Optimal Sliding Mode Control via Penalty Approach for Discrete-Time Linear Systems

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Abstract: This paper introduces a penalty approach and quadratic cost criterion to deal with the optimal sliding mode control problem for linear discrete-time systems. The solution provided is a linear state feedback control law based on a recursive Riccati equation defined in a unified framework. A numerical example is shown to illustrate the effectiveness of this new approach.

Keywords: Sliding-mode control, optimal control, least-squares problem, penalty functions, Riccati equation.

1. INTRODUCTION

This paper deals with optimal sliding mode control of discrete-time systems. The existence of a sliding mode requires the stability of the state trajectories considering a sliding surface. A switching control law can be designed to ensure that the states reach and remain in the sliding surface, see for instance Utkin (1992). This kind of optimal discrete-time sliding mode control is featured with the ability to robustly stabilize linear discrete-time systems in order to minimize a quadratic performance index, see for instance Mu et al. (2007), Xu (2007), and Gao et al. (1995).

We propose in this paper an alternative procedure to find the optimal recursive sliding mode control for discrete-time linear systems proposed in Xu (2007). The new framework developed is based on a combination of weighted least-squares problem Kailath et al. (2000) and penalty functions Albert (1972), Bazaar et al. (1993), and Luemberger (2003). An important issue should be emphasized in the new formulation proposed in this paper: the way that the linear constraints are incorporated via penalty function in the functional to be minimized.

The combination of these techniques to solve constrained optimization problems has made possible to deal, in an appropriate way, with recursive solutions of some control and filtering problems of uncertain linear systems, see for instance Bianco et al. (2008), Cerri et al. (2009), and Cerri et al. (2010). In all of them, the solutions provided have the advantage of depending on the discrete-time Riccati equation, which can be solved recursively. Thus, the motivations to develop this new proposal is that it can be more appropriate to explore later the sliding mode control problem for multivariable systems and systems subject to uncertainties in the parameter matrices. Preliminarily, in order to achieve this goal, we intend in this paper to verify effectiveness of this technique to solve the optimal sliding mode control problem.

This paper is organized as follows: Section 2 introduces the linear system, the sliding surface, and the penalized constrained least-squares problem we are dealing with; Section 3 presents the least-squares solution in an alternative array of matrices and the penalty function technique; Section 4 reviews the optimal sliding mode control problem for nominal discrete-time linear systems under the point of view of the new approach we are proposing; and Section 5 provides a numerical example.

The notation used in this paper is standard: $\mathbb{R}$ is the set of real numbers, $\mathbb{R}^n$ is the set of n-dimensional vectors whose elements are in $\mathbb{R}$, $\mathbb{R}^{m \times n}$ is the set of $m \times n$ real matrices, $A^T$ is the transpose of the matrix $A$, $P > 0$ ($P \succeq 0$) denotes a positive definite (semi definite) matrix, $a > 0$ denotes positive scalar, $\|x\|$ is the Euclidean norm of $x$, $\|x\|_p$ is the weighted norm of $x$ defined by $(x^T P x)^{\frac{1}{2}}$, and $x^T P x$ for lack of space.

2. PROBLEM FORMULATION

Consider the nominal discrete-time system given by:

$$x_{k+1} = F x_k + G u_k, \quad k = 0, ..., N,$$

where $x_k \in \mathbb{R}^n$ is the state vector, $u_k \in \mathbb{R}^m$ is the control input, $F \in \mathbb{R}^{n \times n}$ and $G \in \mathbb{R}^{n \times m}$ are known matrices, $x_0$ is the known initial state vector, and $\{u_k\}_{k=0}^N$ is defined as a sequence of inputs without constraint.

