Pattern Formation in Networks of Nonlinear Systems with Delay Couplings *

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Abstract: This paper considers pattern formation in networks of identical nonlinear systems with time-delays. First of all, applying the harmonic balance methods, we derive the corresponding harmonic balance equations for networks of identical nonlinear systems with delay couplings. Then, solving the equations by reducing to the stability problem of linear retarded systems, we can estimate the oscillation profile such as the frequency, amplitudes and phases of coupled systems. Based on this analysis method, we also develop a design method of networks of nonlinear systems that can achieve a prescribed oscillation profile. The effectiveness of the proposed methods is shown by numerical examples.

Keywords: Delay systems, Nonlinear systems, Pattern generation, Complex systems, Harmonic balance

1. INTRODUCTION

In recent years, network systems have attracted attention in applied physics, mathematical biology, social sciences, control theory and interdisciplinary fields. In particular, synchronization and pattern formation of coupled systems has been the subject of intense study. From the standpoint of control engineering, synchronization and pattern formation of coupled systems are important notions to realize decentralized control techniques and biomimetic control approach in increasingly complex applications. For instance, central pattern generators (CPGs), which produce the rhythmic movements such as locomotion, breathing and scratching, are considered to consist of a group of neurons. The study of CPGs has been carried out not only for understanding of biological behavior but also for producing rhythmic locomotion of multi-legged robots.

The behaviors of coupled identical systems and networks have been studied by a large numbers of researchers in broad areas (Golubitsky and Stewart (2004); Newman et al. (2006) and so on). Golubitsky al. showed that the existence of periodic solutions in symmetric coupled systems and the pattern classification of gait is based on the symmetric network structure of the CPG. Pogromsky et al. (1999, 2002) considered the synchronization problem in symmetric networks of chaotic systems and derived sufficient conditions for partial synchronization in networks and existence for periodic solutions. However these researches focus on only the symmetric structure of networks. Iwasaki (2008) proposed a systematic approach for the analysis and synthesis of CPGs by applying the multivariable harmonic balance method. The proposed approach has no restriction on the network structure.

On the other hand, the research interest on synchronization of coupled systems are shifting to the synchronization problem in delayed networks during recent years (Oguchi et al. (2008, 2009); Neefs et al. (2010)). Since time-delays caused by signal transmission affect the behavior of coupled system in practical situations, it is therefore important to study the effect of time-delay in network systems.

The objective of this paper is to extend the systematic approach based on the harmonic balance method (Iwasaki (2008)) for delayed networks. First, we derive the corresponding harmonic balance equations for networks of identical nonlinear systems with delay couplings. Then, solving the equations by reducing to the stability problem of linear retarded systems, we can estimate the oscillation profile such as the frequency, amplitudes and phases of coupled systems. Based on this analysis method, we also develop a design method of networks of nonlinear systems that can achieve a prescribed oscillation profile. The effectiveness of the proposed methods is shown by numerical examples.

2. NONLINEAR NETWORK SYSTEMS

2.1 Model Description

In this paper, we consider $n$ identical nonlinear oscillators interconnected as follows (Iwasaki (2008)).

$$v_i = \psi_i(q_i), q_i = f(s)u_i, u_i = \sum_{r=1}^{n} \mu_{ir}(s)v_j$$

where $u_i$ and $v_i$ are the input and the output of oscillator $i$, $\psi_i$ is a static nonlinear function and is defined by $\psi_i(x) = \tanh(x)$ and $\mu_{ir}$ means the transfer function denoting a connection from oscillator $r$ to oscillator $i$, which is given by $\mu_{ir}(s) = k_{ir}e^{-\tau_{m}s}$. Here each nonlinear oscillator can be considered as a model of the electrical activity of a neuron, $k_{ir} \in \mathbb{R}$ means a coupling gain between $i$ and $r$, and $\tau_{m}$ is
a commensurate delay of $\tau$, i.e. $\tau_m = m\tau$ for $m = 1, \ldots, h$. $f(s)$ is a linear time-invariant part of oscillator given by

$$f(s) = \frac{\omega_0}{s + \omega_0}.$$  \hspace{1cm} (2)

