Mean Field Analysis of Controlled Cucker-Smale Type Flocking: Linear Analysis and Perturbation Equations

Mojtaba Nourian ∗ Peter E. Caines∗ Roland P. Malhamé∗∗

∗ CIM and Department of Electrical & Computer Engineering, McGill University, 3480 University Street, Montreal, QC H3A 2A7, and GERAD, Montreal, Canada,
e-mail: {mnourian,peterc}@cim.mcgill.ca

∗∗ GERAD, Montreal and Department of Electrical Engineering, École Polytechnique de Montréal, Montreal, QC H3C 3A7, Canada,
e-mail: roland.malhame@polymtl.ca

Abstract: This paper presents a linear analysis and perturbation equations in the synthesis of Cucker-Smale (C-S) type flocking via Mean Field (MF) stochastic control theory. In this model the state of each individual agent consists of both its position and its controlled velocity and all agents have similar stochastic dynamics. The agents are coupled via their nonlinear individual cost functions which are based on the C-S flocking algorithm in its original uncontrolled formulation. The MF system of equations approximates this stochastic system of individual agents as the population size goes to infinity. The key result is that C-S flocking behaviour may be obtained as a Nash dynamic competitive game equilibrium. After reviewing the case with linear cost coupling, we present the perturbation (i.e., linearized) equations of the nonlinear MF system of equations around the Gaussian solution of the linear cost coupling case.

Keywords: Stochastic systems, decentralized and cooperative optimization and control.

1. INTRODUCTION

Collective motion such as the flocking of birds, schooling of fish and swarming of bacteria, is one of the most widespread phenomenon in nature. The study of collective motion in nature is of interest not only to model, analyze and interpret these widespread phenomena, but also because ideas from these behaviours can be used by engineers to develop efficient algorithms for a wide range of applications (see (Perea et al., 2009; Lin et al., 2004), among many other papers).

There are two main classes of models for the flocking and swarming behaviour: (i) individual based models in the form of coupled Ordinary (Stochastic) Differential Equations (O(S)DEs) (see for instance (Cucker and Smale, 2007)), and (ii) continuum models in the form of Partial (or integro-partial) Differential Equations (PDEs) to model the collective motion in the case of systems with large populations (see (Carrillo et al., 2010; Topaz and Bertozzi, 2004; Ha and Liu, 2009) among others).

In (Cucker and Smale, 2007) Cucker and Smale formulated an interesting individual based flocking model for a group of agents. This model is motivated by the collective motion of a group of birds such that each bird updates its velocity as a weighted velocities of all the other birds. The weights in this model are functions of the relative distance of the birds such that as the mutual distance between two birds increases the influence of their velocities on each other decreases.

In this paper we study a controlled flocking model previously introduced in (Nourian et al., 2010b) by use of the Mean Field (MF) (or Nash Certainty Equivalence (NCE)) stochastic control theory. In this model agents have similar dynamics and are coupled via their nonlinear individual cost functions which are based on the Cucker-Smale (C-S) flocking algorithm in its original uncontrolled formulation. Since 2003, the MF framework has been studied for large population dynamic noncooperative games in a series of papers by Huang, Caines and Malhamé (see (Caines, 2009)). The common situation is that the dynamics and costs of any individual agent are influenced by certain averages of the mass multi-agent behavior. The specification of a consistency relationship between each individual decision and the overall effect of the population on that agent, in the population limit, is the key idea of the MF framework.

Based on the MF approach developed in (Huang et al., 2007), we derived an individual based MF equation system of the dynamic game stochastic consensus problem and explicitly computed its unique solution in (Nourian et al., 2010a, 2011c). The resulting MF control strategies steer each individual’s state toward the initial state population mean, and by applying these decentralized strategies, any finite population system reaches mean consensus asymptotically as time goes to infinity.

In an analogous way and based on the approach developed in (Yin et al., 2010), the continuum (i.e., the population size N goes to infinity) based MF equation system of the dynamic game stochastic consensus problem is derived in
(Nourian et al., 2011b). Furthermore, the corresponding stationary solution of the consensus MF equation system and its small perturbation stability analysis (based on the technique in (Guéant, 2009)) are studied in (Nourian et al., 2011b).