Our goal is designing, from techniques based on minimization of a quadratic performance criterion, an optimal sliding mode control that be able to regulate the System

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(1). For this purpose, let the time-varying sliding surface be defined as:
\[ s_k = Cx_k + \phi_k = 0, \quad k = 0, \ldots, N, \]
where \( C \in \mathbb{R}^{1 \times n} \) is a constant row vector such that \( CG \neq 0 \). It is assumed that an ideal sliding mode must satisfy:
\[ s_{k+1} = s_k = 0, \quad \forall k. \]

The optimal sliding mode controller proposed in this paper is obtained from a suitable extension of classical techniques based on minimization of a penalized quadratic performance criterion. Consider for a given penalty parameter \( \mu > 0 \) the following unconstrained minimization problem:
\[
\min_{y_{k+1}, v_k} \left\{ \left( \begin{array}{c}
I & 0 \\
0 & I \\
I & -\mathcal{G}
\end{array} \right) \left[ \begin{array}{c}
y_{k+1} \\
v_k
\end{array} \right] - \left[ \begin{array}{c}
0 \\
-I \\
-I
\end{array} \right] \left[ \begin{array}{c}
y_k \\
0 \\
0
\end{array} \right] \right\}^T \left[ \begin{array}{c}
P & 0 & 0 \\
0 & R & S \mathcal{T} \\
0 & S & Q \end{array} \right] \left[ \begin{array}{c}
y_{k+1} \\
v_k \\
0
\end{array} \right] + \mu P
\]
where the variables \( y_{k+1} \) and \( v_k \) are defined for an extended model of (1),
\[
y_k = \left[ \begin{array}{c}
x_k \\
\phi_k
\end{array} \right] \quad \text{and} \quad v_k = (\phi_{k+1} - \phi_k),
\]
whose weighting matrices \( \mathcal{P}, \mathcal{Q}, \mathcal{R}, \mathcal{S} \), and nominal matrices \( \mathcal{F}, \mathcal{G} \) will be defined in the next sections.

Note that this formulation to solve the optimum sliding mode control problem was established in terms of a penalized weighted cost criterion, see more details of this approach in Cerri et al. (2009) and Cerri et al. (2010).

3. PRELIMINARIES RESULTS

In this section, we present some fundamental results to the solution of a constrained least-squares problem based on techniques of weighted least-squares and penalty functions.

3.1 Weighted Least-Squares Problem

Consider the well-known weighted least-squares problem defined by minimization problem:
\[
\min_{x \in \mathbb{R}^n} \{ J(x) \},
\]
with the quadratic function \( J(x) \) given by
\[
J(x) = ||Ax - b||_V^2 = (Ax - b)^TW(Ax - b),
\]
where \( W \in \mathbb{R}^{n \times n} \) (weighting matrix) is symmetric positive definite, \( A \in \mathbb{R}^{n \times m} \) and \( b \in \mathbb{R}^n \) are assumed known, and \( x \in \mathbb{R}^m \) is the unknown vector. The solution for this problem can be seen in Kailath et al. (2000). As alternative, the next result shows that the solution for this problem admits a simpler and more compact representation.

**Lemma 1.** Suppose that \( W \) is a symmetric non-singular matrix such that \( A^TW \ A \succeq 0 \). Then the following sentences are equivalent:
(i) \( \hat{x} \in \arg \min_x \{ (Ax - b)^TW(Ax - b) \} \).
(ii) \( x = \hat{x} \) is a solution of \( A^TWAx = A^TWh \).
(iii) \( (x, \lambda) = (\hat{x}, \hat{\lambda}) \) is a solution of
\[
\begin{bmatrix}
W^{-1} & \lambda \\
A^T & 0
\end{bmatrix}
\begin{bmatrix}
x \\
\lambda
\end{bmatrix}
= \begin{bmatrix}
b \\
0
\end{bmatrix}.
\]
If \( A \) is full column rank, then
\[
\hat{x} = [0 \ I] \begin{bmatrix}
W^{-1} & \lambda \\
A^T & 0
\end{bmatrix}^{-1} \begin{bmatrix}
b \\
0
\end{bmatrix}.
\]
is the unique solution for equation in (ii) and the minimum value of \( J(x) \) is given by
\[
J(\hat{x}) = [b^T \ 0] \begin{bmatrix}
W^{-1} & \lambda \\
A^T & 0
\end{bmatrix}^{-1} \begin{bmatrix}
b \\
0
\end{bmatrix}.
\]

**Proof.** Omitted.

3.2 Penalty Function

Based on Bazaraa et al. (1993), penalty functions transform constrained in unconstrained optimization problems. The constraints are placed into the objective function via penalty parameter in such way that penalizes any violation of the constraints.

