Extremum seeking-like observer for nonlinear systems *

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Abstract: The observer design problem for nonlinear systems is crucial in most practical applications of control theory, since direct measurements of the entire state of the plant are usually not available. Herein we focus on the sampled-data observer design problem. For nonlinear systems we propose an observer defined within the framework of hybrid multi-rate systems based on a modified version of the static multivariable extremum-seeking algorithm.

Keywords: Nonlinear observer, extremum-seeking, hybrid systems

1. INTRODUCTION

Since the knowledge of the entire state of the plant is crucial in most practical cases to either assess the correct behavior of the plant or to control the system itself, the problem of determining the internal state through available measurements, i.e. the observer design problem, has been extensively studied. For linear time-invariant systems the observer design problem has been thoroughly investigated and solved, see e.g. Luenberger (1971), whereas for nonlinear systems several constructive or conceptually constructive methods have been proposed, although no general solution is available. Moreover, the realistic assumption of availability of only sampled measurements of the output of the plant represents an additional difficulty for the solution of the observer design problem. It has been pointed out in Moraal and Grizzle (1995) that the sampled-data observer design problem may be interpreted as the problem of solving a sequence of nonlinear equations defined by a discrete-time observability mapping. Therefore, to solve the latter problem, Newton’s method has been employed (Newton Observers) firstly in Moraal and Grizzle (1995) and then in Biyik and Arcak (2005), where the explicit definition of an exact discrete-time model is avoided by means of continuous-time filters. However, the explicit evaluation of the Jacobian of the observability mapping, or even its finite difference approximation, represents a serious drawback for the application of the Newton’s method. To circumvent this issue, several modifications of the basic formulation of Newton’s method have been proposed. For instance, in Moraal and Grizzle (1992) Broyden’s method is used to iteratively approximate the Jacobian by means of secant updates such that a bounded deterioration of the approximation is guaranteed at each step. However, therein it is pointed out that Broyden’s method does not guarantee convergence and the recalculation of the exact Jacobian is necessary, since the property of bounded deterioration is not effective when the method is used to solve a sequence of nonlinear equations. In Arcak and Nesic (2004) the approximation of the discrete-time model is studied. In particular, families of observers for the approximate model are defined, depending on specific approximation parameters, and the conditions on the choice of the values for the parameters that guarantees convergence of the observer are given.

Extremum seeking is a powerful mathematical tool to solve minimization problems for static or dynamics mappings, since the method does not require knowledge of the Jacobian of the mapping to minimize. In fact, descent directions are determined by means of a periodic dithering signal, which is added to the input of the mapping to probe the mapping itself. The extremum-seeking approach has gained increasing interest in recent years thanks to the idea of using averaging theory and singular perturbation analysis to study the stability property of the algorithm, see e.g. Ariyur and Krstić (2003) and Krstić and Wang (2000). In Krstić and Wang (2000) the method originally presented in Krstić et al. (1995) is extended to the regulation to unknown set points or reference trajectories, self-optimizing control.

The main contribution of this article is the definition of an observer design methodology for linear and nonlinear systems in the presence of sampled measurements. The proposed observer is defined within the framework of hybrid multi-rate systems and makes use of a modified version of the extremum-seeking algorithm. In particular, the evolution of the observer in continuous-time, namely in the inter-sampling intervals, is determined by a copy of the nonlinear system whereas, in correspondence of the sampling time instants, the internal state of the observer is updated according to the information extracted.
from the sampled output via a modified version of the extremum-seeking algorithm. Finally, it is worth to underline that, even if the observer state evolves continuously, the sampled-data nature of the problem is preserved, since only samples of the output of the plant are employed by the observer.

The rest of the article is organized as follows. In Section 2 preliminary definitions and the basic notation are given. In Section 3 the modified extremum-seeking algorithm is discussed for static quasiconvex, scalar or multivariable, functions. The hybrid multi-rate observer is defined in Section 4. In the last two sections a numerical example is presented and conclusions are drawn, respectively.

