Boundary Feedback Control of Coupled Hyperbolic Linear PDEs Systems with Nonlinear Boundary Conditions

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Abstract: We try to stabilize equilibrium solutions of physical models described by a class of coupled first-order linear partial differential equations with nonlinear boundary conditions. These systems are distributed parameter systems in which spatio-temporal chaos occurs. In order to stabilize the equilibrium solutions of the systems, at first, we reduce the PDEs to discrete-time systems using the method of characteristics. Then, we design control laws for the discrete-time systems and use them for the original PDEs systems. In this report, we propose a method to get corresponding discrete-time systems for coupled PDEs systems with \(2n \geq 2\) state variables. Furthermore, we apply the method to a coupled time-delayed Chua’s circuit, and design a control law using dynamic feedback linearization.

Keywords: Partial differential equation, Boundary value problem, Chaos, Nonlinear system control, Application of nonlinear analysis and design

1. INTRODUCTION

Recently, controlling spatio-temporal chaos has attracted great interest since the dynamics of spatio-temporal systems is often quite complicated and produces rich patterns with respect to time and space, which cannot be captured by low-dimensional dynamics (Hu and Qu [1994], Kocarev et al. [1997], Huang [2004]). In Suzuki and Sakamoto [2010], for a class of physical models described by a linear wave equation with nonlinear boundary conditions, stabilization of equilibrium solutions and periodic solutions, and synchronization of a pair of the circuits have been tried. These systems involve complex behaviors, which is called ideal turbulence. Ideal turbulence is the notion introduced by Sharkovsky et al. (Sharkovsky [2006], Romanenko and Sharkovskii [2007]), and a complex phenomenon that occurs in distributed parameter systems (DPSs) induced, in particular, by boundary value problems for hyperbolic partial differential equations (PDEs). In systems having ideal turbulence, one can often observe cascade processes of emergence of structures of decreasing scales or processes that even lead to stochastization of the systems. From the viewpoint of utilizing such spatio-temporal chaos, we have tried to apply the synchronization methods of spatio-temporal chaotic systems (time-delayed Chua’s circuits: Sharkovsky [2006]) to a multi-channel spread-spectrum communication system (Suzuki and Sakamoto [2009]). On the other hand, for some engineering applications, e.g., power systems and integrated circuits, it is also important to avoid such complicated phenomena and keep steady states so that the long time prediction becomes easier. This report is concerned with stabilization of equilibrium solutions.

This paper extends the results for the PDE systems with two state variables in Suzuki and Sakamoto [2010] to that for coupled PDEs systems with \(2n \geq 2\) state variables, which includes, e.g., coupled time-delayed Chua’s circuits. To this end, at first, using the method of characteristics, we reduce the PDEs system to a discrete-time system under an assumption for the propagation periods of the signal in the media. Then, we design a control law for the discrete-time system, and apply the control law to the original PDEs system. As an application example, we deal with a coupled time-delayed Chua’s circuit, which consists of two transmission lines with nonlinear (piecewise affine) resistances called Chua’s diode (Chua [1994]). For a nonlinear discrete-time system derived from the PDEs of the circuit by the method of characteristics, we design a control law using dynamic feedback linearization (Marino and Tomei [1991]).

The method of characteristics is a geometric solution method for a certain class of DPSs, especially, first-order hyperbolic PDE systems (Arnold [1988]), and considered as a strong tool for the analysis since this method does not include any approximation such as discretization. In literature Greenberg and Tsiwn [1984], Coron et al. [2007], Prieur et al. [2008], boundary conditions to stabilize equi-
Let us consider a circuit consisting of two transmission lines and Chua’s diodes (Chua [1994]) as shown in Fig. 1.