Consider the following constrained optimization problem:
\[
\min_{f(x)} \{ f(x) \}
\]
s.t. \( h(x) = 0 \),
with optimal solution \( x^o \). Suppose that this problem is replaced by:
\[
\min_{f(x)} \{ f(x) + \mu h(x)T h(x) \},
\]
where \( \mu \) is a positive real number. For each \( \mu > 0 \), let \( \hat{x}(\mu) \) be the optimal solution to the problem (7). Then,
\[
x^o = \lim_{\mu \to +\infty} \hat{x}(\mu)
\]
The term \( \mu h(x)^T h(x) \) is referred as penalty function. Details on penalty functions can be seen in Albert (1972), Bazaraa et al. (1993), Luenberger (2003), and Zangwill (1969). For our purpose they are particularly attractive for dealing with quadratic programming problems with linear equality constraint, it transforms constrained in unconstrained optimization problems.

3.3 Constrained Least-Squares Problem

The next lemma shows some important equivalences of this minimization problem we dealing with.

**Lemma 2.** Let \( V \in \mathbb{R}^{n \times n} \) positive definite, \( N \in \mathbb{R}^{k \times m} \) and \( M \in \mathbb{R}^{n \times m} \). Define the following quadratic functional:
\[
J(x) := (Mx - z)^TV(Mx - z)
\]
and consider the constrained minimization problem given by
\[
\min_{x \in \mathbb{R}^m} \{ J(x) \},
\]
s.t. \( N x = w \)
where \( z \in \mathbb{R}^n \), \( x \in \mathbb{R}^m \) and \( w \in \mathbb{R}^k \). Associated with (9), we have for each \( \mu > 0 \) the following unconstrained minimization problem:
\[
\min_{x(\mu) \in \mathbb{R}^m} \{ J(x(\mu)) \},
\]
where
\[
J(x(\mu)) := (Gx(\mu) - B)^TV(\mu)(Gx(\mu) - B).
\]
\[ G = \begin{bmatrix} M \\ N \end{bmatrix}, \quad V(\mu) = \begin{bmatrix} V & 0 \\ 0 & \mu I \end{bmatrix}, \quad B = \begin{bmatrix} z \\ w \end{bmatrix}. \]

Suppose that the matrix \( G \) is full column rank, then the following statements are valid:

(i) for each \( \mu > 0 \), the optimal solution \( \hat{x}(\mu) \) and minimum value \( J(\hat{x}(\mu)) \) associated with (10) are given by

\[
\begin{bmatrix} \hat{x}(\mu) \\ J(\hat{x}(\mu)) \end{bmatrix} = \begin{bmatrix} 0 \\ I \end{bmatrix}^T \begin{bmatrix} V^{-1}(\mu) G^{-1} \\ 0 \end{bmatrix} \begin{bmatrix} B \\ 0 \end{bmatrix},
\]

(ii) \( \lim_{\mu \to +\infty} \hat{x}(\mu) = x^o \) and \( \lim_{\mu \to +\infty} J(\hat{x}(\mu)) = J(x^o) \), where \( x^o \) and \( J(x^o) \) are optimal solution and minimum value, respectively, for (9) and given by

\[
\begin{bmatrix} x^o \\ J(x^o) \end{bmatrix} = \begin{bmatrix} 0 \\ w \end{bmatrix}^T \begin{bmatrix} V^{-1} & 0 \\ 0 & M^{-1} \end{bmatrix} \begin{bmatrix} z \\ w \end{bmatrix}. \tag{11}
\]

\[ \text{Proof. Omitted.} \]

Remark 3. Note that the quadratic term

\[ (N \hat{x}(\mu) - w)^T \mu I (N \hat{x}(\mu) - w) \]

goes to zero when \( \mu \to +\infty \).

The next remark considers the inverse of an array of matrices which is useful to the control approach we are proposing.

Remark 4. Observe the matrixial block

\[ \Gamma = \begin{bmatrix} X & I \\ I & 0 \end{bmatrix}, \]

it is invertible and its inverse is given by

\[ \Gamma^{-1} = \begin{bmatrix} 0 & I \\ I & -X \end{bmatrix}, \]

for any matrix \( X \).