2. PRELIMINARIES

We define the closed ball of radius \( \rho \) centered at \( x_0 \) as

\[
B_\rho(x_0) = \{ x \in \mathbb{R}^n : |x - x_0| \leq \rho \},
\]

where \( |\cdot| \) denotes the standard Euclidean norm on \( \mathbb{R}^n \), and the distance of \( x \) from a set \( \mathcal{A} \) as

\[
dist(A, x) = \inf_{y \in A} \{|x - y|\}.
\]

Moreover, \( co(x_1, \ldots, x_n) \) denotes the convex-hull of the vectors \( \{x_1, \ldots, x_n\} \). Finally, \( C^1(\mathbb{R}^n, \mathbb{R}) \) denotes the set of continuous functions \( f(\cdot) : \mathbb{R}^n \rightarrow \mathbb{R} \) with continuous first-order derivative. We recall that a function \( f(\cdot) : S \subseteq \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0} \) is strictly quasiconvex if and only if for all \( (x, y) \in S \) and all \( \lambda \in [0,1] \) \( f(\lambda x + (1 - \lambda)y) < \max\{f(x), f(y)\} \), see Sydsæter et al. (2008). In particular, a strictly quasiconvex function \( f(\theta) \) has strictly convex lower contour sets, i.e., the lower level sets \( P_{c} \triangleq \{ \theta \in S : f(\theta) \leq c \} \) are strictly convex for all values of \( c \). Finally, note that the set of quasiconvex functions includes the set of convex functions, i.e., the quasiconvexity condition is less constraining than convexity.

3. EXTREMUM-SEEKING ALGORITHM

In this section we present the modified version of the extremum-seeking algorithm, exploited in the following for the nonlinear observer design problem.

**Assumption 1.** The \( C^1(\mathbb{R}^n, \mathbb{R}_{\geq 0}) \) function \( M(\cdot) : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0} \) is strictly quasiconvex and

\[
||\nabla M(\theta)|| \geq \delta > 0,
\]

for all \( \theta \in \mathbb{R}^n \setminus B_a(\theta^*) \) and some \( a > 0 \), with \( \theta^* \) such that \( M(\theta^*) < M(\theta) \) for all \( \theta \in \mathbb{R}^n \).

Note that the continuous differentiability of the function implies that \( M(\cdot) \) is locally Lipschitz continuous.

Several iterative techniques have been proposed in the literature to determine a sequence of values for the variable \( \theta \) that minimizes the function \( M(\theta) \), namely a sequence of values that approximates \( \theta^* \). The extremum-seeking algorithm is a well-known method to solve the minimization problem informally defined above. The algorithm may be used in the more general framework where also dynamics of the variable \( \theta \) have to be taken into account, see e.g. Ariyur and Krstic (2003) and Tan et al. (2006). We exploit here a modified static multivariable discrete-time version of the classical extremum-seeking technique which allows to find an approximation of \( \theta^* \). The amplitude of the probing signal directly influences the accuracy of the approximation, since in particular the former determines the ability of the algorithm to find descent directions.

An appealing feature of the extremum-seeking technique is that it does not require the knowledge of the gradient \( \nabla M(\theta) \), necessary to implement other methods such as Newton-Rapson, gradient, line search, just to new a few. Moreover, it is worth to underline that the extremum-seeking method guarantees global convergence to the optimal value under Assumption 1.

In the classical scheme a sinusoidal (dithering) probing signal is employed to investigate the function and determine the descending directions. However, since the evaluation of \( M(\cdot) \) is in general time consuming, we prefer to consider a sawtooth probing signal \( d(j) \), ranging between \([-1,1]\], with alternating positive and negative slopes generated by

\[
d_1^j = \text{mod}(d_j, \gamma) + 1,
\]

\[
d_2^j = d_2 \text{sign}(\gamma - 1 - d_1),
\]

\[
d = a d_1^j \gamma - 1,
\]

with \( d_1(0) = d_2(0) = 1, \gamma > 1, a > 0 \), where \( \text{mod}(\cdot) \) is the modulo function, and \( \text{sign}(s) \) is equal to 1 for all \( s \geq 0 \), \(-1 \) otherwise. This signal allows to investigate positive and negative increments which are not repeated, as it would be using \( \sin(\cdot) \) as probing function since \( \sin(\omega(\pi/2 + t)) = \sin(\omega(\pi/2 - t)) \) for \( t \in [0, \pi/2] \). This allows to save CPU time. Note that

\[
d(j) = -d(\gamma + j),
\]

which is crucial to guarantee convergence of the proposed algorithm.