In (Nourian et al., 2010b), similar to (Yin et al., 2010; Nourian et al., 2011b), we derive a set of coupled deterministic equations approximating the stochastic model in systems with large populations. This set of equations consists of coupled (backward in time) Hamilton-Jacobi-Bellman (HJB) and (forward in time) Fokker-Planck-Kolmogorov (FPK) equations in the control optimized form, and an infinite population cost coupling. Subject to the existence of a unique solution to the nonlinear MF system of equations, we showed that the set of MF control laws for the system possesses an \( \epsilon_N \)-Nash equilibrium property, where \( \epsilon_N \to 0 \) as the population size \( N \) goes to infinity (Nourian et al., 2011a).

In this paper we review the analysis presented in (Nourian et al., 2011b) of the MF system of equations in the case with linear cost coupling. In this case the agents reach mean-consensus in velocity asymptotically (as time goes to infinity) on the initial velocity population mean. Then, we present the perturbation (i.e., linearized) equations of the nonlinear MF system of equations around the Gaussian solution of the linear cost coupling case (based on the approach developed in (Guéant, 2009; Nourian et al., 2011b)).

The following notation will be used. We use the integer valued subscript as the label for an individual agent. The integer \( N \) is reserved to denote the population size of the system. \( \| \cdot \| \) denotes the 2-norm of vectors and \( \| x \|_Q := (x^TQx)^{1/2} \) for any appropriate dimension vector \( x \) and matrix \( Q \geq 0 \). \( \text{tr}(A) \) denotes the trace of a square matrix \( A \). The gradient of a scalar function \( f \) is denoted by \( \nabla f \) where \( \nabla \) denotes the vector differential operator. \( \nabla_x \) and \( \nabla_v \) are the divergence operators with respect to \( x \) and \( v \), respectively. The Laplacian of a function \( f \) is denoted by the symbol \( \Delta f \). \( \partial_i f \) denotes the partial derivative of \( f \) with respect to the variable \( z \) and \( \partial_z^2 \) denotes the second derivative with respect to \( z \).

### 2. THE UNCONTROLLED CUCKER-SMALE MODEL

The fundamental uncontrolled C-S model for a system of population \( N \) is given by the nonlinear system of ODEs:

\[
\begin{align*}
\begin{cases}
\frac{dx_i(t)}{dt} &= v_i(t)dt, \\
\frac{dv_i(t)}{dt} &= \frac{1}{N} \sum_{j=1}^{N} w_{ij}(v_j(t) - v_i(t))dt,
\end{cases}
\end{align*}
\]

where \( t > 0, x_i \in \mathbb{R}^n \) and \( v_i \in \mathbb{R}^n \) are, respectively, position and velocity vectors of the \( i \)-th agent, \( 1 \leq i \leq N \), and \( x_i(0), v_i(0), 1 \leq i \leq N \), are given. The communication rates \( w_{ij}(\cdot) \) are given by

\[
w_{ij}(t) \equiv w((x_i(t) - x_j(t))) := \frac{1}{(1 + \|x_i(t) - x_j(t)\|^2)^{\beta}},
\]

for some fixed \( \beta \geq 0 \).

It is shown in (Cucker and Smale, 2007) that the agents’ velocities converge to a common value (the average of initial velocities) regardless of the initial configurations when \( \beta < 1/2 \) and also the distance between agents remain fixed and bounded but not necessarily the same. This result was improved in (Ha and Liu, 2009) in the case of \( \beta = 1/2 \).

The corresponding continuum (i.e., the population size \( N \) goes to infinity) model of the individual uncontrolled C-S flocking model is given by

\[
\frac{\partial f(x, v, t)}{\partial t} + \nabla_x f(x, v, t) = \nabla_v \cdot (\xi(f)(x, v, t)f(x, v, t)),
\]

where \( f(x, v, t) \) denotes the density function of particles positioned at \( (x, t) \in \mathbb{R}^n \times \mathbb{R}_+ \) with the velocity \( v \in \mathbb{R}^n \); \( f(x, v, 0) \) is given, and

\[
\xi(f)(x, v, t) := \int_{\mathbb{R}^n} w(||x - x'||)(v - v')f(x', v', t)dx' dv'.
\]

See the comprehensive survey paper (Carrillo et al., 2010) (and the references therein) for the derivation of the continuum C-S flocking model from the individual based C-S algorithm in large populations.

