Adaptive Estimation in Nonlinearly Parameterized Discrete-Time Nonlinear Systems

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Abstract: This paper proposes an uncertainty set update method for a class of discrete-time nonlinearly parameterized systems. The method is based on the sequential application of a method proposed for linearly parameterized uncertain nonlinear systems in which the nonlinearly parameterized systems can be treated as linearly parameterized uncertain systems at each step. It is demonstrated that, given a persistence of excitation condition, convergence of the parameter estimates is guaranteed. The application of this method is applied to highly nonlinear system of a bioreactor operating under Monod kinetics.

1. INTRODUCTION

Parameter estimation in dynamical systems has been a central theme in systems research for decades. Starting from nonlinear regression techniques to the advent of nonlinear adaptive control, the problem has received considerable attention. Adaptive estimation of nonlinear systems remains a relatively unexplored field. Most existing design techniques are restricted to systems that are linear in the unknown (constant) parameters. Representative techniques are discussed in several references such as Krstic and Deng [1998], Krstic et al. [1995], Marino and Tomei [1995].

Several authors have also studied this class of problems for specific applications. One such application is the study of microbial growth kinetics where most models, due to the importance of classical enzyme kinetics models, are nonlinearly parameterized (Boskovic [1998], Zhang and Guay [2002]). The nonlinearity of these models prevents one from using normal techniques to establish parameter convergence. For Monod models, one can show that parameter convergence can be achieved subject to a conservative persistence of excitation condition that can only be derived using highly tailored Lyapunov based arguments. Another leading approach consists of approximating the nonlinearity using neural networks (Zhang et al. [2000], Wang [2009], Ge [2001]). The main drawbacks of these techniques is that such approximations cannot be used to uniquely reconstruct the unknown parameter vector.

Work on nonlinearly parameterized systems remains challenging. The most significant approach to solve this problem can be found in several papers by Annaswamy and co-workers (Cao et al. [2003], Annaswamy et al. [1998], Kojic et al. [1999], Netto et al. [2000]). Their approach exploits convexity of the system dynamics with respect to the parameters to develop a class of min-max adaptive estimation routines. A gradient-based approach is proposed subject to a worst-case parameter set. In Loh et al. [1999], the authors consider the estimation in systems with nonconvex parameterizations. The approach yields a global estimation scheme. The technique relies on a min-max approach formulated as a modification of a standard least-squares parameter estimation approach. Various generalizations are offered to tackle convex and/or concave parameterization. A nonlinear persistency of excitation condition is proposed.

In Adetola and Guay [2010], a novel parameter estimate routine was developed for linearly parameterized systems in the presence of exogenous disturbances. The parameter estimation routine are used to update the parameter uncertainty set, at certain time instants, in a manner that guarantees non-expansion of the set leading to a gradual reduction in the conservativeness or computational demands of the algorithms. The main contribution of this work is the generalization of an uncertainty set-update method to nonlinearly parameterized discrete-time nonlinear systems. The design proposed elects to treat nonlinearly parameterized systems as linearly parameterized systems subject to bounded uncertainties, where the nonlinearity is treated as a bounded disturbance. To deal with the presence of this uncertainty, we propose an extension of the set-based estimation technique proposed in Adetola and Guay [2010] (recently extended to linearly parameterized discrete-time system in Lehrer et al. [2010]) for nonlinearly parameterized nonlinear discrete-time dynamical systems. Conditions under which the adaptive estimation approach converges to the true unknown value is akin to a persistency of excitation condition.

In contrast to the approach proposed in Loh et al. [1999], the approach presented here focusses on a set-based approach. The main requirement is the local convexity of the estimation problem over a user-defined compact set. The main assumption is that this set must be assumed to contains a unique parameter value. Under the assumption of local convexity, the routine can uniquely identify the

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parameter. The key here is that one can estimate not only the parameter but also a parameter uncertainty set known to contain to true value of the parameters.

The paper is organized as follows. The problem statement is given in Section 2. Section 3 presents the parameter and uncertainty set estimation routines. A finite difference approximation of a bioreactor model is considered to demonstrate the effectiveness of the proposed technique in Section 4. This is followed by short conclusions in Section 5.