3.1 The scalar case, i.e. \( n = 1 \).

To begin with, the proposed approach is introduced in the case \( n = 1, i.e. \theta \in \mathbb{R} \). Different sampling intervals, multi-rate system, are considered for the variables \( \theta(k) \) and \( d(j) \). This implementation choice greatly simplifies the proof of convergence of the algorithm, and prevents sinusoidal oscillations induced by the probing signal to affect directly \( \theta(k) \). Therefore the variable \( \theta \) is updated exactly \( 2\gamma \) times slower than the update of \( d \), namely

\[
\theta(k + 1) = \theta(k) - a \text{sign} \left( \sum_{j=1}^{2\gamma} d(j) M(\theta(k) + d(j)) \right),
\]

\( a > 0 \), with initial condition \( \theta(0) = \theta_0 \). The sum in (4) collects the information given by the probing signal \( d \) about the descending direction of the function \( M \) and, as in the classical extremum-seeking, it is given by the function \( M(\cdot) \), evaluated at \( \theta + d \), modulated by the signal \( d \). In fact, since we are interested in the minimization of \( M \), a negative sum in (4) implies \( \theta^* \leq \theta(k) \) and viceversa.

**Proposition 1.** Consider system (2) and system (4). Let Assumption 1 hold. Then, the set \( B_a(\theta^*) \) is globally asymptotically stable for any \( a > 0 \) and there exists a finite
Proof: To begin with note that by Assumption 1 it follows that \( |M(\theta^* + s) - M(\theta^* + hs)| \geq \delta(h - 1)s \) for all \(|s| > a\) and \(h > 1\). We consider two possible cases, namely if \(\theta(k) \notin B_a(\theta^*)\), then \(\theta^* + a < \theta(k)\) or \(\theta(k) < \theta^* - a\). In the former it can be easily shown that

\[
\sum_{j=1}^{2\gamma} d(j)M(\theta(k) + d(j)) \geq d(1)(M(\theta(k) + d(1)) - M(\theta(k) - d(1))) \geq \delta d(1)^2 > 0, \tag{5}
\]
yielding by the update (4) and the Assumption 1, that \(M(\theta(k + 1)) \leq M(\theta(k)) - \delta a\), which trivially holds even with \(\theta(k) < \theta^* - a\). This implies stability of the set \(B_a(\theta^*)\) and, since \(d \neq 0\), it also guarantees that \(\text{dist}(B_a(\theta^*), \theta(k)) = 0\) for all \(k \geq k_a\).

Remark 1. The sign(\(\cdot\)) function in (4) allows to conclude the forward invariance and finite-time attractivity of \(B_a(\theta^*)\), since \(\theta(k + 1) - \theta(k) < a\). However, if a small set \(B_a(\theta^*)\) is selected, i.e. small \(a > 0\), then the convergence of \(\theta\) to \(\theta^*\) might be slow and could be improved considering a gain function \(\sigma(\cdot)\), which may also evolve over time according to additional dynamics, i.e.

\[
\theta(k + 1) = \theta(k) - \sigma \left( \sum_{j=1}^{\gamma - 1} d(j)M(\theta(k) + d(j)) \right),
\]
exploiting, for instance, the Armijo and Wolfe conditions and the backtracking algorithm, see Nocedal and Wright (1999).

3.2 The \(n\)-dimensional case

In the multidimensional case the classical extremum-seeking employs \(n\) different probing signals, with different frequencies, exploring an \(n\)-dimensional hyper-cube to find descent directions, Ariyur and Krstic (2003). To reduce the number of points where \(M(\cdot)\) is evaluated and speed-up the update of \(\theta\) – which, however, may not imply fast convergence – we consider the \(n\) directions along each \(\theta_i\) independently. The approach described above represents a mean value approximation of \(\nabla M\). Since here we propose to find descent directions along each direction \(\theta_i\) independently, in a discretized framework, at the points in \(\mathbb{R}^n\) where the contour of the level sets of \(M(\cdot)\) is not differentiable there might not exist orthogonal directions, e.g \(e_1\) and \(e_2\) for \(n = 2\) (\(\{e_1, e_2\}\) is the standard orthonormal basis of \(\mathbb{R}^2\)), that locally belong to the cone of descent directions.