### 3. PROBLEM FORMULATION

Consider a system of \( N \) agents. The dynamics of the agents are given by controlled SDEs:

\[
\begin{align*}
\begin{cases}
\frac{dx_i(t)}{dt} &= v_i(t)dt, \\
\frac{dv_i(t)}{dt} &= u_i(t)dt + Cd\omega_i(t),
\end{cases}
\end{align*}
\]

where \( x_i(\cdot) \in \mathbb{R}^n \) is the position, \( v_i(\cdot) \in \mathbb{R}^n \) is the velocity, \( u_i(\cdot) \in \mathbb{R}^n \) is the control input, and \{\omega_i : 1 \leq i \leq N\} denotes a set of \( N \) independent \( p \)-dimensional standard Wiener processes. The set of initial data \{\( x_i(0), v_i(0) \) : \( 1 \leq i \leq N \)\} are assumed to be mutually independent and also independent of \{\omega_i : 1 \leq i \leq N\} with finite second moments. The matrix \( C \) has a compatible dimension.

The objective of each individual agent \( i \), \( 1 \leq i \leq N \), is to minimize (over the admissible control set \( U_i \)) its ergodic or Long Run Average (LRA) cost function given by

\[
J_i^{(N)}(u_i, u_{-i}) := \limsup_{T \to \infty} \frac{1}{T} \int_0^T \phi^{(N)}((x_i, v_i); (x, v)_{-i}) + \|u_i\|_R^2 dt,
\]

where

\[
\phi^{(N)}((x_i, v_i); (x, v)_{-i}) := \frac{1}{N} \sum_{j=1}^{N} w(x_i - x_j)(v_i - v_j) Q\|v_i - v_j\|^2,
\]

where the functions \( w_{ij}(\cdot) \) are defined in (2). In (4), \( x, v, \cdot := (x, v,\cdot,\cdot,\cdot,\cdot) \), and the matrices \( Q \) and \( R \) are symmetric positive definite with compatible dimensions. To indicate the dependence of \( J_i \) on \( u_i, u_{-i} := (u_1,\cdot,\cdot,\cdot,\cdot,\cdot) \) and the population size \( N \), we write it as \( J_i^{(N)}(u_i, u_{-i}) \).

It is important to note that the generic agent \( i \) is coupled to all other agents via the nonlinear function \( \phi^{(N)}((x_i, v_i); (x, v)_{-i}) \) in its cost function.

Hence, the model (3)-(4) may be regarded as a controlled game theoretic formulation of the C-S flocking model.
For each i, let $z_i := [x_i, v_i]^T$ and rewrite (3) as
dzi(t) = (Fz(t) + Gu(t))dt + Ddω(t), 1 ≤ i ≤ N, (5)
where

$F = \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix}, \quad G = \begin{pmatrix} 0 \\ I \end{pmatrix}, \quad D = \begin{pmatrix} 0 \\ C \end{pmatrix}.$

The LRA cost function (4) for an individual agent i, 1 ≤ i ≤ N, may be rewritten as

$J_1(N) := \limsup_{T \to \infty} \frac{1}{T} \int_0^T \left( \phi(z_i; z_{-i}) + \|u(t)\|_R^2 \right) dt,$ (6)

where

$\phi(z_i; z_{-i}) := \left\| \frac{1}{N} \sum_{j=1}^N w(\|x_i - x_j\|)(v_j - v_i) \right\|_Q^2.$

4. MEAN FIELD STOCHASTIC CONTROL THEORY

We take the following steps to the dynamic game flocking model (5)-(6) based on the Mean Field (MF) control approach (developed in [Yin et al., 2010] after [Huang et al., 2007]):

1. The continuum (infinite population) limit: In this step a Nash equilibrium for the model (5)-(6) in the continuum population limit (as N goes to infinity) is characterized by a “consistency relationship” between the individual strategies and the mass effect (i.e., the overall effect of the population on a given agent). This consistency relationship is described by a so-called MF equation system (see (13)-(15) below).