2. PROBLEM DESCRIPTION AND ASSUMPTIONS

Consider the system:

\[ x_{k+1} = x_k + F(x_k, u_k, \theta), \tag{1} \]

where \( x_k \in \mathbb{R}^n \) is a state at some time step \( k \), \( u_k \in \mathbb{R}^m \) is the control input at some time step \( k \), and \( \theta \in \mathbb{R}^p \) is a vector of system parameters.

**Assumption 1.** The state of the system, \( x_k \), is known at all time steps \( k \).

**Assumption 2.** The state and input variables evolve on a compact set, \( x_k \in \mathcal{X} \subset \mathbb{R}^n \), \( u_k \in \mathcal{U} \subset \mathbb{R}^m \).

**Assumption 3.** \( \theta \) is uniquely identifiable and lies within an initially known compact set defined by the ball function \( \Theta^0 = B(\theta_0, z_0) \), where \( \theta_0 \) is some initial estimate of the parameters and \( z_0 \) is the initial radius of the uncertainty set.

3. PARAMETER ESTIMATION SCHEME

3.1 Parameter Estimate Update

The problem is first reparameterized by describing the true parameters values by

\[ \theta = \theta_0 + \delta, \tag{2} \]

and the parameters estimates by

\[ \hat{\theta}_k = \theta_0 + \hat{\delta}_k, \tag{3} \]

where \( \theta_0 \) represents the centre of the parameter uncertainty set defined by the ball \( \Theta \). The nonlinearity is rewritten using a bounded nonlinearity and a linearly parameterized term about the centre of the uncertainty set, \( \theta_0 \). This reconstruction is motivated by a particular choice of state-predictor as demonstrated below. By the mean-value theorem and (2), it follows that one can write:

\[ F(x_k, \theta_0 + \delta) - F(x_k, \theta_0) = \int_0^1 \frac{\partial F}{\partial \theta}(x_k, \theta_0 + \lambda \delta) d\lambda \delta \tag{4} \]

where \( \lambda \) is the integrating factor. Now, let

\[ \Psi(x_k, \theta_0, \delta) = \int_0^1 \frac{\partial F}{\partial \theta}(x_k, \theta_0 + \lambda \delta) d\lambda, \tag{5} \]

and

\[ \Delta \Psi(x_k, \theta_0, \delta) = \Psi(x_k, \theta_0, \delta) - \Psi(x_k, \theta_0, 0). \tag{6} \]

In the remainder, we will make the following assumption concerning the nonlinearity \( \Delta \Psi(x_k, \theta_0, \delta) \).

**Assumption 4.** The nonlinearity \( \Delta \Psi(x_k, \theta_0, \delta) \) is such that there exists a strictly positive constant \( L > 0 \)

\[ \| \Delta \Psi(x_k, \theta_0, \delta) \| \leq L \| \delta \| \leq L z_0 \tag{7} \]

\( \forall \theta_0 \in \Theta_0 \).

Based on (1) and (4), the following state predictor is proposed:

\[ \hat{x}_{k+1} = \hat{x}_k + F(x_k, \theta_0, \hat{\delta}_k) + K c_k \]

\[ + (c_k^T - K c_k^T + \Psi(x_k, \theta_0, 0)) \hat{\delta}_{k+1} - \hat{\delta}_k \]

\( \tag{8} \)

where \( K \in \mathbb{R}^{n \times n} \) is the observer gain matrix. Defining the prediction error, \( e_k = x_k - \hat{x}_k \), the error dynamics are given by

\[ e_{k+1} = e_k + \Delta \Psi(x_k, \theta_0, \delta) - \Psi(x_k, \theta_0, 0) \hat{\delta}_k - K c_k \]

\[ - (c_k^T - K c_k^T + \Psi(x_k, \theta_0, 0)) (\hat{\delta}_{k+1} - \hat{\delta}_k) \]

\[ - \Delta \Psi(x_k, \theta_0, \hat{\delta}_k). \tag{9} \]