These transmission lines are denoted by wave equations

\[
\begin{align*}
L_1 \partial_t i_1 + \partial_x v_1 = 0, & \quad x \in [0, 2l], \ t \in \mathbb{R}_+, \\
C_1 \partial_t v_1 + \partial_x i_1 = 0, & \quad x \in [0, 2l], \ t \in \mathbb{R}_+, \\
L_2 \partial_t i_2 + \partial_x v_2 = 0, & \quad x \in [0, l], \ t \in \mathbb{R}_+, \\
C_2 \partial_t v_2 + \partial_x i_2 = 0, & \quad x \in [0, l], \ t \in \mathbb{R}_+, \\
\end{align*}
\]

and nonlinear boundary conditions

\[
\begin{align*}
v_1(0, t) = 0, & \quad \text{where } G_i \text{ represents the voltage-current characteristic of Chua’s diode given as follows (see Fig. 2).} \\
i_1(2l, t) - i_2(0, t) = G_1(v_1(2l, t) - R_1(i_1(2l, t) - i_2(0, t)) - E), & \quad \text{(4)} \\
v_2(0, t) = v_1(2l, t), & \quad \text{(5)} \\
i_2(l, t) = G_2(v_2(l, t) - R_2i_2(l, t) - E), & \quad \text{(6)}
\end{align*}
\]

where \( G_1 \) and \( G_2 \) are the characteristic impedance and the propagation velocity of the transmission line, and given by \( G_1 = \sqrt{L_1/C_1} \) and \( G_2 = 1/\sqrt{L_2/C_2} \), respectively.

3. STABILIZING EQUILIBRIUM SOLUTIONS

3.1 Problem Statement

Following the above example, let us consider an initial and boundary-value problem of coupled first-order linear PDEs

\[
\begin{align*}
& \partial_t p_i + \nu_i \partial_x p_i = 0, \quad x \in [0, l_i], \ i = 1, \ldots, n, \\
& \partial_t q_i - \nu_i \partial_x q_i = 0, \quad x \in [0, l_i], \ i = 1, \ldots, n
\end{align*}
\]

with boundary conditions having control inputs

\[
\begin{align*}
p_i(0, t) = q_i(l_i(t)), & \quad \text{where } f_i, \ i = 1, 2, 3 \text{ are given in the appendix.}
\end{align*}
\]

Although the wave forms in Fig. 3 are complicated, they are continuous functions.

The definition of equilibrium solution will be given at the next section.
Here, $\nu_i (> 0)$ denotes the propagation velocity of signals in the $i$th medium, and $u_i$ is a control input. In fact, (9) and (10) for given an initial function has a unique weak solution in a function space consisting of functions in $H^1(\prod_i [0,l_i], \mathbb{R}^2)$ which satisfy boundary conditions (10). Here $H^1(\prod_i [0,l_i], \mathbb{R}^2)$ is the Sobolev space of $2n$-dimensional vector functions whose derivatives are defined in the sense of distributions. In particular, if the initial condition satisfies the boundary conditions, the solution exists in $C(\prod_i [0,l_i], \mathbb{R}^2)$ (i.e., the solution is a continuous function). We suppose that the initial function satisfies (10) in this paper.

In addition, we consider the following assumption for (9) with (10):

**Assumption:** For each $i$, let $\tau_i := \tau_i/\nu_i$, which denotes the propagation period of signals in the medium. We suppose that, for any pair of $i$ and $j$, $\tau_i/\tau_j$ is rational.\[\square\]

This assumption is equivalent to a condition that there exists a time interval $\Delta T \in \mathbb{R}$ and positive integers $\{N_i\}$ such that each $\tau_i$ can be described as $\tau_i = N_i \Delta T$, $N_i \in \mathbb{N}$.

As mentioned above, our purpose is stabilizing equilibrium solutions, which is defined as follows.

**Definition 1.** A solution of (9) with (10) is said to be an **equilibrium solution** if it takes temporally-constant-distributed-values, that is, a solution $^{4} (p^*, q^*)(p^*, q^*)$ satisfying that $(p^*, q^*)(x, t) = (p^*, q^*) (x, 0)$ for all $t \geq 0$.