To avoid the previous issue the following assumption is needed.

Assumption 2. The function \(M(\cdot)\) is twice continuously differentiable and

\[
\left| \frac{\partial^2 M(\theta)}{\partial \theta^2} \right| \leq K.
\]

If Assumption 2 holds then the mean curvature of the level sets contour is bounded, see Carman (1976), and it is always possible to find sufficiently small increments \(a_i\), positive or negative, along at least one direction \(e_i\), \(i = 1, \ldots, n\), such that \(M(\theta + a_i e_i) - M(\theta) < 0\).

The following fact can be deduced from Assumptions 1 and 2.

Fact 2. There exists \(a > 0\) such that any increment vector \(v \in \{a|r_1 e_1, \ldots, r_n e_n|\}\), for some sequence \(\{r_1, r_2, \ldots, r_n\}\), \(r_i \in \{1, -1\}\), belongs to the cone of descent directions of \(M(\cdot)\), hence

\[
M(\theta + v) - M(\theta) \leq -\delta a. \tag{6}
\]

Note that any convex selection of the vectors \(a|r_1 e_1, \ldots, r_n e_n|\), with the correct \(r_i\)’s, satisfies inequality (6), since the level sets of \(M(\cdot)\) are convex. It remains to show that we are able to determine, via an extremum-seeking-like algorithm (EXSL), the sequence \(r_1, r_2, \ldots, r_n\) to minimize the function \(M\). To this aim, consider

\[
\Delta_{i,i}(k) = \sum_{j=1}^{\gamma - 1} d(j)M(\theta(k) + e_i d(j)), \tag{7a}
\]

\[
\theta(k + 1) = \theta(k) - a \sum_{i=1}^{n} e_i \frac{\Delta_{i,i}(k)}{\text{trac}(\Delta(k))}, \tag{7b}
\]
with \(\theta(0) = \theta_0\), where \(\Delta(\cdot) : \mathbb{N} \mapsto \mathbb{R}^{n \times n}\) is a diagonal matrix.

Proposition 3. Suppose that Assumptions 1 and 2 are satisfied and consider system (2) and system (7a)-(7b). Then the set \(B_a(\theta^*)\) is globally asymptotically stable for any \(a > 0\) and there exists a finite \(k_a\), depending on \(|\theta^* - \theta_0|\), such that \(\text{dist}(B_a(\theta^*), \theta(k)) = 0\) for all \(k \geq k_a\).

Proof: The proof easily follows from (7a) that selects the descent directions on each coordinate \(\theta_i\), namely it defines the sequence \(\{r_1, r_2, \ldots, r_n\}\). Then, given that (7b) yields a convex combination of the different descent directions along the \(e_i\)’s, the claims follow from Fact 2.

Remark 4. The speed of convergence of the algorithm in Proposition 3 could be improved selecting independently dynamic step sizes \(a_i(k)\), allowing the step size to increase up to certain value \(a_{\text{max}}\) dynamically, or decrease down to \(a_{\text{min}}\) in case of oscillation detection, namely if along the direction \(e_i\) a high curvature of the level sets contour is encountered. As an example, given \(a_{\text{max}} \geq a_i > 1\), and \(a_i(0) > 0\),

\[
a_i(k + 1) = \min\{a_{\text{max}}, \alpha(\Delta_{i,i}(k)\Delta_{i,i}(k - 1))a_i(k)\}, \tag{8a}
\]

\[
\alpha(s) = \begin{cases} \lambda_1 & \text{if } s > 0, \\ \lambda_{-1} & \text{otherwise.} \end{cases} \tag{8b}
\]

Note that in this definition we implicitly assumed \(a_{\text{min}} = 0\). With this selection only the existence of \(K\) in Assumption 2 has to be guaranteed, whereas the actual value of \(K\) may be unknown.