Since the term \( \Delta \Psi(x_k, \theta_0, \hat{\delta}_k) \hat{\delta}_k \) is known at all time steps, the state predictor is modified to be

\[ \hat{x}_{k+1} = \hat{x}_k + F(x_k, \theta_0, \hat{\delta}_k) + K c_k \]

\[ + (c_k^T - K c_k^T + \Psi(x_k, \theta_0, 0)) (\hat{\delta}_{k+1} - \hat{\delta}_k) \]

\[ - \Delta \Psi(x_k, \theta_0, \hat{\delta}_k), \tag{10} \]

and the error dynamics become

\[ e_{k+1} = e_k + \Delta \Psi(x_k, \theta_0, \delta) + \Psi(x_k, \theta_0, 0) \hat{\delta}_k \]

\[ - K c_k - (c_k^T - K c_k^T + \Psi(x_k, \theta_0, 0)) (\hat{\delta}_{k+1} - \hat{\delta}_k). \]

We define an auxiliary variable \( \eta_k \) as follows:

\[ \eta_k = e_k - c_k^T \hat{\delta}_k. \tag{11} \]

An output filter is also defined as

\[ c_{k+1} = c_k^T + \Psi(x_k, \theta_0, 0) - K c_k \]

\[ c_0 = 0. \tag{12} \]

The auxiliary variable dynamics are given by

\[ \eta_{k+1} = \Delta \Psi(x_k, \theta_0, \delta) + \Psi(x_k, \theta_0, 0) \hat{\delta}_k \]

\[ - K c_k - (c_k^T - K c_k^T + \Psi(x_k, \theta_0, 0)) (\hat{\delta}_{k+1} - \hat{\delta}_k). \tag{13} \]

Since the \( \eta_k \) dynamics depend on the unknown model parameters, it is necessary to use an estimate of the true value of \( \eta_k \). This estimate is generated by the recursion

\[ \hat{\eta}_{k+1} = \hat{\eta}_k - K \eta_k. \tag{14} \]

Let the identifier matrix \( \Sigma_k \) be defined as

\[ \Sigma_{k+1} = \Sigma_k + c_k c_k^T \]

\[ \Sigma_0 = \alpha I > 0, \tag{15} \]

with an inverse generated by the recursion

\[ \Sigma_k^{-1} = \Sigma_k^{-1} - \Sigma_k^{-1} c_k (I + c_k \Sigma_k^{-1} c_k^T)^{-1} c_k^T \Sigma_k^{-1} \]

\[ \Sigma_0^{-1} = \frac{1}{\alpha} I \geq 0. \tag{16} \]

The parameter update is performed indirectly by updating the variable \( \hat{\delta} \),

\[ \hat{\delta}_{k+1} = \hat{\delta}_k + \Sigma_k^{-1} c_k (I + c_k^T \Sigma_k^{-1} c_k)^{-1} (e_k - \hat{\eta}_k). \tag{17} \]
To ensure that the parameter estimates remain within the constraint set \( \Theta_k \), we propose to use a projection operator of the form:

\[
\hat{\delta}_{k+1} = \text{Proj}(\delta_k + \Sigma^{-1}c_k (I + c_k^T \Sigma^{-1} c_k)^{-1} (e_k - \hat{\eta}_k), \Theta_k).
\]  

(18)

The operator \( \text{Proj} \) represents an orthogonal projection onto the surface of the uncertainty set applied to the parameter estimate. The parameter uncertainty set is defined by the ball function \( B(\hat{\theta}_k, \tilde{z}_k) \), where \( \hat{\theta}_k \) and \( \tilde{z}_k \) are the parameter estimate and set radius found at the latest set update. Following Goodwin and Sin [1984], the projection operator is designed such that

- \( \hat{\theta}_{k+1} \in \Theta_k \)
- \( \delta_{k+1}^T \Sigma_k \delta_{k+1} \leq \delta_k^T \Sigma_k \delta_k \)

and the variable \( \hat{\delta}_{k+1} \) is recovered by

\[
\hat{\delta}_{k+1} = \hat{\delta}_{k+1} - \hat{\theta}_0.
\]  

(19)

It can be shown that the parameter update law (18) guarantees convergence of the parameter estimation error \( \hat{\theta}_k = (\theta_k + \hat{\delta}) - (\theta_k + \hat{\delta}_k) \) to a small neighbourhood of the origin.