Defining a norm $\| \cdot \|$ on $C(\prod_i [0,l_i], \mathbb{R}^2)$ by

$$\|(p, q)(\cdot, t)\| := \sum_{i=1}^{n} \sup_{x \in [0, l_i]} (|p_i(x, t)| + |q_i(x, t)|).$$

we give the definition of the stability of the equilibrium solution as follows.

**Definition 2.** An equilibrium solution $r^* = (p^*, q^*)$ of the system (9) is said to be stable if, for all $\epsilon > 0$, there exists a $\delta > 0$ such that, for any solutions satisfying $\|(p, q)(\cdot, 0) - r^*\| < \delta$, we have $\|(p, q)(\cdot, t) - r^*\| < \epsilon$ for all $t > 0$.

**Definition 3.** An equilibrium solution $r^* = (p^*, q^*)$ of the system (9) is said to be globally asymptotically stable if it is stable and, for any initial condition, we have

$$\lim_{t \to \infty} \|(p, q)(\cdot, t) - r^*\| = 0.$$

Now, we can state the problem which we tackle in this section.

**Problem:** For the system (9) with (10), design a control law

$$u(t) = u(p_1(0, t), q_1(0, t), p_1(l_1, t), \cdots, q_n(l_n, t))$$

that (globally asymptotically) stabilizes an equilibrium solution $r^* = (p^*, q^*) = 0$ by using only the boundary values.

### 3.2 Stabilizing the equilibrium solution

In general, first-order linear PDEs are solvable by using the method of characteristics. For (9), introducing parameters $r, s \in \mathbb{R}$ and integrating the following coupled ordinary differential equations (ODEs) with appropriate initial conditions

$$\frac{dt}{dr} = 1, \quad \frac{dz}{dr} = \nu_i, \quad \frac{dp_i}{dr} = 0, \quad \frac{dq_i}{dr} = 0,$$

we can find solutions of the PDE (9). The ODEs (11) and (12) have trajectories each of which takes constant values along lines $t = x/\nu_i$ (const. and $t + x/\nu_i = \text{const.}$, respectively. Note that, for each $i$, the time interval, which a state value spends propagating from one of the boundaries (e.g., $x = 0$) to the opposite boundary ($x = l_i$), is $\tau_i$. Therefore, $p_i(0, t)$ and $q_i(l_i, t)$ can be represented by

$$p_i(0, t) = p_i(l_i, t + \tau_i), \quad q_i(l_i, t) = q_i(0, t + \tau_i).$$

Then, a set of the boundary conditions (10) becomes

$$\begin{pmatrix} p_i(l_i, t + \tau_i) \\ q_i(0, t + \tau_i) \end{pmatrix} = H_i(p_i(l_i, t), \cdots, q_i(l_i, t), q_i(0, t), \cdots, q_n(0, t), u_i(t)), \quad i = 1, \cdots, n,$$

which means a difference equation with continuous arguments. This equation can be regarded as an evolution equation. Although (13) is well-posed for some appropriate initial condition, capturing the dynamics is difficult since the time differences $\tau_i$ generally take different values with respect to $i$. If these time differences take a same single value, it is easier to analyze the system (13) and design a control law.

To overcome this matter (i.e., to make the time differences the same size), for each $i$, divide the medium with the length $l_i$ into $N_i$ equal parts

$$\begin{cases} \frac{\partial p_{i,1}}{\partial t} + \nu_i \frac{\partial p_{i,1}}{\partial x} = 0, & x \in [0, l_i/N_i] \\ \frac{\partial q_{i,1}}{\partial t} - \nu_i \frac{\partial q_{i,1}}{\partial x} = 0, & x \in [0, l_i/N_i] \end{cases}$$