4. OBSERVER DESIGN

Consider a nonlinear system described by equations of the form

\[
\dot{x} = f(x, u),
\]

\[
y = h(x, u), \tag{9}
\]

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where $x(t) \in \mathbb{R}^n$ is the state variable, $u(t) \in \mathbb{R}^m$ denotes the input and $y(t) \in \mathbb{R}^p$ is the output of the system. We assume that the input $u$ acts on the system via a sample-and-hold device with sampling time $T$, such that $u$ is constant in the interval between two consecutive samplings. Herein we focus on the problem of designing an observer for the system (9) in the presence of sampled measurements exploiting the extremum-seeking-like algorithm presented in the previous section. To this end, it is useful to introduce the notation concerning the sampling intervals and the time instants. Let $t_0 = 0$, the sampling time between two consecutive measurements of the system (9) is denoted by $T$, i.e. the $i$-th measurements is defined as $y(iT) = h(x(iT), u(iT))$, or shortly $y_i = h(x_i, u_i)$. Since we propose an observer with hybrid dynamics, we briefly recall the general hybrid model introduced in Goebel et al. (2009). An hybrid system is described by ordinary differential equations and difference equations of the form

$$
\dot{\xi}(t,i) = F(\xi(t,i)), \quad \xi(t,i) \in C,
$$

$$
\chi(t,i + 1) = G(\chi(t,i + 1)), \quad \chi(t,i + 1) \in D,
$$

where $F(\cdot)$ and $G(\cdot)$ are usually called flow map and jump map, respectively. An hybrid arc $\chi(t,i)$ is a mapping $\chi: \text{dom}(\chi) \rightarrow \mathbb{R}^n$, where $\text{dom}(\chi)$ is the hybrid time domain. The interested reader is referred to Goebel et al. (2009) for further details. In the following we omit the time arguments when possible, i.e.

$$
\dot{\chi} = F(\chi), \quad \chi \in C, \quad \chi^+ = G(\chi) \in D,
$$

denoting $\chi(t,i + 1)$ as $\chi^+$ in the difference equation.

To estimate the value of $x(t)$ we propose a hybrid observer with state $\chi = [\xi^T, \tau]^T \in \mathbb{R}^n \times \mathbb{R}_{\geq 0}$, inputs $u \in \mathbb{R}^m$, and $w \in \mathbb{R}^n$, and

$$
\dot{\chi} = \begin{bmatrix} f(\xi, u) \\ 0 \end{bmatrix}, \quad \chi \in C, \quad \chi^+ = \begin{bmatrix} w \\ 0 \end{bmatrix}, \quad \chi \in D,
$$

(11a)

with flow set $C = \{ \tau \in \mathbb{R}_{\geq 0} : 0 \leq \tau \leq T \}$ and jump set $D = \{ \tau \in \mathbb{R}_{\geq 0} : \tau \geq T \}$. The time regularization via $\tau$ allows to define the times $t_i$, where new samples of $y$ are available, $y(t_i) = y(iT) = y_i$. We do not introduce here the complete hybrid model with the EXSL to avoid burden of notation, but the signal $w$ is actually the EXSL output, which is used to reset the continuous time estimate $\chi(t,i)$.

Let $Y^q_i \in \mathbb{R}^{np}$ and $U^q_i \in \mathbb{R}^{qm}$ be defined as

$$
Y^q_i = \begin{bmatrix} y_{i-N+1} \\ \vdots \\ y_i \\ \vdots \\ y_{i+N} \end{bmatrix}, \quad U^q_i = \begin{bmatrix} u_{i-N+1} \\ \vdots \\ u_i \\ \vdots \\ u_{i+N} \end{bmatrix}.
$$

Moreover, define $H(x_{i-N+1}, U^q_i) : \mathbb{R}^{n \times m} \mapsto \mathbb{R}^{np}$ as

$$
H(x_{i-N+1}, U^q_i) \triangleq \begin{bmatrix} h(F(x_{i-N+1}, U^q_i), u_{i-N+2}) \\ \vdots \\ h(F_N(x_{i-N+1}, U^q_i), U^q_i) \end{bmatrix},
$$

(12)

with $Y^q_i = H(x_{i-N+1}, U^q_i)$, where $F^q_i$ is such that $x_{i-N+1+q} = F_T^q(x_{i-N+1}, U^q_i)$. Note that in general the analytical expression of $F$ is unknown.