4. NOMINAL SLIDING MODE CONTROL PROBLEM

In this section, we revisit the sliding mode control problem for a nominal discrete-time linear system. We show that the same classical recursive solution provided in Xu (2007) can be obtained through the new approach we are proposing.

4.1 Problem Formulation

Consider again the nominal discrete-time system given by

\[ x_{k+1} = Fx_k + G\phi_k, \quad k = 0, \ldots, N. \tag{12} \]

As it was defined previously, let the time-varying sliding surface be defined as \( s_k = Cx_k + \phi_k = 0, \quad k = 0, \ldots, N, \) where \( C \in \mathbb{R}^{1 \times n} \) with \( CG \neq 0 \). In addition, the ideal sliding mode must satisfy \( s_{k+1} = s_k = 0, \quad \forall k \).

Based on Xu (2007), for the case without disturbance, the sliding mode control is given by:

\[ u_k = u_k^e + u_k^s, \tag{13} \]

where

\[ u_k^s = -(CG)^{-1}(CFx_k + \phi_{k+1}) \]

and

\[ u_k^e = -(CG)^{-1}(rs_k - eT\text{sign}(s_k)), \quad 0 < r < 1, \quad e > |CG| \]

are the equivalent control and switching control, respectively.

Based on standard techniques of quadratic cost criterion, Xu (2007) defines the following quadratic performance index:

\[ J = x_N^T P_{N+1} x_{N+1} + \sum_{j=0}^{N} \Omega_j(x_j, u_j), \tag{14} \]

where the auxiliary term \( \Omega_j(x_j, u_j) \) is given by

\[ \Omega_j(x_j, u_j) := x_j^T Q x_j + u_j^T u_j, \tag{15} \]

with \( P_{N+1} \geq 0 \) and \( Q > 0 \) of appropriated dimensions.

The feedback System (12) with the control law (13) is written as

\[ x_{k+1} = (F - G(CG)^{-1}(CF - rC))x_k - G(CG)^{-1}(\phi_{k+1} - r\phi_k). \tag{16} \]

From a suitable rearrangement of (16), an equivalent linear discrete-time system can be considered as:

\[ y_{k+1} = Fy_k + G\phi_k, \quad k = 0, \ldots, N, \tag{17} \]

with

\[ F = \begin{bmatrix} F - G(CG)^{-1}(CF - rC) \quad G(CG)^{-1}(r - 1) \end{bmatrix}, \]

\[ G = \begin{bmatrix} -G(CG)^{-1} \\ 0 \end{bmatrix}, \]

where the new state variable \( y_k \) and the new input control \( \phi_k \) are defined by

\[ y_k = x_k \quad \text{and} \quad \phi_k = (\phi_{k+1} - \phi_k), \]

respectively. Furthermore, the quadratic performance index (14)-(15) written in this new variables becomes:

\[ J = y_N^T P_{N+1} y_{N+1} + \sum_{j=0}^{N} \tilde{\Omega}(y_j, v_j), \tag{18} \]

where the auxiliary quadratic term \( \tilde{\Omega}(y_j, v_j) \) is given by

\[ \tilde{\Omega}(y_j, v_j) = y_j^T Q y_j + 2y_j^T S v_j + v_j^T R v_j \]

with

\[ P_{N+1} = \begin{bmatrix} P_{N+1}^1 & 0 \\ 0 & 0 \end{bmatrix}, \quad S = \begin{bmatrix} (CG)^{-2} U^T \\ (CG)^{-2}(1 - r) \end{bmatrix}, \]

\[ Q = \begin{bmatrix} Q + (CG)^{-2} U^T U \quad (CG)^{-2} U^T (1 - r) \\ (CG)^{-2}(1 - r) U \quad (CG)^{-2}(1 - r)^2 \end{bmatrix}, \]

\[ R = (CG)^{-2} \quad \text{and} \quad U = (CF - rC). \]

It is well-known that a constrained minimization problem used to deduce optimal regulators is defined to determine an optimal control sequence

\[ v^* = \{ v_0^*, \ldots, v_{N-1}^* \} \]

that minimizes the quadratic cost functional (18) subject to (12), i.e.