$$\begin{cases} \frac{\partial p_{i,2}}{\partial t} + \nu_i \frac{\partial p_{i,2}}{\partial x} = 0, & x \in [l_i/N_i, 2l_i/N_i] \\ \frac{\partial q_{i,2}}{\partial t} - \nu_i \frac{\partial q_{i,2}}{\partial x} = 0, & x \in [l_i/N_i, 2l_i/N_i] \end{cases}$$

$$\vdots$$

$$\begin{cases} \frac{\partial p_{i,N_i}}{\partial t} + \nu_i \frac{\partial p_{i,N_i}}{\partial x} = 0, & x \in [(N_i - 1)l_i/N_i, l_i] \\ \frac{\partial q_{i,N_i}}{\partial t} - \nu_i \frac{\partial q_{i,N_i}}{\partial x} = 0, & x \in [(N_i - 1)l_i/N_i, l_i] \end{cases}$$

and set a new boundary conditions by adding artificial boundary conditions at the dividing points, which mean that state values pass through these points without change:

$$p_{i,1}(0, t) = H_i^p(p_{i,1,N_i}(l_i, t), \cdots, p_{i,N_i}(l_i, t), q_{i,1}(0, t), \cdots, q_{i,N_i}(0, t), u_i(t))$$

$$q_{i,1}(0, t) = q_{i,2}(l_i/N_i, t)$$

$$p_{i,2}(l_i/N_i, t) = p_{i,1}(l_i/N_i, t)$$

$$q_{i,2}(l_i/N_i, t) = q_{i,3}(2l_i/N_i, t)$$

$$\vdots$$

$$p_{i,N_i}(l_i, t) = p_{i,N_i-1}(l_i, t)$$

$$q_{i,N_i}(l_i, t) = H_i^q(p_{i,N_i}(l_i, t), \cdots, p_{i,1,N_i}(l_i, t), q_{i,1}(0, t), \cdots, q_{i,N_i}(0, t), u_i(t)).$$

Here, $H_i = [H_i^p \ H_i^q]^T$. From the method of characteristics, we have $\Delta H = [\Delta H_i^p \ \Delta H_i^q]^T$.
\[\begin{align*}
\alpha_{i,1}(t+\Delta T) &= H_p^i(\alpha_{1,N_1}(t), \ldots, \alpha_{n,N_n}(t), \\
\beta_{1,1}(t+\Delta T) &= \beta_{1,2}(t) \\
\alpha_{i,2}(t+\Delta T) &= \alpha_{i,1}(t) \\
\beta_{2,2}(t+\Delta T) &= \beta_{2,3}(t) \\
&\vdots \\
\alpha_{i,N_i}(t+\Delta T) &= \alpha_{i,N_i-1}(t) \\
\beta_{N_i,N_i}(t+\Delta T) &= H_h^i(\alpha_{1,N_1}(t), \ldots, \alpha_{n,N_n}(t), \\
&\beta_{1,1}(t), \ldots, \beta_{n,1}(t), u_i(t))
\end{align*}\]

(14)

where, for the sake of simplicity, we let

\[\begin{align*}
\alpha_{i,j}(t) &:= p_{i,j}(t/N_i, t), \\
\beta_{i,j}(t) &:= q_{i,j}(t) + \Delta T/\nu_i, \\
j &= 1, \ldots, N_i, \quad i = 1, \ldots, n.
\end{align*}\]

Thus, expanding the system, we derive the difference equation whose step time is given by the value \(\Delta T\) not depending on the index \(i\). Then, the behavior of (14) can be characterized by the following discrete-time dynamical system:

\[\begin{align*}
\alpha_{i,j}[k+1] &= H_p^i(\alpha_{1,N_1}[k], \ldots, \alpha_{n,N_n}[k], \\
\beta_{1,j}[k+1] &= \beta_{1,2}[k] \\
&\vdots \\
\alpha_{i,j}[k+1] &= \alpha_{i,N_i}[k] \\
\beta_{N_i,j}[k+1] &= H_h^i(\alpha_{1,N_1}[k], \ldots, \alpha_{n,N_n}[k], \\
&\beta_{1,1}[k], \ldots, \beta_{n,1}[k], u_i[k])
\end{align*}\]

(15)

The following fact can be confirmed easily.