**Lemma 1.** The identifier (16) and the parameter update law (18) are such that \( \hat{\delta} = \hat{\delta} - \hat{\delta} \) and therefore \( \hat{\theta}_k = \theta_k - \hat{\theta}_k \) are bounded. Furthermore, if the trajectories of the system of such that the following PE condition is met:

\[
\sum_{i=0}^{k+T-1} c_i c_i^T > \beta_T I, \quad \forall k > T
\]

for some \( T > 0 \) and \( \beta_T > 0 \) then the parameter estimation error \( \delta_k \) converges to a neighbourhood of the origin as \( k \to \infty \).

**Proof** Let \( V_{\hat{\delta}_k} = \hat{\delta}_k^T \Sigma_k \hat{\delta}_k \). It follows from the properties of the projection operator that:

\[
V_{\hat{\delta}_{k+1}} - V_{\hat{\delta}_k} = \hat{\delta}_{k+1}^T \Sigma_k \hat{\delta}_{k+1} - \hat{\delta}_k^T \Sigma_k \hat{\delta}_k \leq \hat{\delta}_{k+1}^T \Sigma_k \hat{\delta}_{k+1} - \hat{\delta}_k^T \Sigma_k \hat{\delta}_k
\]  

(20)

Furthermore, substituting for \( \Sigma_k \), one obtains:

\[
V_{\hat{\delta}_{k+1}} - V_{\hat{\delta}_k} \leq \hat{\delta}_{k+1}^T \Sigma_k \hat{\delta}_{k+1} + \hat{\delta}_k^T \Sigma_k \hat{\delta}_k + \hat{\delta}_k^T c_k^T c_k \hat{\delta}_k - \hat{\delta}_k^T \Sigma_k \hat{\delta}_k
\]

By virtue of the projection algorithm, the parameter estimates \( \hat{\delta}_k \) and the parameter estimation error \( \delta_k \) are bounded. Since the system dynamics are assumed to be stable, it follows that the system states are bounded. As a result, one can conclude that the output filter parameters \( c_k \) are also bounded. As a result, \( \Sigma_k \) is bounded on any finite interval and it follows that, \( V_{\hat{\delta}_{k+1}} - V_{\hat{\delta}_k} \) is bounded.

Let us define the \( \eta \) estimation error as, \( \tilde{\eta}_k = \eta_k - \hat{\eta}_k \), with dynamics:

\[
\tilde{\eta}_k = \Delta \Psi(x_k, \theta_0, \delta) \delta + \tilde{\eta}_k - K \tilde{\eta}_k.
\]

Using the parameter update law and letting

\[
e_k - \tilde{\eta}_k = e_k - \eta_k - \hat{\eta}_k = c_k^T \hat{\delta}_k + \tilde{\eta}_k,
\]

one can write \( \tilde{\delta}_{k+1} \) as:

\[
\tilde{\delta}_{k+1} = \tilde{\delta}_k - \Sigma_k^{-1} c_k (I + c_k^T \Sigma_k^{-1} c_k)^{-1} (e_k - \tilde{\eta}_k)
\]

or:

\[
\tilde{\delta}_{k+1} = \Sigma_k^{-1} \Sigma_k \tilde{\delta}_k - \Sigma_k^{-1} c_k (I + c_k^T \Sigma_k^{-1} c_k)^{-1} \tilde{\eta}_k.
\]  

(21)

By induction, it is easy to see that the parameter estimation error is such that,

\[
\tilde{\delta}_{k+1} = \Sigma_k^{-1} \Sigma_k \tilde{\delta}_0 - \Sigma_k^{-1} \sum_{i=0}^{k} c_i \tilde{\eta}_k
\]  

(22)

Note that we can always rewrite the last equation as:

\[
\tilde{\delta}_{k+1} = \left( \Sigma_0 + \sum_{i=0}^{k} c_i c_i^T \right)^{-1} \left( \Sigma_0 \tilde{\delta}_0 - \sum_{i=0}^{k} c_i \tilde{\eta}_k \right)
\]  

(23)

If \( \lambda_{\min} \left[ \sum_{i=0}^{k} c_i c_i^T \right] = \beta(k) \) and \( \Sigma_0 = \alpha I \), then

\[
\lambda_{\min} \left[ \Sigma_0 + \sum_{i=0}^{k} c_i c_i^T \right] = \alpha + \beta(k).
\]