Then the network of $N$ coupled oscillators is summarized as the following equations.

$$v = \Psi(q) = \begin{bmatrix} \psi_1 \\ \vdots \\ \psi_n \end{bmatrix}, \quad F(s) = \begin{bmatrix} (f(s) & \cdots & f(s)) \end{bmatrix} = f(s)I_n$$

$$q = F(s)M(s)\Psi(q)$$

$$= \begin{bmatrix} f(s)\mu_{11}(s)\psi_1(q_1) + \cdots + f(s)\mu_{1n}(s)\psi_n(q_n) \\ \vdots \\ f(s)\mu_{n1}(s)\psi_1(q_1) + \cdots + f(s)\mu_{nn}(s)\psi_n(q_n) \end{bmatrix}$$

where $M(s)$ is the transfer matrix whose $(i,j)$ entry is $\mu_{ij}(s)$. For the above system, we deal with the following two problems:

(I) **Analysis problem**: Given a network system with delays, determine whether the coupled systems have oscillatory trajectories, and if so, estimate the oscillation profile (frequency, amplitudes, phase) without actually simulating the differential equations.

(II) **Synthesis problem**: Given a network structure with delays and a desired oscillation profile, determine the coupling strength $k_{ij}$ in the coupling transfer function $\mu_{ij}(s)$ so that the resulting network system achieves the given profile.

### 2.2 Analysis of periodic solutions

We assume that the system (1) has a periodic solution. Applying Fourier series expansion, $q$ is described by

$$q(t) = \sum_{k=0}^{\infty} a_k \sin(\omega_k t) + b_k \cos(\omega_k t)$$  \hspace{1cm} (3)

where $a_k, b_k \in \mathbb{R}^n$ and $\omega \in \mathbb{R}$. Supposing that $\omega$ is sufficiently higher than the band-pass frequency of $f(s)$, $q_i(t)$ can be approximated by

$$q_i(t) \approx \alpha_i \sin(\omega t + \phi_i), i = 1, \ldots, n$$

where $\alpha_i$ and $\phi_i$ denote the amplitude and phase, respectively. Furthermore, using a describing function $\kappa_i$, $\psi_i$ can be approximated by

$$\psi_i(q_i) \approx \kappa_i(\alpha_i)q_i, \quad \psi_i(x) := \frac{2}{\pi} \int_{0}^{\pi} \psi_i(x \sin \theta) \sin \theta d\theta.$$  

The describing function $\kappa_i(x)$ is a monotonically decreasing function satisfying $\kappa_i(0) = 1$ and $\kappa_i(\infty) = 0$ since $\psi_i(x) = \tanh(x)$. Using these approximations, we obtain the corresponding multivariable harmonic balance equation for the nonlinear oscillator delay network as follows.

$$(M(j\omega)K(\alpha) - 1/f(j\omega)I_n)q = 0$$  \hspace{1cm} (4)

where $j$ denotes the imaginary unit, $I_n$ the $n \times n$ identity matrix, $q_i := \alpha_i e^{j\phi_i}$ and $K(\alpha) := \text{diag}(\kappa_i(\alpha_i))$. The triplet $(\omega, \alpha, \phi)$ satisfying the equation (4) is the oscillation profile to be solved. In this case, however, not only are the solution unique in general, there may exist infinite number of solutions due to the existence of time-delays. Among the solutions, we have to choose a triplet $(\omega, \alpha, \pi)$ so that the estimated oscillation is stable. Here, replacing $K(\alpha)$ with a constant matrix $K(\alpha) := K[q]$, the stability of oscillation can be expected by checking if the following characteristic quasi-polynomial of linear system has a pair of solutions $s = \pm j\omega$ on the imaginary axis and the rest in the open right half plane.

$$\det(M(s)K(\alpha) - 1/f(s)I_n) = 0.$$  \hspace{1cm} (5)

If the couplings have no delay, the stability analysis can be reduced to the eigenvalue problem of the constant matrix $MK(\alpha)$ since the matrix $M$ is a constant matrix. However, if the couplings have delays, then the entries of $M(s)$ are not constant but functions of $e^{-s\tau_m}$. Therefore the stability analysis cannot be done in the same way.