**Proposition 4.** For an equilibrium point of the discrete-time system (15), \((\alpha^*, \beta^*)\), we define a set of functions as follows:

\[p_i^*(x,t) \equiv \alpha_{i,1}^*, q_i^*(x,t) \equiv \beta_{i,1}^*, \quad i = 1, \ldots, n.\]

Then, this is an equilibrium solution for (9), (10) without inputs. Conversely, each equilibrium solution of (9), (10) is represented by an equilibrium point of (15) in the form (16).

Now, we are in the stage to design a control law to stabilize an equilibrium point \((p^*, q^*)\) of (9) with (10). The basic idea is not to design a control law for controlling the state \((p,q)\) directly, but to keep \((\alpha, \beta)(t)\) in the neighborhood of \((\alpha^*, \beta^*)\). At first, for (15), we design a control law to converge \((\alpha, \beta)(t)\) to \((\alpha^*, \beta^*)\), and then, apply the control law to the original system (9), (10). The following two propositions guarantee the validity of this strategy.

**Proposition 5.** Assume that, for the discrete-time system (15) there exists a state feedback law \(u[k] = \hat{u}(\alpha, \beta)[k]\) such that, for the equilibrium point \((\alpha^*, \beta^*)\), \(\hat{u}\) locally stabilizes \((\alpha^*, \beta^*)\) (that is, for all \(\varepsilon > 0\), there exist \(\delta > 0\) and a state feedback input \(u[k] = \hat{u}(\alpha, \beta)[k]\) such that, if \(|(\alpha, \beta)[0] - (\alpha^*, \beta^*)| < \delta\), we have \(|(\alpha, \beta)[k] - (\alpha^*, \beta^*)| < \varepsilon\) for all \(k > 0\). Then, using this control law \(\hat{u}\) for the IBVP (9) as follows:

\[u(t) := \hat{u}((\alpha, \beta)(t)), \quad t \in \mathbb{R},\]

the equilibrium solution \((p^*, q^*)\) of the IBVP is locally stabilized.

\[\text{Proof.}\] Let \(V_\varepsilon(x) \subseteq \mathbb{R}^n, \quad m = \sum_{i=1}^n N_i\) be \(\varepsilon\)-neighborhood around \(x \in \mathbb{R}^m\). If the initial sets of (14) satisfy

\[\{(\alpha, \beta)(t)\}_{t \in [0,\Delta T]} \subseteq V_\delta((\alpha^*, \beta^*)),\]

then, under the control law \(\hat{u}\), it turns out that

\[\{(\alpha, \beta(k\Delta T + t))\}_{t \in [0,\Delta T]} \subseteq V_\delta((\alpha^*, \beta^*)), \quad k = 1, 2, \ldots,\]

Therefore, for all \(t \geq 0\), we have

\[\{(\alpha, \beta(t))\}_{t \geq 0} \subseteq V_\varepsilon((\alpha^*, \beta^*)).\]

The states of the interior of the distributed system are considered as follows. Using the method of characteristics, we find that, for each \(i, j\)

\[p_{i,j}(x,t) = \alpha_{i,j}(t + \Delta T - x/\nu_i), \quad x \in [0, l_i/N_i]\]

It turns out that, for all \(t\),

\[\|(p,q)(t) - (p^*, q^*)\| < 2\varepsilon.\]

**Proposition 6.** If the equilibrium point of (15) can be globally asymptotically stabilized by a state feedback law \(u[k] = \hat{u}(\alpha, \beta)[k]\), then, by letting \(u(t) = \hat{u}((\alpha, \beta)(t))\), the equilibrium solution of the IBVP (9) with (10) can be globally asymptotically stabilized.