\[
\begin{array}{l}
\min_{v_k} \left\{ y_N^T P_{N+1} y_{N+1} + \sum_{j=0}^{N} \tilde{\Omega}(y_j, v_j) \right\} \\
\text{s.t. } y_{k+1} = Fy_k + Gv_k, \quad k = 0, \ldots, N.
\end{array}
\tag{19}
\]
Then, after finding the optimal input sequence $v^*$, from the optimization problem (19), we have the optimal switching function given by:

$$ s_k^* = Cx_k^* + \phi_k^*, $$

$$ \phi_{k+1}^* = \phi_k^* - v_k^*, \phi_0 = -Cx_0. \tag{20} $$

Consequently, the optimal control input sequence that regulates the System (12) will be given from a combination of (13) with (20). See more details in Xu (2007).

### 4.2 Reformulated Problem

In the problem (19), the minimization was done only in terms of the control variable $v_k$. Thus, the optimal trajectory

$$ y^* = \{y_N^*, \ldots, y_{N+1}^*\} $$

is completely determined from (17), by the knowledge of the optimal control sequence $v^*$.

Consider now the minimization problem (19) reformulated as:

$$ \min_{y_{k+1}, v_k} \left\{ y_{N+1}^T P_{N+1} y_{N+1} + \sum_{j=0}^{N} \Omega_j (y_j, v_j) \right\}, \tag{21} $$

subject to $y_{k+1} = Fy_k + Gu_k$, $k = 0, \ldots, N$

which provides an optimal sequence $\{(y_{k+1}^*, v_k^*)\}_{k=0}^N$. Note that there is no problem to consider the minimization variables, state $y_{k+1}^*$ and control $v_k^*$, as a unique variable of constrained minimization problem. In each minimization step $k$, the optimal control law $v_k^*$ and the optimal state $y_{k+1}^*$ can be obtained at the same time and given in terms of the known state vector $y_k^*$.

Both minimization problems proposed, (19) and (21), consist in a particular case of the well-known Linear Quadratic Regulator (LQR) problem. The solution we propose in the following is based on the combination of least-squares problem and penalty functions, following the line of Cerri et al. (2009) and Cerri et al. (2010). This approach follows the idea of quadratic optimization subject to equality constraints which include the nominal system (12) and the optimal sliding surface (2). After these considerations, we have the solution of (21) easily obtained from classical arguments of dynamic programming, see for instance Bertsekas (2005). The next lemma presents a useful procedure to deal with the minimization problem (21).

**Lemma 5.** The constrained minimization problem (21) can be solved recursively through the following procedure:

$$ \min_{y_N, v_N} \left\{ \Omega_0 (y_0, v_0) + \min_{y_{N-1}, v_{N-1}} \left\{ \Omega_1 (y_1, v_1) + \cdots + \right. \right. $$

$$ \left. \left. + \min_{y_{N+1}, v_{N+1}} \left\{ \Omega_N (y_N, v_N) + y_{N+1}^T P_{N+1} y_{N+1} \right\} \right\} \right\}, $$

subject to $y_{k+1} = Fy_k + Gu_k$ for all $k = 0, \ldots, N$.

**Proof.** Omitted.

According to the optimality principle, we have for each step $k = N-1, \ldots, 0$ the following constrained minimization problem:

$$ \min_{y_{k+1}, v_k} \left\{ J_k (y_{k+1}, v_k) \right\}, $$

subject to $y_{k+1} = Fy_k + Gu_k$.

$$ \min_{y_{k+1}, v_k} \left\{ J_k (y_{k+1}, v_k) \right\}, $$

where $J_k (y_{k+1}, v_k)$ is the partial cost defined by

$$ J_k (y_{k+1}, v_k) = y_{k+1}^T P_{k+1} y_{k+1} + \Omega_k (y_k, v_k). \tag{23} $$

Since $y_{k+1}$ and $v_k$ are the unknown variables of the reformulated problem, some algebraic manipulations in expression (23) allow us to rewrite it as the following weighted quadratic functional:

$$ J_k (y_{k+1}, v_k) = $$(20)

and of such way the constrained minimization problem (22) reduces to a typical constrained least-squares problem following the minimization problem defined in (9).