As a result, one obtains:

\[
\|\tilde{\delta}_{k+1}\| \leq \frac{1}{\alpha + \beta(k)} \left( \alpha \|\tilde{\delta}_0\| + \left\| \sum_{i=0}^{k} c_i \tilde{\eta}_k \right\| \right)
\]  

(24)

If we add the requirement that the sequence of matrices \( c_i \) is PE,

\[
\sum_{i=k}^{k+T-1} c_i c_i^T > \beta_T I, \quad k > T
\]

then it follows that

\[
\sum_{i=0}^{k-1} c_i c_i^T > \frac{\beta_T}{2T} I, \quad k > T
\]

such that \( \beta(k) > \frac{\beta_T}{2T} k \). As a result, one gets:

\[
\|\tilde{\delta}_{k+1}\| \leq \frac{1}{\alpha + \frac{\beta_T}{2T} k} \left( \alpha \|\tilde{\delta}_0\| + \left\| \sum_{i=0}^{k} c_i \tilde{\eta}_k \right\| \right)
\]  

(25)

For the \( \tilde{\eta} \) dynamics, we pose the Lyapunov function,

\[
V_{\tilde{\eta}} = \frac{1}{2} \|\tilde{\eta}\|^2.
\]

This function is such that:

\[
V_{\tilde{\eta}_{k+1}} - V_{\tilde{\eta}_k} = (\Delta \Psi(x_k, \theta_0, \delta) \delta + \tilde{\eta}_k - K \tilde{\eta}_k)^T \times (\Delta \Psi(x_k, \theta_0, \delta) \delta + \tilde{\eta}_k - K \tilde{\eta}_k) - \tilde{\eta}_k^T \tilde{\eta}_k
\]

It can be shown that:

\[
V_{\tilde{\eta}_{k+1}} - V_{\tilde{\eta}_k} \leq (2(1 - K)^2 - 1) \|\tilde{\eta}\|^2 + 2 \|\Delta \Psi(x_k, \theta_0, \delta) \delta\|^2
\]

where \( K \) is chosen such that \( (2(1 - K)^2 - 1) < 0 \). Using the known bound on the unknown values of the parameters and using Assumption 4, it follows that:

\[
V_{\tilde{\eta}_{k+1}} - V_{\tilde{\eta}_k} \leq (2(1 - K)^2 - 1) \|\tilde{\eta}\|^2 + 2L^2 \tilde{\eta}_0^2
\]

Thus, \( V_{\tilde{\eta}_{k+1}} - V_{\tilde{\eta}_k} < 0 \) if

\[
\|\tilde{\eta}\|^2 > \frac{2L^2 \tilde{\eta}_0^2}{(2(1 - K)^2 - 1)}.
\]

Simple calculation confirms that there exists a value of the gain \( K \) for which this conditions holds. By standard arguments, it follows that since \( \tilde{\eta}_0 = 0 \) then
∀k ≥ 0. Substitution of (26) in (25) yields
\[ \|\tilde{\delta}_{k+1}\| \leq \frac{1}{\alpha + \frac{\sigma_k}{\sqrt{2L^2z^4_{\theta_0}}}} \left( \frac{2L^2z^4_{\theta_0} - 1}{\sqrt{2(1 - K)^2 - 1}} \right) \]
where \( \sup_k |c_k| = M \) for some positive constant \( M > 0 \).
Taking the limit, it follows that:
\[ \lim_{k \to \infty} \| \tilde{\delta}_k \| \leq \frac{2\sqrt{2TM^2z^4_{\theta_0}}}{\beta \sqrt{2(1 - K)^2 - 1}} \]
The parameter estimation error converges to a neighbourhood of the origin. This completes the proof.