The EXSL in Proposition 3 is now used to estimate the solution $x_{i-N+1}$ of the nonlinear equations

$$
Y^q_i - H(x_{i-N+1}, U^q_i) = 0,
$$

substituting the Newton algorithm considered in Moraal and Grizzle (1995) and Biyik and Arcak (2005). Note that the vector of samples $Y^q_i$ describes a moving window of $N$ consecutive measurements $y_i$.

Within each inter-sampling interval $[t_i, t_{i+1}]$ the EXSL performs the following operations.

1. Considers the latest $N$ samples $Y^q_i$ and $U^q_i$ with $\theta_0 = \xi(t_{i-N}, i - N)$.
2. The dynamics of the system (7) are evaluated iteratively $c$ times yielding $\theta$ that is an estimate of $x_{i-N}$.
3. The estimate $\xi$ is reset as

$$
\xi(t_{i+1}, i + 1) = w(t_{i+1}) \triangleq F_{NT}(\theta_0, U^q_i).
$$

(13)

Note that each update of $\theta$ at step (2) requires the evaluation of $H(\theta + d)$ exactly $2c$ times, corresponding to the $2c$ different values of the probing signal $d$ given by (2).

Moreover, each element $H_i = h(F_{(i-N)c}^q(\theta + d, U^q_i, u_i))$ of $H$ requires a numerical integration to be evaluated (to save CPU time the $j$-th integration uses the values yielded by the $j-1$-th integration).

The previous considerations imply that we are assuming the CPU of the controller to be able to perform numerical integrations of $(2c + 1)NT$ time of the system (9), supposing that the computation time of the updates of $\theta$ and $d$ is negligible. This means that, depending on the numerical integration algorithm, we have to carefully choose $c$ and $\gamma$ to evaluate $w(t_{i+1})$ within a “real” time interval $T$.

**Theorem 1.** Consider the system (9) where $w$ is the output (13) of the EXSL algorithm (7). Define $1. M(\xi, x) \triangleq \Gamma(Y^q_i - H(\xi, U^q_i))$ and let $\Gamma(\cdot)$ be such that Assumptions 1 and 2 are satisfied. Moreover, assume that

i) there exist compact sets $U \subset \mathbb{R}^m$ and $X \subset \mathbb{R}^n$ such that $u_i \in U$ and $x(t_i) \in X$ for any $k \in \mathbb{N}$ and $t_i \geq t_0$;

ii) the system (9) is $N$-observable uniformly with respect to $X$ and $U$.

Then there exist $a > 0$ and $c > 0$ such that all the trajectories of the interconnected system (9)-(11) are such that

$$
\lim_{t \to \infty} \|\xi(t, j) - x(t)\| \leq ace^LT_N,
$$

were $L$ satisfies the inequality

$$
f(\xi, u) - f(x, u) \leq L\|\xi - x\|,
$$

for any $x \in X$ and $u \in U$.

**Proof:** To begin with let $i \geq N - 1$ (note that $i$ starts from zero), such that a first set of data $Y^q_i$ is available and EXSL can perform the first $c$ iterations yielding, for $i = N$, $\xi(t_N, N) = w(t_N)$. To simplify the notation of the proof, let $\theta(i)$ be the estimate of $x_{i-N}$, such that $\xi(t_i, i) = w(t_i) = F_{NT}(\theta(i), i, U^q_i)$. Assume that $\gamma(i) \notin B_a(x(t_i-N))$ and define $V(\xi(t,i), x(t)) = M(\xi(t,i), x(t))$.

The properties of $M(\cdot)$ in Assumptions 1 and 2 yield $V(\xi, x) \geq 0$, $V(\xi, x) = 0$ if and only if $\xi = x$. Since $x$...
and $\xi$ belong to compact sets for all $t \geq 0$, then there exists a constant $m$ such that the continuous-time evolution, i.e. during flows,

$$V(\xi(t,i), x(t)) \leq m,$$

which implies that the maximum growth of $V(\cdot)$ during the flow time is $mT$. The EXSL, between each sample, performs $c$ iterations yielding

$$V(\xi(t_{i+1}, i+1), x(t_{i+1})) - V(\xi(t_{i+1}, i), x(t_{i+1})) \leq -c\sigma a,$$

given that $\theta_i(i) \notin B_\delta(x(t_{i-N}))$. Then, there exist $c > 0$ and $a > 0$ such that