**Proof.** The local stability of the equilibrium solution is guaranteed by Proposition 5. Here, we confirm that each trajectory with an arbitrary initial condition converges to the equilibrium solution.

From the global asymptotically stability of (15), for each element \((\alpha, \beta)[0]\) and all \(\varepsilon > 0\) there exists a \(N = N((\alpha, \beta)[0])\) such that \(|(\alpha, \beta)[k] - (\alpha^*, \beta^*)| < \varepsilon\) for all \(k \geq N\). When we give a smooth function as an initial function of the PDE (9) with (10), an initial set for the difference equation (13) is determined. Since this initial set is bounded, closed and connected, there exists the finite supremum of \(N, \bar{N} < \infty\), when one fixes \(\varepsilon\) and varies the value \((\alpha, \beta)[0]\) in the initial set \(\delta\). Therefore, we have

\[|(\alpha, \beta)(t) - (\alpha^*, \beta^*)| < \varepsilon, \quad t \geq \bar{N}.\]

Consequently, it turns out that

\[\|(p,q)(t) - (p^*, q^*)\| < 0, \quad (t \to \infty),\]

\[\text{as} \quad \text{t} \to \infty.\]

\[\text{Proof.}\] Actually, it is possible by taking the initial condition of (9) close to the equilibrium solution \((p^*, q^*)\). We can confirm this using a proof by contradiction: If we assume that \(\sup_{(\alpha, \beta)[0]} N = \infty\), then it turns out that there exist points in the initial set that do not converge to the \(\varepsilon\)-neighborhood around \((\alpha^*, \beta^*)\).
and hence, the equilibrium solution of (9) with (10) is globally asymptotically stable.

In this way, the existing methods on global stabilization of discrete-time nonlinear systems can be applied to global stabilization of coupled hyperbolic linear PDEs systems.

Remark 7. If designed feedback law for (15) uses all of the state values \((\alpha, \beta)\), one has to use the states at the artificial boundary points for controlling the original PDEs system. It is always possible to estimate these values by storing the state values at real boundaries, which surround the hypothetical ones, since the signals propagate with constant values.

3.3 An application: Stabilization of an equilibrium solution of the coupled time-delayed Chua’s circuit

For the coupled time-delayed Chua’s circuit introduced in Sec. 2, consider stabilizing an equilibrium solution of the system with control inputs at the left boundary and the right boundary. Then, the system is described by

\[
\begin{align*}
\frac{\partial u_1}{\partial t} + \frac{\partial v_1}{\partial x} &= f_1(\xi_1, \xi_6) - \xi_6, \\
\frac{\partial v_1}{\partial t} &= g_1(\eta_1, \eta_6) \\
\end{align*}
\]

with boundary conditions

\[
\begin{align*}
v_1(0, t) &= u_1(t), \\
v_1(2l, t) &= i_2(t) \\
\end{align*}
\]

where \(\{\xi^*_i\} \) is the equilibrium point \(^8\) of (19), and \(g_1, g_3\) is given by

\[
g_1(\eta_3, \eta_6) = f_1(\xi_1, \xi_6) - \xi_6, \\
g_3(\eta_5 - \eta_4) = f_3(\xi_5 - \xi_6).
\]

For (20), using the dynamic state feedback linearization (Marino and Tomei [1991]), we design a control law \(u_1\) and \(u_2\) to stabilize an equilibrium point \(\eta = 0\) of (20). At first, to erase the nonlinearity at the last row in (20), we find \(v[k]\) as follows.

\[
u_2[k] = ((Z_2 + R_2)/Z_2)(-g_3(\eta_5[k - \eta_4[k] + \bar{u}_2[k])),
\]

where \(\bar{u}_2\) is an input variable we can give arbitrarily. Next, based on the technique of the dynamic state feedback linearization, extending the system (20) by adding a dynamics with respect to input \(u_2\)

\[
u_2[k + 1] = w[k].
\]