2. $\epsilon_N$-Nash equilibrium for the finite N model: The distributed continuum based MF control law (derived from the MF equation system in Step 1) establishes an $\epsilon_N$-Nash equilibrium for the finite N population model (5)-(6) where $\epsilon_N$ goes to zero asymptotically (as N approaches infinity).

4.1 Mean Field Approximation

In a large N population system, the mean field approach suggests that the cost-coupling function for a “generic” agent i (1 ≤ i ≤ N) in (6),

$\phi(N)(z_i; z_{-i}) := \left\| \frac{1}{N} \sum_{j=1}^N w(\|x_i - x_j\|)(v_j - v_i) \right\|_Q^2,$

be approximated by a deterministic function $\phi(z, \cdot)$ which only depends on $z = z_i$.

Replacing the function $\phi(N)(z_i; z_{-i})$ with the deterministic function $\phi(z, \cdot)$ in the i-th agent’s LRA cost function (6) reduces the dynamic game flocking model (5)-(6) to a set of N independent optimal control problems.

4.2 Preliminary Optimal Control of a Single Agent

We consider a “single agent” optimal control problem:
dz(t) = (Fz(t) + Gu(t))dt + Ddω(t),
\[ \inf_{u \in U} J(u) := \inf_{u \in U} \limsup_{T \to \infty} \frac{1}{T} \int_0^T \left( \phi(z, t) + \|u(t)\|_R \right) dt, \] (8)

where $z(\cdot) \in \mathbb{R}^{2n}, u(\cdot) \in \mathbb{R}^m$ are the state and the control input, respectively; $z(0)$ is given; $\omega$ denotes a p-dimensional standard Wiener processes; $\phi(z)$ is a known positive function, and $U$ is the corresponding admissible control set of the generic agent (see [Nourian et al., 2011a]).

An admissible control $u^o(\cdot) \in U$ is called almost surely (a.s.) optimal if there exists a constant $\rho^o$ such that

$J(u^o) = \limsup_{T \to \infty} \frac{1}{T} \int_0^T \left( \phi(z^o, t) + \|u^o(t)\|_R \right) dt = \rho^o,$ a.s.,

where $z^o(\cdot)$ is the solution of (7) under $u^o(\cdot)$, and for any other admissible control $u(\cdot) \in U$, we have a.s. $J(u) ≥ \rho^o$.

The associated Hamilton-Jacobian-Bellman (HJB) equation of the optimal control problem (7)-(8) is given by

$H(z, u, t) := \min_{u \in U} H(z, u, t) = \rho^o,$ (9)

where $H(z, u, \cdot) := \int_{\omega \in \Omega} H(z, u, t, \omega) dt$, $H(z, u, t, \omega)$ is the solution of (7) under $u(\cdot)$, $\omega$ denotes a denoted by a so-called MF equation system (see (13)-(15) below) in (9).

The solution of the optimal control problem (7)-(8) is

$u^o(t) := \arg \min_{u \in U} \int_{\omega \in \Omega} H(z, u, t, \omega) dt$ which

4.3 Evolution of the Population Density

We emnucate the following assumption:

(A1) We assume that the sequence of initial conditions $\{(x_i(0), v_i(0)) : 1 ≤ i ≤ N\}$ has a compactly supported probability density $f_0(x, v)$ such that $\int_A f_0(x, v) dx dv = 1$ where $A$ is a compact interval containing all $(x(0), v(0))$, 1 ≤ i ≤ N. Let

$f_N(x, v, t) := \frac{1}{N} \sum_{i=1}^N \delta(x - x_i(t))\delta(v - v_i(t)),$

be the empirical density associated with N agents where $\delta$ is the Dirac delta. We assume that $f_N(x, v, 0 : N ≥ 1)$ converges weakly to $f_0(x, v)$, i.e., for any $\psi(x, v) \in C_b^2$ (the space of bounded continuous functions on $\mathbb{R}^2$),

$\lim_{N \to \infty} \int \psi(x, v) f_N(x, v, 0) dx dv = \int \psi(x, v) f_0(x, v) dx dv.$

For any function $\psi(x, v) \in C_b^2$ we have

$\int \psi(x, v) f_N(x, v, t) dx dv = \frac{1}{N} \sum_{i=1}^N \psi(x_i(t), v_i(t)).$

Since the processes $\{(x_i(t), v_i(t)) : 1 ≤ i ≤ N\}$ are independent and identically distributed (i.i.d.), by the ergodic theorem we have a.s.