Now we decompose the transfer function matrix $M(s)$ into a constant matrix part and the rest, i.e.

$$M(s) = M_0 + \sum_{m=1}^{h} M_m e^{-s\tau_m}$$  \hspace{1cm} (6)

where $M_0$ and $M_m$ are constant coupling matrices with delay $\tau_1, \ldots, \tau_h$. Substituting (2) and (6) into the quasi-polynomial (5) and setting $M_c = -I_n + M_0 K(\alpha)$, we obtain

$$\det \left( \sum_{m=1}^{h} M_m e^{-s\tau_m} K(\alpha) + M_c - \lambda I_n \right) = 0$$  \hspace{1cm} (7)

where $\lambda := s/\omega_0$. The root $\lambda$ satisfying the equation (7) is identical to the pole of the linear retarded system

$$\dot{x}(t) = M_c x(t) + \sum_{m=1}^{h} M_m K(\alpha) x(t - \tau_m).$$  \hspace{1cm} (8)

Therefore, seeking the rightmost characteristic roots of the system (8), we can consider whether the characteristic quasi-polynomial (5) has at least one pair of solutions on the imaginary axis and the rest in the open right half plane. If there exists a solution $\lambda$ on the imaginary axis, it is a solution of the modified harmonic balance equation given by

$$\det \left( \sum_{m=1}^{h} M_m e^{-s\tau_m} K(\alpha) + M_c - \lambda I_n \right) q = 0.$$  \hspace{1cm} (9)

Assuming that there exists a $q \in \mathbb{C}^n$ satisfying (9), we can expect that the trajectory of each oscillator $q_i(t)$ has a phase $\phi_i$ and frequency $\omega$, i.e. $q_i(t) \approx \alpha_i \sin(\omega t + \phi_i)$, where a triplet $(\omega, \alpha, \phi)$ is determined by

$$j\omega/\omega_0 = \lambda, \quad \alpha_i = \alpha_i e^{j\phi_i}, \omega > 0.$$  \hspace{1cm} (10)

The analysis on the oscillation profile can be accomplished by finding $q \in \mathbb{C}^n$ satisfying the equation (5), but $K(\alpha)$ depends on $q$. So, in this paper, extending the algorithm introduced by Iwasaki (2008) for delay systems, we propose a heuristic calculation method.

**Algorithm:**

**Step 1** Set the initial values $k = 0$ and $v_k = [1, \ldots, 1]^T$.

**Step 2** Let $K_k := K[q_k]$ and find the rightmost root $\lambda_k$ which has the maximum imaginary part and satisfies the characteristic quasi polynomial

$$\det \left( \sum_{m=1}^{h} M_m e^{-\lambda_k \omega_0 m \tau} K_k + M_c - \lambda_k I_n \right) = 0.$$
Step 3 Find the corresponding eigenvector \( x_k \) to \( \lambda_k \) obtained in Step 2:
\[
\sum_{m=1}^{h} M_{mn} e^{-\lambda_k \omega m \tau} K_k + M_{c} - \lambda_k I_n \right) x_k = 0
\]
\[\|x_k\| = 1\]

Step 4 Update the eigenvector
\[
v_{k+1} = e^{\tau} y_k, \sigma_k := \Re(\lambda_k), y_k := x_k^* v_k
\]

Step 5 If \[\|v_{k+1} - v_k\| \leq \epsilon\], then the algorithm terminates, and \( q := v_{k+1} \) is the solution of the equation (9). So, we obtain a triplet \((\omega, \alpha, \phi)\) from (10). Otherwise go to Step 2 and iterate.

Remark 1. For computing the rightmost roots of the characteristic quasi polynomial, there are several numerical techniques and some useful softwares like DDE-BIFTOOL are also developed.