Now, with Lemmas 2 and 5 in mind, we can define the following alternative unconstrained minimization problem for each step $k$:

$$ \min_{y_{k+1}, v_k} \left\{ J_k^\mu (y_{k+1}, v_k) \right\}, $$

where

$$ J_k^\mu (y_{k+1}, v_k) = $$

and the constraints were incorporated in the quadratic form, through of weighting matrix $\mu I$ with $\mu > 0$.

With these considerations in hand we are in position to establish a new framework to design a LQR for sliding mode systems.

**Theorem 6.** Considering the optimization problem (21), the optimal recursive solution is given through the following linear regulator:

$$ \begin{bmatrix} y_{k+1}^* \\ v_k^* \end{bmatrix} = \begin{bmatrix} L_k \\ K_k \end{bmatrix} \begin{bmatrix} y_k^* \\ v_k \end{bmatrix}, k = 0, \ldots, N, \tag{24} $$

where $L_k, K_k$ and $P_k$ are obtained from (25).

**Proof.** This proof can be obtained based on the Lemmas 5, 2, and Remark 4. ✓

**Remark 7.** Combining the optimal input sequence

$$ v^* = \{v_0^*, \ldots, v_{N-1}\} $$

obtained from Theorem 6 with the expressions given in (20), then the optimal sliding mode controller to the System (12) is given by:

$$ \begin{bmatrix} y_{k+1}^* \\ v_k^* \end{bmatrix} = \begin{bmatrix} L_k \\ K_k \end{bmatrix} \begin{bmatrix} y_k^* \\ v_k \end{bmatrix}, $$

$$ s_k^* = Cx_k^* + \phi_k^*, $$

$$ \phi_{k+1}^* = \phi_k^* - v_k^*, \phi_0 = -Cx_0. $$

for each $k = 0, \ldots, N$ where $y_k = \begin{bmatrix} x_k \\ \phi_k \end{bmatrix}$. The state of the closed-loop system is described by the first entry of the vector $y_k^*$. ✓
As it was pointed out in this section we propose an alternative procedure to solve the optimal sliding mode control problem. In the following we show that this procedure is equivalent to the solution presented in Xu (2007).

**Lemma 8.** The recursive equation (24)-(25) can be rewritten as:

\[
y_{k+1}^\ast = \left[ F - G(R + G^T P_{k+1} G) (S^T + G^T P_{k+1} F) \right] y_k^\ast \\
v_k^\ast = G(R + G^T P_{k+1} G)^{-1} (S^T + G^T P_{k+1} F) y_k^\ast,
\]

for all \( k = 0, \ldots, N \), where

\[
\begin{align*}
P_k &= (F - G R^{-1} S^T)^T P_{k+1} \\
&\quad - P_{k+1} G(R + G^T P_{k+1} G)^{-1} G^T P_{k+1} \\
&\quad + (Q - S R^{-1} S^T),
\end{align*}
\]

for all \( k = N, \ldots, 0 \).

**Proof.** The optimal solution \( \begin{bmatrix} y_k^\ast \\ v_k^\ast \end{bmatrix} \) for each \( k = 0, \ldots, N \) can be found through the following linear system:

\[
\begin{bmatrix}
0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
I - P_{k+1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & I & 0 & -R & -S^T & 0 & 0 \\
0 & 0 & I & -S & -Q & 0 & 0 & 0 & 0 \\
I & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 \\
0 & 0 & I & 0 & 0 & 0 & 0 & -G^T & 0
\end{bmatrix}
\begin{bmatrix}
\lambda_1 \\
\lambda_2 \\
\lambda_3 \\
\lambda_4 \\
\lambda_5 \\
\lambda_6 \\
\lambda_7 \\
\lambda_8 \\
\lambda_9 \\
\end{bmatrix}
= \begin{bmatrix}
y_k^\ast \\
r_{k+1}^\ast
\end{bmatrix}
\]

which, in turn, can be rewritten as set of equations

\[
\begin{align*}
I \lambda_2 + I y_{k+1}^\ast &= 0 \\
I \lambda_1 - P_{k+1} \lambda_2 &= 0 \\
I \lambda_3 + 1 v_{k}^\ast &= 0 \tag{28} \\
\lambda_6 &= -I y_k^\ast \tag{29} \\
I \lambda_3 - R \lambda_5 - S^T \lambda_6 &= 0 \tag{30} \\
I \lambda_4 - S \lambda_5 - \lambda_6 &= 0 \tag{31} \\
l y_{k+1}^\ast + G v_k^\ast &= F y_k^\ast \tag{32} \\
I \lambda_1 + I \lambda_7 &= 0 \tag{33} \\
I \lambda_3 - G^T \lambda_7 &= 0. \tag{34}
\end{align*}
\]