$\lim_{N \to \infty} \int \psi(x, v) f_N(x, v, t) dx dv = \int \psi(x, v) f(x, v, t) dx dv,$ (11)
where \( f(z, \cdot) \equiv f^u(z, \cdot) \) is the density of the generic agent’s state which evolves according to the SDE (7) with control law \( u(\cdot) \in \mathcal{U} \).

The evolution of \( f^u(z, \cdot) \) (where \( z = [x, v]^T \)) is defined by the Fokker-Planck-Kolmogorov (FPK) equation:

\[
\partial_t f^u(z, t) + \nabla_z \cdot \left( (Fz + Gu) f^u(z, t) \right) = \frac{1}{2} \text{tr}(DD^T \triangle f^u(z, t)),
\]

where we use \( f^u \) to indicate the dependence of \( f \) on the control law \( u \).

Now by substituting the optimal control \( u^\circ(\cdot) \) into the above FPK equation we get the (forward in time) nonlinear deterministic PDE:

\[
\partial_t f(z, t) + \nabla_z \cdot \left( (Fz - \frac{1}{2} GR^{-1}G^T \nabla_z h(z, t)) f(z, t) \right) = \frac{1}{2} \text{tr}(DD^T \triangle f(z, t)),
\]

where \( f(z, 0) = f_0(x, v) \), and \( h(z, \cdot) \) is the solution of the equation (10). Moreover, we have the optimal cost \( \rho^o = \int_{\mathbb{R}^n} \left( \phi(z(u^\circ)) + (u^\circ)^T Ru^\circ \right) f_\infty(z) dz < \infty \), where \( f_\infty(z) \) is the steady-state solution of the PDE (12).

### 4.4 Mean Field Cost-Coupling Function

For a generic agent \( i \) \( (1 \leq i \leq N) \) the ergodic equation in (11) suggests the approximation of \( \phi^N(z_i; z^o_i) \) (where \( z^o_i \) is the state of all agents \( \{j : 1 \leq j \leq N\} \) distinct from agent \( i \) which evolve according to the SDEs (5) with optimal control law \( u^\circ|_{z=z_i}(\cdot) \equiv u^\circ_i(\cdot) \) for a large \( N \) population system by

\[
\bar{\phi}(z, t) = \int_{\mathbb{R}^n} w(\|x - x'\|(v' - v)f(x', v', t)dx'dv')d\mathcal{Q},
\]

where \( f(x, v, \cdot) \) is the solution of the equation (12) (i.e., the population density under the optimal control \( u^\circ(\cdot) \)).

### 4.5 Mean Field System of Equations

In this section we aim to construct the consistency relationship (between the individual strategies and the mass influence effect) in the stochastic MF control theory (based on the approach developed in (Yin et al., 2010) after (Huang et al., 2007)). The key idea is to prescribe a spatially averaged mass function \( \bar{\phi}(z, \cdot) \) characterized by the property that it is reproduced as the average of all agents’ states in the continuum of agents whenever each individual agent optimally tracks the same mass function \( \phi(z, \cdot) \).

Considering the continuum population limit (i.e., as \( N \) approaches \( \infty \)) of the dynamic game flocking model (5)-(6) we obtain the following Non-Linear continuum based Mean Field (NLMF) equation system:

\[
\partial_t h(z, t) + \left( Fz - \frac{1}{4} GR^{-1}G^T \nabla_z h(z, t) \right) \cdot \nabla_z h(z, t) + \phi(z, t) + \frac{1}{2} \text{tr}(DD^T \triangle h(z, t)) = \rho^o, \quad \text{(13)}
\]

\[
\partial_t f(z, t) + \nabla_z \cdot \left( (Fz - \frac{1}{2} GR^{-1}G^T \nabla_z h(z, t)) f(z, t) \right) = \frac{1}{2} \text{tr}(DD^T \triangle f(z, t)), \quad \text{(14)}
\]

\[
\bar{\phi}(z, t) = \int_{\mathbb{R}^n} w(\|x - x'\|(v' - v)f(x', v', t)dx'dv')d\mathcal{Q}, \quad \text{(15)}
\]

where \( f(z, 0) = f_0(z) \) is the initial population density and \( \int_{\mathbb{R}^n} f(z, t) dz = 1 \) for any \( t \geq 0 \). We also assume the boundary conditions:

\[
\lim_{|x| \text{ or } |v| \to \infty} f(x, v, t) = 0 \quad \text{or} \quad h(x, v, t) = 0, \quad \forall t \geq 0.
\]