We treat \(\bar{u}_2\) like a state variable. Then, changing variables by a nonlinear transformation \(^9\) \(\Phi: \eta \mapsto \zeta\)

\[
\zeta_1 = \eta_1, \quad \zeta_2 = g_1(\eta_2, \eta_6) + \eta_3, \quad \zeta_3 = \phi^{-1}(\eta_1, u_2) + \eta_4
\]

we have

\[
\begin{bmatrix}
\zeta_1 \\
\zeta_2 \\
\zeta_3 \\
\zeta_4 \\
\zeta_5 \\
\zeta_6 \\
\zeta_7
\end{bmatrix}_{k+1}
= \begin{bmatrix}
\zeta_2 \\
\zeta_3 \\
\zeta_4 \\
\zeta_5 \\
\zeta_6 \\
\Phi^{-1}_3(\zeta_2, \zeta_6) + \zeta_8 \\
\Phi^{-1}_3(\zeta_2, \zeta_6) + \zeta_8
\end{bmatrix}_{k} + \begin{bmatrix}
\zeta_1 \\
\zeta_2 \\
\zeta_3 \\
\zeta_4 \\
\zeta_5 \\
\zeta_6 \\
\zeta_7
\end{bmatrix}_{k}
\]

The fifth row in (21) does not affect the other rows and converges to 0 as \(\zeta_6\) and \(\zeta_8\) go to the origin. Therefore, the fifth row can be neglected. Then, for a new variable \(v\), which is arbitrarily given as an input, \(u_1\) is chosen so that

\[
v[k] = g_1(\zeta_4[k] + u_1[k], w[k]) + \zeta_4[k] + u_1[k]
\]

is satisfied. Actually, \(u_1\) is given as follows.

\[
u_1[k] = U_1(\zeta_4[k], v[k], w[k])
\]

\[
\begin{bmatrix}
\zeta_4 \\
\zeta_5 \\
\zeta_6 \\
\zeta_7
\end{bmatrix}_{k+1}
= \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\zeta_4 \\
\zeta_5 \\
\zeta_6 \\
\zeta_7
\end{bmatrix}_{k}
\]

Here, \(\Phi^{-1}\) is well-defined, and the inverse transformation is

\[
\eta_1 = \phi^{-1}(\zeta_1, \zeta_7), \quad \eta_2 = \zeta_4, \quad \eta_4 = \phi^{-1}(\zeta_4, \zeta_8)
\]

\[
\begin{bmatrix}
\zeta_4 \\
\zeta_5 \\
\zeta_6 \\
\zeta_7
\end{bmatrix}_{k+1}
= \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\zeta_4 \\
\zeta_5 \\
\zeta_6 \\
\zeta_7
\end{bmatrix}_{k}
\]

\(^8\) \(\xi^*_i\) is the equilibrium point \(\eta = 0\) of (19), and \(g_1, g_3\) is given by

\[
g_1(\eta_3, \eta_6) = f_1(\xi_1, \xi_6) - \xi_6, \\
g_3(\eta_5 - \eta_4) = f_3(\xi_5 - \xi_6).
\]

\(^9\) In fact, this transformation is well-defined, and the inverse transformation is
This system is not only controllable, but also stable. Therefore, for example, we can design control inputs $v$ and $w$ as follows.

$$v[k] = 0, \quad w[k] = 0.$$  

We give a summary of our control law for (19):

$$u_1[k] = U_1(\xi_2[k] - \xi_2^*, 0, 0) = -\xi_2[k] + \xi_2^*$$

$$u_2[k] = \left((-Z_2 + R_2)/Z_2\right)(-f_3(p_2(t), t)) + \xi_2^*.$$  

For the original PDE system (17), (18), we use this control law as follows.

$$u_1(t) = -q_1(0, t) + \xi_2^*$$

$$u_2(t) = \left((-Z_2 + R_2)/Z_2\right)(-f_3(p_2(t), t)) + \xi_2^*.$$  

Fig. 4 shows a simulation result using the coupled time-delayed Chua’s circuit. From Fig. 4, we can see that a complicated initial state converges to the equilibrium solution.