2.3 Numerical Examples

Example 1 In this analytical method, we do not assume the symmetric structure of the network. To show the availability of this method for complex network systems, we consider a network of \( n = 15 \) oscillators and the network structure is assumed to be given in Fig. 1. Each vertex denotes a oscillator and the number \( i \) inside of each vertex indicates the \( i \)-th oscillator. The edge \((i, j)\) represents the existence of coupling between the \( i \)-th oscillator and the \( j \)-th oscillator. Now we assume that all coupling has identical delay \( \tau \) and the matrix \( M(s) \) in (6) is given by
\[
M(s) = M_1 e^{-\tau s}
\]
where
\[
M_1 = \begin{bmatrix}
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
-1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & -1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & -1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & -1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & -1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & -1 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 1 & 1 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 1 & 1 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 1 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 \\
-1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1
\end{bmatrix}
\]

For this network system, we estimate the profiles of oscillators for the following two cases:
(i) \( \tau = 0.2 \), (ii) \( \tau = 0.3 \). Figures 2 and 3 show the comparison between the estimated amplitudes and phases for \( q_i \) and those obtained by numerical simulations for \( \tau = 0.2 \) and \( \tau = 0.3 \), respectively. Here the phase \( \phi_i \) means the phase difference between the 1st oscillator and the \( i \)-th oscillator. In addition, the estimated frequency \( \omega \) for each delay is summarized in Table 1.

Table 1. Comparison between the estimation and simulation results of the frequency of coupled oscillator
<table>
<thead>
<tr>
<th>( \tau = 0.2 )</th>
<th>( \tau = 0.3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Estimation</td>
<td>2.26</td>
</tr>
<tr>
<td>Simulation</td>
<td>2.23</td>
</tr>
</tbody>
</table>

Example 2 Consider \( n = 5 \) identical nonlinear systems with delay couplings. The network structure is shown in Fig. 6. Each vertex denotes an oscillator, and the directed edge \((i, j)\) means that there exists a coupling from the \( i \)-th oscillator to the \( j \)-th oscillator. In particular, the red edges mean couplings without delay and others are couplings with delay \( \tau = 0.3 \). Now the transfer function matrix of the coupling \( M(s) \) is assumed to be given by
\[
M(s) = M_0 + M_1 e^{-\tau s}
\]
where
Sim
Est
Phase φi [deg]
Oscillator index i
Fig. 4. Estimated and simulated phase of each oscillator (τ = 0.3)

Amplitude αi
Oscillator index i
Fig. 5. Estimated and simulated amplitude of each oscillator (τ = 0.3)

Fig. 6. Network structure in Example 2

\[
M_0 = \begin{bmatrix}
0 & 0 & -1.5991 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
-0.6566 & 0 & 0 & 0 & 0 \\
0 & 0 & 1.6206 & 0 & 0
\end{bmatrix}
\]

and

\[
M_1 = \begin{bmatrix}
0 & 1.2728 & 0 & 0 & 0 \\
-1.2875 & 0 & -0.5615 & 0 & 0 \\
0 & -1.7732 & 0 & 0.0875 & 0 \\
0 & 0 & -1.2973 & 0 & -1.1642 \\
0 & 0 & 0 & 0.7016 & 0
\end{bmatrix}
\]

The nonzero entries of these matrices are given by random numbers. Figure 7 shows the steady state behaviors of coupled oscillators \(q(t)\) and the oscillation of each system is stable. Then, applying the foregoing algorithm, we estimate the profile of each oscillator. The comparison between the estimated values and the simulation results is summarized in Table 2. From Table 2, we can conclude that the estimation of a triplet \((ω, α, φ)\) can be almost perfectly accomplished.

### 3. SYNTHESIS OF NETWORKS WITH DELAYS

#### 3.1 Design method

In the previous section, we considered how to estimate the profile of the coupled oscillators under given network structure and couplings. In this section, we consider how to design the coupling gain \(k_{ij}\) to achieve given specifications of oscillation profile.

From the discussion in the foregoing section, it is necessary that the coupling gain matrices \(M_0, \ldots, M_h\) satisfy the equation (5) for a given triplet \((ω, α, φ)\). Furthermore, the roots of eq. (7) or equivalently the poles of the linear retarded system (8) have to be on the imaginary axis and in the left half-plane of the complex plane only.

Based on the above idea, we obtain the following results on the design of coupling gains.