The NLMF system of (coupled) equations consists of:

- The nonlinear (backward in time) MF-HJB equation (10) which describes the HJB equation of a generic agent’s ergodic optimal problem (7)-(8) with cost coupling \( \phi(z, \cdot) \);
- The the nonlinear (forward in time) MF-FFPK equation (12) which describes the evolution of the population density with the optimal control law

\[
u^\circ(\cdot) := -\frac{1}{2} R^{-1}G^T \nabla_z h(z, \cdot), \quad \text{(16)}
\]

- The (spatially averaged) MF-CC (Cost-Coupling) (15) which is the cost coupling aggregate effect of the agents in the infinite population limit.

### 4.6 ε-Nash Equilibrium Property

In this section we assume that there exists a unique solution to the NLMF system of equations (13)-(15). In a finite \( N \) population system we assume that the \( i^{th} \) agent applies the continuum (i.e., infinite population) based MF control input:

\[
u^\circ_i(\cdot) := u^\circ|_{z=z_i}(\cdot) = -\frac{1}{2} R^{-1}G^T \nabla_z h(z, \cdot)|_{z=z_i}, \quad \text{(17)}
\]

where \( h(z, \cdot) \) is the solution of the MF-HJB equation (13). Hence, the closed-loop dynamics of the \( i^{th} \) in the finite \( N \) population system is:

\[
dz^o_i = (Fz^o_i - \frac{1}{2} GR^{-1}G^T \nabla_z h|_{z=z_i}) dt + Dd\omega_i, \quad \text{(18)}
\]

or

\[
\begin{align*}
dz^o_i(t) &= v^o_i(t) dt, \\
dv^o_i(t) &= -\frac{1}{2} R^{-1} \nabla_v h(v, t)|_{v=v_i} dt + C d\omega_i(t),
\end{align*}
\]

where \( x_i(0) \) and \( v_i(0) \) are given.

Under assumption (A1) we showed in (Nourian et al., 2011a) that the set of the continuum based control laws \( \{u^\circ_i : 1 \leq i \leq N\} \) in (17) obtained from (13)-(15) is an almost sure \( \epsilon_N \)-Nash equilibrium for the stochastic system in the finite population, (6)-(18). That is to say,

\[
J_i^N(u^\circ_i, u^\circ_{-i}) - \epsilon_N \leq \inf_{u_i \in \mathcal{U}_i} J_i^N(u_i, u^\circ_{-i}) \leq J_i^N(u^\circ_i, u^\circ_{-i}),
\]

almost surely, where \( u^\circ_{-i} := (u^\circ_1, \cdots, u^\circ_{i-1}, u^\circ_{i+1}, \cdots, u^\circ_N) \) and \( \mathcal{U}_i \) is the admissible control set of the \( i^{th} \) agent. Moreover, \( \lim_{N \to \infty} \epsilon_N = 0 \), almost surely.
5. ANALYSIS OF LINEAR COST COUPLING CASE

In (4) we let $\beta = 0$ (i.e., linear coupling cost case) in the definition of $w_{ij}(\cdot)$ in (2). For simplicity of analysis we consider the scalar system case (3) and we let $C = \sigma$, a positive constant, and in (4) we let $R = Q = I$.