From (29), (35), and (36), \( \lambda_3 \) can be considered as:

\[
\lambda_3 = -G^T P_{k+1} \lambda_2. \tag{37}
\]

Combining (30), (31), (32), and (37) we obtain:

\[
v_k^\ast = R^{-1} G^T P_{k+1} \lambda_2 - R^{-1} S y_k^\ast. \tag{38}
\]

Now, considering (28), (34), and (38), \( \lambda_2 \) can be calculated as:

\[
\lambda_2 = (I + G R^{-1} G^T P_{k+1})^{-1} (G R^{-1} S T - F) y_k^\ast. \tag{39}
\]

Replacing (39) in (38) and applying the matrix inversion lemma, we conclude that:

\[
v_k^\ast = -(R + G^T P_{k+1} G)^{-1} (S^T + G^T P_{k+1} F) y_k^\ast. \tag{40}
\]

For the equivalence about closed-loop system, replacing (40) in (33), results

\[
y_{k+1}^\ast = \left[ F - G(R + G^T P_{k+1} G)^{-1} (S^T + G^T P_{k+1} F) \right] y_k^\ast. \tag{41}
\]

The equivalence of the Riccati equation is easily obtained from (23) when we replace (40) and (41), i.e.:

\[
J_k^T (y_{k+1}, v_k) = y_k^\ast P_k y_k^\ast,
\]

where

\[
P_k = (F - G R^{-1} S^T)^T P_{k+1} \\
&\quad - P_{k+1} G(R + G^T P_{k+1} G)^{-1} G^T P_{k+1} \\
&\quad + (Q - S R^{-1} S^T),
\]

\[\diamond\]

## 5. NUMERICAL EXAMPLE

Consider the finite horizon System (12) with \( N = 50 \), characterized by the following parameter and weighting matrices:

\[
\begin{bmatrix}
1.2 & 0.1 \\
-0.5 & 2
\end{bmatrix}, \quad
\begin{bmatrix}
0 \\
1
\end{bmatrix}, \quad
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}, \quad R = 1,
\]

setting \( \epsilon = 0.4 \), \( T = 0.5 \) and \( r = 0.25 \), the initial state

\[
x_0 = \begin{bmatrix}
0 \\
1
\end{bmatrix}, \quad C = \begin{bmatrix}
2 & 1
\end{bmatrix}.
\]

The simulations we perform in this example are based on Theorem 6 and Remark 7. We present in Figures 1 and 2 the behavior of the closed-loop system and sliding mode control law, respectively.
Fig. 1. Closed-loop system.

Fig. 2. Sliding mode control law.

When the horizon \( N \to +\infty \), we observe that \( P_k \) converges to
\[
P = \begin{bmatrix} 180.1939 & 20.4717 & 0 \\ 20.4717 & 6.8016 & 0 \\ 0 & 0 & 0 \end{bmatrix},
\]
and the respective matrix gain \( K \) is given by
\[
K = \begin{bmatrix} 1.3129 & 0.0560 & -0.7500 \end{bmatrix}.
\]

The same result is obtained when we consider the structure presented in Lemma 8.

6. CONCLUSION

This paper proposed an alternative framework to compute optimal sliding mode controllers for discrete-time linear systems. The main feature of this approach is that the parameter matrix of the feedback system, the controller gain and the solution of the Riccati equation are presented in a unified array of matrices. Equivalent to the standard solution provided in Xu (2007), the recursiveness of the algorithm remains valid. It is worth emphasize that the redefinition of this problem is useful to deal with multivariable cases and with the System (12) subject to uncertainties in the parameter matrices.

REFERENCES