**Theorem 2.** Let \(ω ∈ \mathbb{R}, α ∈ \mathbb{R}^n, φ ∈ \mathbb{R}^n\) and \(mτ ∈ \mathbb{R}\) for \(m = 1, \ldots, h\) be given, where \(ω > 0, α_i > 0\) for \(i = 1, \ldots, n\) and \(0 ≤ τ < 2π/ω\). Furthermore, assume that \(φ\) is chosen such that there exists at least one \(φ_i\) satisfying \(φ_i - φ_j ≠ kπ\) for any \(j ≠ i\) and \(k ∈ \mathbb{Z}\). Then the delay-coupled oscillator is expected to have oscillatory behavior if there exist \(M_i ∈ \mathbb{R}^{n×n}\) for \(i = 0, \ldots, h\) and a positive symmetric matrix \(P\) satisfying

\[
[ℜ(Aq) \ Ω(Aq)] = RΩ
\quad (11)
\]

\[
B^T P + PB < 0
\quad (12)
\]

where

### Table 2. Estimated and simulated oscillation profiles in Example 1

<table>
<thead>
<tr>
<th></th>
<th>(q_1)</th>
<th>(q_2)</th>
<th>(q_3)</th>
<th>(q_4)</th>
<th>(q_5)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Estimation</td>
<td>0.73</td>
<td>0.40</td>
<td>0.46</td>
<td>0.47</td>
<td>0.56</td>
</tr>
<tr>
<td>Phase</td>
<td>0.0</td>
<td>94.0</td>
<td>-151.0</td>
<td>-58.1</td>
<td>180.6</td>
</tr>
<tr>
<td>Simulation</td>
<td>0.70</td>
<td>0.39</td>
<td>0.46</td>
<td>0.47</td>
<td>0.56</td>
</tr>
<tr>
<td>Phase</td>
<td>0.0</td>
<td>95.1</td>
<td>-151.1</td>
<td>-57.1</td>
<td>180.7</td>
</tr>
</tbody>
</table>
\[ A := \sum_{m=1}^{h} M_m e^{-j\omega \tau_m} K(\alpha) + M_0 K(\alpha) - I \]
\[ B := \sum_{m=1}^{h} M_m K(\alpha) + M_0 K(\alpha) - I_n \]
\[ R := [c \ s] \in \mathbb{R}^{n \times 2}, \ c_i := \alpha_i \cos \phi_i, \ s_i := \alpha_i \sin \phi_i \]
\[ \Omega := \begin{bmatrix} -\omega/\omega_0 & 0 \\ -\omega/\omega_0 & 0 \end{bmatrix} \]

**Proof.** For given \( \omega, \alpha_i, \phi_i \) and \( \tau_m \), the coupling gain matrices \( M_i \) for \( i = 0, \ldots, h \) must satisfy the following harmonic balance equation:
\[ \left( \sum_{m=1}^{h} M_m e^{-j\omega \tau_m} K(\alpha) + M_c \right) q = \lambda q \tag{13} \]
where \( \lambda = j\omega/\omega_0 \) and \( M_c = -I_n + M_0 K(\alpha) \). Now if we suppose that the characteristic quasi-polynomial has the characteristic roots at \( \pm j\omega \), the following equation equivalently holds:
\[ \left( \sum_{m=1}^{h} M_m e^{-j\omega \tau_m} K(\alpha) + M_c \right) q = j\omega q \tag{14} \]
Paying attention to that \( q_i = \alpha_i e^{j\phi_i} = \alpha_i \cos \phi_i + j\alpha_i \sin \phi_i \) and rewriting the above equation, we obtain eq. (11).

On the other hand, if the system (8) with \( \tau = 0 \), i.e. \( \dot{x}(t) = (M_c + \sum_{m=1}^{h} A_m) x(t) \), is asymptotically stable, the corresponding matrix pencil \( \Lambda \) is regular. Then Theorem 3 in Appendix A shows a necessary and sufficient condition for
\[ \lambda I_n - M_c - \sum_{m=1}^{h} M_m e^{-\lambda_0 \omega_0 m^2} K(\alpha) = 0 \]
to have at least one nonzero root on the imaginary axis and the corresponding delay values. From this result, if we choose the minimum value in the delay values, the characteristic quasi polynomial has at least one pair of roots \( \pm j\omega \) on the imaginary axis and the rest on the left half plane. Therefore it is necessary that \( \tau < \frac{\pi}{\omega_0} \) and there exists a positive definite matrix \( P \) satisfying \( B^T P + PB < 0 \). \( \square \)

### 3.2 Examples

Consider network systems with the network structure given in Figure 6 again.