4. CONCLUSION

In this report, for a class of infinite-dimensional dynamical systems described by coupled first-order linear PDEs with nonlinear boundary conditions, a control method to stabilize equilibrium solutions has been proposed by using the method of characteristics. We have applied the proposed method to a coupled time-delayed Chua’s circuit, and confirmed that the stabilization is accomplished only by boundary inputs even if spatio-temporal chaos occurs in such a system.

REFERENCES


Appendix A

The maps $f_i$, $i = 1, 2, 3$ in Sec.2 are given as follows.

$$f_1(x, y) = \begin{cases} 
    a_1 x + a_2 y + b_1, & x - y - E < -\delta_1, \\
    a_1 x + a_2 y + b_2, & |x - y - E| \leq \delta_1, \\
    a_1 x + a_2 y + b_3, & x - y > \delta_1,
\end{cases}$$

$$f_2(x, y) = x + y - f_1(x, y),$$

$$f_3(x) = \begin{cases} 
    a_5 x + b_1, & |x - E| < -\delta_2, \\
    a_6 x + b_2, & |x - E| \leq \delta_2, \\
    a_7 x + b_3, & x - E > \delta_2,
\end{cases}$$

where

$$a_1 = \frac{m_1, Z_1}{m_1, (Z_1 + 2R_1)}; \quad a_2 = \frac{2m_1, R_1 + 2}{m_1, (Z_1 + 2R_1)};$$

$$b_1 = \frac{m_1, B_1}{m_1, (Z_1 + 2R_1)};$$

$$a_3 = \frac{m_1, Z_1}{m_1, (Z_1 + 2R_1)}; \quad a_4 = \frac{2m_1, R_1 + 2}{m_1, (Z_1 + 2R_1)};$$

$$b_2 = \frac{m_1, B_1}{m_1, (Z_1 + 2R_1)};$$

$$\delta_1 = \frac{m_1, (Z_1 - R_1)}{m_1, (Z_1 + 2R_1) + 1};$$

$$a_5 = \frac{m_2, (Z_1 - R_1)}{m_2, (Z_2 + R_1) + 1}; \quad b_4 = \frac{-m_2, E - (m_2, a - m_2, b_2) Z_2}{m_2, (Z_2 + R_1) + 1};$$

$$a_6 = \frac{m_2, (Z_2 - R_1)}{m_2, (Z_2 + R_1) + 1}; \quad b_5 = \frac{m_2, E - m_2, b Z_2}{m_2, (Z_2 + R_1) + 1};$$

$$\delta_2 = \frac{m_2, (Z_2 + R_1) + 1}{m_2, Z_2};$$

Appendix B

$$\phi^{-1}(x, y) = \begin{cases} 
    \frac{x - a y + (-m_1, Z_1 - m_2, Z_2 - b_1 + \xi)}{1 + a_1}, & x \in (-\delta_1, -E - \delta_1), \\
    \frac{x - a y + (-m_1, Z_1 - m_2, Z_2 - b_2 + \xi)}{1 + a_1}, & x \in (-E - \delta_1, -E), \\
    \frac{x - a y + (-m_1, Z_1 - m_2, Z_2 - b_3 + \xi)}{1 + a_1}, & x \in (-E, \delta_1), \\
    \frac{x - a y + (-m_1, Z_1 - m_2, Z_2 - b_4 + \xi)}{1 + a_1}, & x \in (\delta_1, E - \delta_1), \\
    \frac{x - a y + (-m_1, Z_1 - m_2, Z_2 - b_5 + \xi)}{1 + a_1}, & x \in (E - \delta_1, E), \\
    \frac{x - a y + (-m_1, Z_1 - m_2, Z_2 - b_6 + \xi)}{1 + a_1}, & x \in (E, \delta_1).
\end{cases}$$