\
- A(\delta)^{t}U^{t}Q(\delta)U A(\delta))x(k) \leq 0. \tag{15}
$$

Note that $V(k+1, \delta) - V(k, \delta) = 0$ occurs only at $x(k) = 0$. If we restrict the variables $P_i = P$ for all $i$ (i.e. common $P$, but not necessary for common $Q$) in (11) and the feasible solutions are obtained, the quadratic-like stability of system (1) can be guaranteed. In this case, the Lyapunov function can be simply set

$$
V(k, \delta) = x^{t}(k)E(\delta)\Delta E(\delta)x(k).
$$

The application of parameter-dependent Lyapunov function to robust controller synthesis can be referred to (Feretal et al., 1996).

**Remark 5.** If there is no variation in the elements of $E(\delta)$ (i.e. $E_i = E$ for all $i$), condition (11) can be modified to a simpler version

$$
\begin{bmatrix}
A^{t}F + F^{t}A_i - E^{t}P_iE \\
GA_i - F \\
A^{t}G^{t} - F^{t} \\
P_i - U^{t}Q_iU - G - G^{t}
\end{bmatrix} < 0 \\
i = 1, 2, \cdots, N \tag{16}
$$

with $P_i > 0$ and $Q_i > 0$ for all $i$. The proof is similar to that of Theorem 1. Premultiplying (16) by $[I_n \ A(\delta)]$ and postmultiplying by $[I_n \ A(\delta)]^{t}$ yields

$$
A^t(\delta)P(\delta)A(\delta) - E^tP(\delta)E \\
- A^t(\delta)U^tQ(\delta)UA(\delta) < 0 \quad \forall \delta \in \Delta. \tag{17}
$$

When the case is further restricted to the standard state variable systems, (i.e $E = I$), condition (16) is then simplified to (without $U^{t}Q_iU$ term)

$$
\begin{bmatrix}
A^{t}F + F^{t}A_i - P_i - A^{t}G^{t} - F^{t} \\
GA_i - F \\
P_i - G - G^{t}
\end{bmatrix} < 0, \\
i = 1, 2, \cdots, N. \tag{18}
$$

with $P_i > 0$. The result coincides with a corollary in (Peaucelle et al., 2000). While setting $E = I$ and $F = 0$, the LMIs of (16) reduce further to

$$
\begin{bmatrix}
-P_i - A^{t}G^{t} \\
GA_i - P_i - G - G^{t}
\end{bmatrix} < 0, \quad i = 1, 2, \cdots, N. \tag{19}
$$

which is the same as main result of (Oliveira, 1999).

**Remark 6.** Due to the singularity of $E_1$, the pencil matrix of each vertex system, denoted by $|E_i - A_i|$, contains zeros at infinity. This excludes the application of (Henrion et al., 2001) to our problem. Actually, the present result provides an important complement to (Henrion et al., 2001). Q.E.D.

4. A NUMERICAL EXAMPLE

Consider an uncertain discrete-time descriptor system described by

$$
E_1 = \begin{bmatrix}
1.8 & 0 & 0 & 1.08 \\
-0.4 & 1.8 & 0 & -0.2 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix},
$$
According to Theorem 2, the whole families of descriptor system are regular, causal, and Schur stable.

5. CONCLUSION

In this paper, the LMI-based sufficient conditions for checking robust stability of uncertain discrete-time descriptor systems are developed under some constraints on the uncertain advanced state matrix. It has been demonstrated that some results existing in the literature can be viewed as special cases of the proposed conditions. For the general case (i.e. without any constraints on the advanced state matrix), the problem is quite involved and still remains open.

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7. REFERENCES