3.2 Set Update

The parameter uncertainty set, defined by the ball function \( B(\hat{\theta}_c, z_c) \) is updated using the parameter update law (18) and the error bound (28) according to the following algorithm:

Algorithm 1. beginning at time step \( k = 0 \), the set is adapted according to the following iterative process

1. **Initialize** \( z_c = z_{\theta_0}, \hat{\theta}_c = \hat{\theta}_0 \)
2. at time step \( k \), using equations (18) and (28) perform the update
   \[ \left\{ \begin{array}{l l} (\hat{\theta}_c, z_c) & \text{if } z_{\theta_0} \leq z_c - \| \tilde{\theta}_k - \hat{\theta}_c \| \\ (\hat{\theta}_c, z_c) & \text{otherwise} \end{array} \right. \]
3. in the case when the uncertainty set is updated in step (2), the following values are reset as follows \( c_k^T = 0 \) and \( \tilde{\eta}_k = e_k \)
4. Return to step (2) and iterate, incrementing to time step \( k + 1 \)

An update law that measures the worst-case progress of the parameter update law is adapted from the one proposed in Adetola and Gui [2009]
\[ z_{\theta_k} = \frac{V_{\tilde{\theta}_k}}{4\lambda_{\min}(\Sigma_k)} \]  

\[ V_{\tilde{\theta}_{k+1}} = V_{\tilde{\theta}_k} + \left( \frac{2L^2z^4_{\theta_0}}{\sqrt{2(1 - K)^2 - 1}} \right) - (e_k - \tilde{\eta}_k)^T (I + c_k^T \Sigma_k^{-1} c_k)^{-1} (e_k - \tilde{\eta}_k) \]  

\[ V_{\tilde{\theta}_{k+1}} = \begin{cases} \begin{array}{l} V_{\tilde{\theta}_k} & \text{if } V_{\tilde{\theta}_{k+1}} \geq V_{\tilde{\theta}_k} \\ V_{\tilde{\theta}_{k+1}} & \text{otherwise} \end{array} \end{cases} \]

\[ V_{\tilde{\theta}_0} = 4\lambda_{\max}(\Sigma_0)(z_{\theta_0})^2. \]  

The following lemma ensures that the non-exclusion property of Algorithm 1 which implies that \( \Theta_{k+1} \subseteq \Theta_k \). The main difference arises from the choice of update of \( V_{\tilde{\theta}_{k+1}} \) described by (30). In this case, the potential increase in \( V_{\tilde{\theta}_k} \) is introduced to remove some conservativeness associated with the nominal uncertain set update based on (29).

**Lemma 2.** The algorithm ensures that

1. the set is only updated when updating will yield a contraction,
2. the dynamics of the set error bound described in (28) are such that they ensure the non-exclusion of the true value \( \theta \in \Theta_k \), \( \forall k \) if \( \theta_0 \in \Theta_0 \).

**Proof**

1. If \( \Theta_{k+1} \not\subseteq \Theta_k \) then
   \[ \sup_{\Theta_k \in \Theta_{k+1}} \| s - \hat{\theta}_k \| \geq z_{\theta_k} \]  

   However, it is guaranteed by the set update algorithm presented, that \( \Theta_k \), at update times, obeys the following
   \[ \sup_{\Theta_k \in \Theta_{k+1}} \| s - \hat{\theta}_k \| \leq \sup_{\Theta_k \in \Theta_{k+1}} \| s - \hat{\theta}_{k+1} \| + \| \hat{\theta}_{k+1} - \hat{\theta}_k \| \leq z_{\theta_{k+1}} + \| \hat{\theta}_{k+1} - \hat{\theta}_k \| \leq z_{\theta_k}. \]

   This contradicts (32). Therefore, \( \Theta_{k+1} \subseteq \Theta_k \) at times where \( \Theta \) is updated.

2. In addition to the non-exclusion property, one needs to ensure that the true value of the parameters, \( \theta = \theta_0 + \delta \) belongs to the uncertainty set at each step. First, one assumes that, at step \( k \),
   \[ V_{\tilde{\theta}_k} \geq V_{\hat{\theta}_k} \]  

   (Note that, by construction, this can always be fulfilled at \( k = 0 \). Then if,
   \[ V_{\tilde{\theta}_{k+1}} - V_{\tilde{\theta}_k} \geq V_{\hat{\theta}_{k+1}} - V_{\hat{\theta}_k}, \]
   it follows that
   \[ V_{\tilde{\theta}_{k+1}} \geq V_{\hat{\theta}_{k+1}}. \]