Therefore, the NLMF system of equations (13)-(15) takes the form,
\[
\partial_t h(x,v,t) + v\partial_x h(x,v,t) - \frac{1}{4}(\partial_v h(x,v,t))^2 + \phi(v,t) + \frac{1}{2}\sigma^2 \partial_v^2 h(x,v,t) = \rho^\sigma,
\]
\[
\partial_t f(x,v,t) + v\partial_x f(x,v,t) - \frac{1}{2}\sigma^2 \partial_v^2 f(x,v,t) = \frac{1}{2}\partial_v (\partial_v h(x,v,t)) f(x,v,t),
\]
\[
\phi(v,t) = \left( \int_{\mathbb{R}^2} (v' - v) f(x,v',t) dxdv' \right)^2.
\]

Since the (spatially averaged) MF-CC function $\bar{\phi}(v,\cdot)$ in (21) is independent of the variable $x$, the solutions of the MF-HJB equation (19) and the MF-FPK equation (20) (and $h$ and $f$ respectively) are independent of variable $x$. Therefore, the terms $\partial_x h$ and $\partial_x f$ are zero, and the optimal control is
\[
u^\sigma(v,\cdot) := -\frac{1}{2}\partial_h h(v,\cdot).
\]

5.1 Stationary Solution

In the stationary setting, the NLMF equation system (19)-(20) takes the form:
\[
\frac{1}{4}(\partial_v h_\infty(v))^2 - \frac{\sigma^2}{2} \partial_v^2 h_\infty(v) = \bar{\phi}(v) - \rho^\sigma,
\]
\[
\frac{1}{2} \partial_v \left( \partial_v h_\infty(v) \right) f_\infty(v) = -\frac{\sigma^2}{2} \partial_v^2 f_\infty(v),
\]
\[
\bar{\phi}(v) = \left( \int_{\mathbb{R}} (v' - v) f_\infty(v') dv' \right)^2.
\]

Theorem 1. (Nourian et al., 2011b) For any arbitrary $\mu \in \mathbb{R}$, there exists the following solution of the stationary equation system (23)-(25):
\[
h_\infty(v) = (v - \mu)^2,
\]
\[
f_\infty(v) = \frac{1}{\sqrt{2\pi}s^2} \exp \left( -\frac{(v - \mu)^2}{2s^2} \right), s^2 := \frac{\sigma^2}{2},
\]
\[
\bar{\phi}_\infty(v) = (v - \mu)^2,
\]
where $h_\infty(v)$ is defined up to a constant.

The asymptotically linearized stability of the NLMF equation system (19)-(20) around the stationary solution (26)-(28) determines the solution of equation (19) uniquely as
\[
h(v,t) = h_\infty(v) = (v - \mu)^2, t \geq 0,
\]
where
\[
\mu := \int_{\mathbb{R}} v f_0(v) dv,
\]
the initial state population mean (see Nourian et al., 2011b)).

5.2 Mean-Consensus

Definition: (Nourian et al., 2010a) Mean-consensus is said to be achieved asymptotically for a group of $N$ agents if
\[
\lim_{t \to \infty} \|Ez_i(t) - Ez_j(t)\| = 0 \text{ for any } i \text{ and } j, 1 \leq i \neq j \leq N.
\]

Using (29) for a finite $N$ population system (5)-(6) yields the set of control laws ($1 \leq i \leq N$)
\[
u^\sigma_i(t) := -\frac{1}{2}\partial_v h(v_i(t)) |_{v = v_i} = -(v_i(t) - \mu),
\]
where $\mu$ is given in (30).

Applying the MF control laws (31) to the agents' dynamics (5) yields ($1 \leq i \leq N$)
\[
dx_i^\sigma(t) = v_i^\sigma(t) dt,
\]
\[
2v_i^\sigma(t) = -(v_i^\sigma(t) - \mu) dt + \sigma d\omega_i(t), t \geq 0,
\]
which give us the solutions:
\[
x_i^\sigma(t) = x_i(0) + \int_0^t v_i^\sigma(\tau) d\tau,
\]
\[
v_i^\sigma(t) = \mu + e^{-t}(v_i(0) - \mu) + \sigma \int_0^t e^{-(t-\tau)} d\omega_i(\tau),
\]
for $t \geq 0$ and $1 \leq i \leq N < \infty$.

Theorem 2. (Nourian et al., 2011b) By applying the continuum based MF control laws (31) in a finite population DBCM (5)-(6), a mean-consensus in velocity is reached asymptotically (as time goes to infinity) with individual asymptotic variance $\sigma^2/2$.