$$ca \geq \frac{mT + h}{\delta},$$

for some $h > 0$, yielding

$$V(\xi(t_{i+1}, i+1), x(t_{i+1})) - V(\xi(t_{i}, i), x(t_{i})) \leq -h.$$}

This implies that $\text{dist}(B_\delta(x(t_i)), \theta_i(i))$ goes to zero as $i$ goes to infinity, and that also $\text{dist}(B_{aeLTN}(x(t)), \xi(t, i))$ tends to zero, for all $t \in [t_i, t_{i+1}]$, proving the claim. In fact, the bound on the error $||\theta_i(i) - x(t_{i-N})||$ a increases to $aeLTN$ when $\theta_i(i)$ is propagated forward of $(N + 1)T$ seconds since, for any $t \in [t_i + \tau, \tau \in (0, T]$,

$$||\xi(t, i) - x(t)|| \leq \epsilon_{LTN}||\theta_i(i) - x(t_{i-N})|| \leq aeLTN.$$}

□

Remark 2. The selection of $\Gamma(\cdot)$ modifies the convergence property of the algorithm as well as the satisfaction of the Assumptions 1 and 2. Note that a necessary condition to satisfy Assumption 1 is the invertibility of the Jacobian of $H$ which is guaranteed by $ii)$ of Theorem 1.

Remark 3. Due to practical result of Theorem 1, it would be possible to combine the proposed approach with a local observer, possibly a classical high-gain observer, which is able to circumvent the issue of the explicit knowledge of an exact discrete-time model. A hybrid multi-rate observer, defined as in (11), is designed herein and its performances are discussed in details for different values of the parameters $T$ and $c$, whereas the length of the moving window is fixed, namely $N = 3$. We let the function $\Gamma(\cdot)$ be defined as the square of the Euclidean norm, hence the function to minimize at the $t$-th sampling instant is given by

$$M(x) = ||Y_i^N - H(x_{i-N+1}, U_i^N)||^2,$$

where $x(t) = (x_1(t), x_2(t), x_3(t)) \in \mathbb{R}^3$ denotes the state, $y(t) \in \mathbb{R}$ is the output of the system and note that the trajectories of the system (15) exhibit a periodic behavior. The dynamical system (15) is considered also in Biyik and Arcak (2005), where a Newton observer is proposed together with the implementation of continuous-time filters to circumvent the issue of the explicit knowledge of an exact discrete-time model. A hybrid multi-rate observer, defined as in (11), is designed herein and its performances are discussed in details for different values of the parameters $T$ and $c$, whereas the length of the moving window is fixed, namely $N = 3$. We let the function $\Gamma(\cdot)$ be defined as the square of the Euclidean norm, hence the function to minimize at the $t$-th sampling instant is given by

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where $x(t) = (x_1(t), x_2(t), x_3(t)) \in \mathbb{R}^3$ denotes the state, $y(t) \in \mathbb{R}$ is the output of the system and note that the trajectories of the system (15) exhibit a periodic behavior. The dynamical system (15) is considered also in Biyik and Arcak (2005), where a Newton observer is proposed together with the implementation of continuous-time filters to circumvent the issue of the explicit knowledge of an exact discrete-time model. A hybrid multi-rate observer, defined as in (11), is designed herein and its performances are discussed in details for different values of the parameters $T$ and $c$, whereas the length of the moving window is fixed, namely $N = 3$. We let the function $\Gamma(\cdot)$ be defined as the square of the Euclidean norm, hence the function to minimize at the $t$-th sampling instant is given by

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Fig. 2. Time histories of $x_1(t)$ (dashed line) together with the estimate $\xi_1(t)$ (solid line) and of $x_3(t)$ together with $\xi_3(t)$ (top and bottom graph, respectively) with $T = 0.2s$ and $c = 20$.

Fig. 3. Time histories of $x_1(t)$ (top), and $x_3(t)$ (bottom), with $T = 0.2s$ and $c = 200$.

time instants. The continuous-time evolution of the observer is described by a copy of the nonlinear system whereas, in correspondence of the sampling time instants, the internal state of the observer is updated according to the information extracted from the sampled output by the extremum-seeking algorithm. To conclude the article, the performances of the proposed approach are tested in a numerical example.

REFERENCES