**Example 3** Firstly, we consider a case in which all couplings have the same length of time-delay \( \tau_1 \). Then we design the coupling gain matrix \( M_1 \) such that the coupled oscillators meet the specification given in Table 3.

<table>
<thead>
<tr>
<th>( \omega )</th>
<th>( \alpha_1 )</th>
<th>( \phi_1 )</th>
<th>( \phi_2 )</th>
<th>( \phi_3 )</th>
<th>( \phi_4 )</th>
<th>( \phi_5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Spec.</td>
<td>3.14</td>
<td>1.0</td>
<td>0</td>
<td>72.9</td>
<td>144.0</td>
<td>216.0</td>
</tr>
<tr>
<td>Case 1</td>
<td>3.43</td>
<td>1.00</td>
<td>0</td>
<td>72.2</td>
<td>143.2</td>
<td>216.6</td>
</tr>
<tr>
<td>Case 2</td>
<td>3.43</td>
<td>1.01</td>
<td>0</td>
<td>69.8</td>
<td>142.1</td>
<td>215.5</td>
</tr>
</tbody>
</table>

Table 3. Specifications and simulation results (Example 3)

Fig. 8. Behaviors of coupled oscillators (\( \tau_1 = 0.2 \))

(12) with respect to \( M_1 \) subject to the specific structure and the equation (11), we can obtain the following matrix \( M_1 \) for \( \tau_1 = 0.2 \) and 0.3, respectively.

1) Case \( \tau_1 = 0.2 \):

\[
M_1 = \begin{bmatrix}
0 & 2.4473 & 0.7195 & 0 & 0 \\
-1.1642 & 0 & 1.7278 & 0 & 0 \\
0 & -1.1642 & 0 & 1.7278 & 0 \\
0.0337 & 0 & -1.1095 & 0 & 1.7615 \\
0 & 0 & -1.0678 & -2.232 & 0
\end{bmatrix}
\]

2) Case \( \tau_1 = 0.3 \):

\[
M_1 = \begin{bmatrix}
0 & 2.09 & 1.2631 & 0 & 0 \\
-2.0437 & 0 & 0.8269 & 0 & 0 \\
0 & -2.0437 & 0 & 0.8269 & 0 \\
0.537 & 0 & -1.1748 & 0 & 1.3629 \\
0 & 0 & -0.5111 & -2.5548 & 0
\end{bmatrix}
\]

The comparison between the given specification and the simulation results with \( M_1 \) mentioned above is summarized in Table 3. Figure 8 shows the behavior of each oscillator in case of \( \tau_1 = 0.2 \) From these results, we see that the design specifications are almost fulfilled.

**Example 4** Consider a case in which the network has both delay free couplings and time-delay couplings with delay \( \tau_1 = 0.2 \). As in Example 2, the red edges indicate delay free couplings and others time-delay couplings. Based on the specifications given in Table 4, the coupling gain matrices \( M_0 \) and \( M_1 \) are obtained by applying Theorem 2 as follows.

\[
M_0 = \begin{bmatrix}
0 & 0 & 1.2062 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
-0.5250 & 0 & 0 & 0 & 0 \\
0 & 0 & -1.2412 & 0 & 0
\end{bmatrix}
\]

\[
M_1 = \begin{bmatrix}
0 & 1.1601 & 0 & 0 & 0 \\
-1.4632 & 0 & 1.8025 & 0 & 0 \\
0 & -1.1858 & 0 & 1.785 & 0 \\
0 & 0 & -1.2572 & 0 & 1.7656 \\
0 & 0 & 0 & -2.6024 & 0
\end{bmatrix}
\]