   Hence, one concludes that
   \[ \| \tilde{\theta}_{k+1} \|^2 \leq \frac{V_{\tilde{\theta}_{k+1}}}{\lambda_{\min}(\Sigma_{k+1})} = 4z^2_{\theta_{k+1}}. \]

   For the set update of \( V_{\tilde{\theta}_k} \), given by (30), it is important to guarantee that the non-exclusion property is still preserved. The projection algorithm always guarantees that the estimation errors are as follows:
   \[ \| \hat{\delta}_k \|^2 \leq 4z^2_{\theta_k} \]

   Therefore, one can always assume that
   \[ \| \tilde{\delta}_k \|^2 \leq \frac{V_{\tilde{\theta}_k}}{\lambda_{\min}(\Sigma_k)} \leq 4z^2_{\theta_k}. \]

   The rate of change of \( V_{\tilde{\theta}_k} \) is given by
   \[ V_{\tilde{\theta}_{k+1}} \leq V_{\tilde{\theta}_k} - (e_k - \tilde{\eta}_k)^T (I + c_k^T \Sigma_k^{-1} c_k)^{-1} (e_k - \tilde{\eta}_k) + \tilde{\eta}_k^T (I + c_k^T \Sigma_k^{-1} c_k)^{-1} \tilde{\eta}_k. \]

   One can also write,
   \[ V_{\tilde{\theta}_{k+1}} \leq V_{\tilde{\theta}_k} - (e_k - \tilde{\eta}_k)^T (I + c_k^T \Sigma_k^{-1} c_k)^{-1} (e_k - \tilde{\eta}_k) + \tilde{\eta}_k^T \tilde{\eta}_k. \]

   By construction,
\[\lambda_{\min} [\Sigma_{k+1}] \|\delta_{k+1}\|^2 \leq V z \hat{\theta}_k + \frac{2 L^2 z^4 \mu_{\min}}{|(2(1-K)^2 - 1)|} \]

\[-(e_k - \hat{\eta}_k)^T (I + c_k^T \Sigma_k^{-1} c_k)^{-1} (e_k - \hat{\eta}_k)\]

Hence,

\[\|\tilde{\delta}_{k+1}\|^2 \leq \frac{1}{\lambda_{\min} [\Sigma_{k+1}]} \left( V z \hat{\theta}_k + \frac{2 L^2 z^4 \mu_{\min}}{|(2(1-K)^2 - 1)|} \right) \]

\[-(e_k - \hat{\eta}_k)^T (I + c_k^T \Sigma_k^{-1} c_k)^{-1} (e_k - \hat{\eta}_k)\]

Since,

\[\lambda_{\min} [\Sigma_{k+1}] \geq \lambda_{\min} [\Sigma_k],\]

then we have that

\[\|\tilde{\delta}_{k+1}\|^2 \leq \frac{1}{\lambda_{\min} [\Sigma_k]} \left( V z \hat{\theta}_k \right) \]

\[-(e_k - \hat{\eta}_k)^T (I + c_k^T \Sigma_k^{-1} c_k)^{-1} (e_k - \hat{\eta}_k)\]

\[\leq 4 \frac{x^2 - \mu_{\min} [\Sigma_k]}{\lambda_{\min} [\Sigma_k]} \]

\[\times \left( (e_k - \hat{\eta}_k)^T (I + c_k^T \Sigma_k^{-1} c_k)^{-1} (e_k - \hat{\eta}_k) \right)\]

\[-\frac{2 L^2 z^4 \mu_{\min}}{|(2(1-K)^2 - 1)|}\]

From the last inequality, we see that, by the action of the projection algorithm, the set update will only lead to potential exclusion of the true value of the parameters if

\[
\left( (e_k - \hat{\eta}_k)^T (I + c_k^T \Sigma_k^{-1} c_k)^{-1} (e_k - \hat{\eta}_k) \right) \geq 0.
\]

(38)

Therefore, one only needs to update the value of \( V z_{k+1} \) when it leads to a decrease, or alternatively, when the inequality (38) is met.

We are now ready to state the main result of this paper.