6. PERTURBATION EQUATIONS OF THE NON-LINEAR MEAN FIELD EQUATION SYSTEM

In this section we study the full NLMF equation system (13)-(15) with the choice of parameters in (19)-(21) but with $\beta$ in (2) not necessarily equal to zero:
\[
\partial_t h(x,v,t) + v\partial_x h(x,v,t) - \frac{1}{4}(\partial_v h(x,v,t))^2 + \phi(v,t) + \frac{1}{2}\sigma^2 \partial_v^2 h(x,v,t) = \rho^\sigma,
\]
\[
\partial_t f(x,v,t) + v\partial_x f(x,v,t) - \frac{1}{2}\sigma^2 \partial_v^2 f(x,v,t) = \frac{1}{2}\partial_v (\partial_v h(x,v,t)) f(x,v,t),
\]
\[
\phi(v,t) = \left( \int_{\mathbb{R}^2} (v' - v) f(x,v',t) dxdv' \right)^2.
\]

By taking the approach of (Nourian et al., 2011b) (after Guéant, 2009) we present the perturbation equations of the NLMF system of equations (32)-(34) around the Gaussian stationary solution (26)-(28). In this nonlinear system we let the perturbation of the solution be
\[
h(x,v,t) = h_\infty(v) + \epsilon h_z(x,v,t),
\]
\[
f(x,v,t) = f_\infty(v) + \epsilon f_z(x,v,t),
\]
\[
\phi(x,v,t) = \bar{\phi}_\infty(v) + \epsilon \bar{\phi}_z(x,v,t),
\]
for $x, v \in \mathbb{R}$ and $t \geq 0$, where $h_\infty$, $f_\infty$ and $\bar{\phi}_\infty$ are defined in (26)-(28), and $f_z(x,v,0)$ and $h_z(x,v,0)$ represent the perturbations on $f_\infty(v)$ and $h_\infty(v)$.

Remark: The reason why we take the relative perturbation form of the density function $f$ in (36) is to employ the Hermite series expansion for the resulting linearized equation system (see Nourian et al., 2011b)).
Since $f$ is a probability density, by (36) we have
\[
\int_{\mathbb{R}^2} f_\infty(v) f_\epsilon(x, v, t) \, dx \, dv = 0, \quad t \geq 0,
\]
\[
\int_{\mathbb{R}^2} v f(x, v, 0) \, dx \, dv = \mu + \ldots.
\]

Theorem 3. (Nourian et al., 2011a) The linearization of the NLMF equation system (32)-(34) around the stationary Gaussian solution (26)-(28) takes the form
\[
\begin{align*}
\partial_h h(x, v, t) + v \partial_v h(x, v, t) &= -\phi_h(x, v, t), \\
\partial_t f(x, v, t) + v \partial_v f(x, v, t) &= 0,
\end{align*}
\]
where $f(x, v, 0)$ is given.

We define the operator $\mathcal{L}_v$ as
\[
\mathcal{L}_v g(v) := (v - \mu) \partial_v g(v) - \frac{\sigma^2}{2} \partial^2_{vv} g(v).
\]
It is known that the Hermite polynomials $\{H_n(v) : n \in \mathbb{N}_0\}$ are the eigenfunctions of the operator $\mathcal{L}_v$ in such a way that $\mathcal{L}_v H_n(v) = n H_{n-1}(v)$ for any $n \in \mathbb{N}_0$ (Nourian et al., 2011b) after (Guéant, 2009).

By using the operator $\mathcal{L}_v$ we can rewrite the PDEs (40)-(41) as
\[
\begin{align*}
\partial_h h(x, v, t) + v \partial_v h(x, v, t) &= \mathcal{L}_v h(x, v, t) - \phi_h(x, v, t), \\
\partial_t f(x, v, t) + v \partial_v f(x, v, t) &= \frac{1}{\sqrt{2}} \mathcal{L}_v h(x, v, t) - \mathcal{L}_v f(x, v, t),
\end{align*}
\]
where $f(x, v, 0)$ is given.

The results concerning the analysis of the perturbed coupled equation system ((42) and (43)-(44)) via Hermite-Fourier polynomials expansion will be reported in future work.

REFERENCES