The comparison between the given specification and the design result is summarized in Table 4 and the simulation result is shown in Fig. 9. Although the phases are short by about 5 degrees in whole, the amplitudes almost coincide with the specification.
Table 4. Specifications and simulation results (Example 4)

<table>
<thead>
<tr>
<th>Spec</th>
<th>ω</th>
<th>α1</th>
<th>α2</th>
<th>α3</th>
<th>α4</th>
<th>α5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sim</td>
<td>3.12</td>
<td>0.60</td>
<td>0.80</td>
<td>1.00</td>
<td>1.20</td>
<td>1.40</td>
</tr>
<tr>
<td>Spec</td>
<td>3.14</td>
<td>0.60</td>
<td>0.80</td>
<td>1.00</td>
<td>1.20</td>
<td>1.40</td>
</tr>
<tr>
<td>Sim</td>
<td>0</td>
<td>40.0</td>
<td>100.0</td>
<td>170.0</td>
<td>250.0</td>
<td></td>
</tr>
</tbody>
</table>

Fig. 9. Behaviors of coupled oscillators (Example 4)

4. CONCLUSIONS

In this paper, we considered pattern formation in networks of nonlinear systems with delays. The approach employed in this paper is based on the harmonic balance method and the obtained results are extensions of Iwasaki (2008) for delayed networks. Although these methods contain approximations and the proposed algorithms are not guaranteed to converge, the validity of the proposed methods were supported by numerical examples.

REFERENCES


Appendix A. STABILITY FOR LINEAR RETARDED SYSTEMS

Regarding to the characteristic roots on the imaginary, the following result holds regarding to the stability of linear retarded systems (Chen et al. (1995), Niculescu et al. (1998)).

Consider the following linear delay system:

\[ \dot{x}(t) = A_0 x(t) + \sum_{i=1}^{m} A_i x(t - i \tau) \] (A.1)

The corresponding characteristic function is given by

\[ p(\lambda; \tau) = \det \left( \lambda I_n - A_0 - \sum_{i=1}^{m} A_i e^{-\lambda i \tau} \right) \] (A.2)

For the system (A.1), the matrix pencil is defined by

\[ \Lambda(z) := zM + N \] (A.3)

where \( M, N \in \mathbb{R}^{(2mn^2) \times (2mn^2)} \) are given by

\[
M := \begin{bmatrix}
I_{n^2} & 0 & \cdots & 0 \\
0 & I_{n^2} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & I_{n^2} \\
0 & 0 & \cdots & 0 \\
\end{bmatrix}
\]

\[
N := \begin{bmatrix}
0 & I_{n^2} & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & -I_{n^2} \\
0 & 0 & \cdots & 0 \\
\end{bmatrix}
\]

and \( B_{-k} \) for \( k = 1, \ldots, m \) and \( B_i \) for \( i = 1, \ldots, m \) are defined as

\[ B_{-k} = I_n \otimes A^T_k, B_i = A_i \otimes I_n, B_0 = A_0 \otimes A^T_0 \]

The operator \( \otimes \) and \( \oplus \) denote the Kronecker product and sum. Then the following theorem holds.

Theorem 3. (Niculescu et al. (1998)) Assume that the matrix pencil \( \Lambda \) is regular. Then the quasi polynomial \( p(\lambda; \tau) = 0 \) has a crossing root on the imaginary axis for some positive delay value \( \tau \) if and only if there exists a complex number \( z_0 \in \sigma(\Lambda) \cap \mathbb{C}(0,1) \) such that

\[ j\omega_0 \in \sigma \left( A_0 + \sum_{i=1}^{m} A_i z_0^i \right) \]

where \( \sigma(\Lambda) \) denotes a point spectrum of \( \Lambda \). Furthermore, for some \( z_0 \) satisfying the condition (ii) above, the set of delays corresponding to the induced crossing is given by,

\[ \mathcal{T}(z_0) = \left\{ \frac{\omega_0}{\omega_0} + \frac{2\pi \ell}{\omega_0} > 0 : j\omega_0 \in \left( A_0 + \sum_{i=1}^{m} A_i \right), \ell \in \mathbb{Z} \right\} \]