**Theorem 3.** The identifier (16), the parameter update law (18), and the set update procedure algorithm 1 are such that \( \tilde{\delta}_k = \delta - \hat{\delta}_k \) and therefore \( \tilde{\theta}_k = \theta - \hat{\theta}_k \) are bounded. Furthermore, if

\[
\lim_{k \to \infty} \lambda_{\min} [\Sigma_k] = \infty
\]

is satisfied, then \( \tilde{\theta}_k \) converges to 0 asymptotically.

**Proof** Since, \( V z_{\hat{\theta}_k} = \delta^T \Sigma \delta \), then it follows by the non exclusion property of the set update algorithm that,

\[
\|\tilde{\delta}_k\| \leq \frac{V z_{\hat{\theta}_k}}{\lambda_{\min} [\Sigma_k]} = 4 z^2 \mu_{\min} \leq 4 z^2, \quad \forall k \geq 0.
\]

Since \( V z_{\hat{\theta}_k} \) is non-increasing, by construction, and since \( \lambda_{\min} [\Sigma_k] \) is increasing, there exists a finite integer \( N \) such that

\[
z_{\hat{\theta}_N} \leq z_{\Delta t} - \|\tilde{\delta}_N\|
\]

leading to a shrinking of the set. Applying the set update yields to a re-centered uncertainty set with a smaller uncertainty radius containing the unknown value of the parameters.

Repeating sequentially, it follows that, as \( k \to \infty \), the properties of the set update are such that the uncertainty radius \( z_{\Delta t} \) will tend asymptotically to zero. This is, in turn, guarantees that \( \lim_{k \to \infty} \|\tilde{\delta}_k\| = 0 \) and, by the non-exclusion property, that the center of the uncertainty set \( \tilde{\theta}_c \) converges to the true value of the parameters \( \theta \).

4. SIMULATION EXAMPLE

Consider the following system representing a chemostat operating under Monod kinetics, discretized using a finite differences method.

\[
x_1(k + 1) = x_1(k) + \Delta t \left\{ \theta_1 x_1(k) x_2(k) - D x_1(k) \right\} \]

\[
x_2(k + 1) = x_2(k) + \Delta t \left\{ \frac{-\theta_3 x_1(k) x_2(k)}{\theta_2 + x_2(k)} + S_0 - D x_2(k) \right\}
\]

where \( \theta = [\mu_{\max}, K_s, \frac{1}{Y_{sx}}]^T \), with values \( \theta = [0.33, 0.5, 0.66]^T \). The control input \( S_0 = 5 \) is kept constant and the input \( D \) is oscillated such that \( 0.07 \leq D \leq 0.13 \). \( \Delta t \) is the size of the time step defined as \( \Delta t = \frac{1}{60} hrs \). The choice of \( \Delta t \) is such that the discrete-time nonlinear system as stated above provides a good approximation of the corresponding continuous-time dynamics of the bioreactor.

It is shown in fig. 1 that the parameter estimates converge to their true values, further, the state estimation error converges to a neighborhood around zero. The radius of the parameter uncertainty set is demonstrated to shrink each instance the set is updated (3), further this plot demonstrates that the magnitude of the \( \delta \) variable error is always smaller than the radius of the set, as expected. Since the \( \delta \) variable error is equal to the parameter estimation error, it is possible to verify that the true parameters remain within the uncertainty set throughout the simulation.

5. CONCLUSION

In this paper, we have demonstrated, that the parameter uncertainty set method developed by Adetola and Guay [2009], generalized to discrete-time systems in Adetola and Guay [2009], can be applied to nonlinearly parameterized systems. The approach of Adetola and Guay [2009] is applied in a sequential manner such that nonlinearly parameterized systems can be treated as linearly parameterized uncertain systems at each step. It is demonstrated that, given a persistence of excitation condition, convergence of
Fig. 1. Parameter estimates and true values under the parameter uncertainty set algorithm, the dashed lines (- -) represent the true parameter values, the solid lines (−) represent the parameter estimates.

Fig. 2. State prediction error $e_k = x_k - \hat{x}_k$

Fig. 3. Radius of the parameter uncertainty set and the magnitude of the variable $\delta$ at time steps when the set is updated. The parameter estimates is guaranteed. The application of this method has been applied to highly nonlinear system of a bioreactor operating under Monod kinetics.

REFERENCES


