Output Feedback LPV Control Strategies for Flexible Robot Arms

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Abstract: This paper deals with flexible manipulators and aims at developing a general LPV (Linear Parameter Varying) methodology for modeling and control. As an example, the case of a 2-links manipulator with one flexible link is considered. By the use of the assumed modes technique, the effects of link deflections are accounted for when developing the dynamic model from Euler-Lagrange equations. This nonlinear dynamic model is reformulated into a quasi-LPV model based on the function substitution approach. The LPV model serves as a basis for the development of parameter-dependent control laws involving the joint variables only and minimizing the end-effector deflection. The relevance of the proposed control approaches is demonstrated by simulations.

Keywords: robotic manipulators, LPV modeling, LPV control, flexible arms, LMI conditions

1. INTRODUCTION

The control of robot manipulators is a challenging research area that has attracted a large attention from scientists and engineers for several decades (see Canudas de Wit et al. (1996)). In the most basic approaches, the mechanical structure of such articulated systems is supposed to be completely rigid and the control algorithms are developed based on this assumption. The rigidity can be reinforced by an appropriate choice of building materials, or by an a posteriori treatment of the existing structure. However, this strengthening of the structure is not always possible and the flexibility effects may remain significant. Lightweight robots that are used in aerospace (Hirzinger et al., 2004) and medical (Hagn et al., 2008) applications typically exhibit such a phenomenon.

When multiple-links robot manipulators are considered, the significance of the nonlinear effects in the dynamic model, the presence of lightly damped oscillatory modes, as well as the underactuated nature of the control system (the deflection variables are not actuated) make the problem of designing controllers with performance guarantees become very difficult. In this context, the end-effector trajectory tracking problem is definitely the toughest. Several works on the subject consider a linearization of the model around a nominal operating condition, or some a priori knowledge of the reference trajectory (see Camudas de Wit et al. 1996, De Luca and Book 2008 and references therein). Herein, the proposed methodology allows to account for some nonlinear effects such as the variation of the inertia matrix with respect to the operating point and the Coriolis and centripetal torques.

In order to synthesize control laws that guarantee the internal stability of the system, and to reduce the effect of the flexibilities for an accurate trajectory tracking of the reference signals, we propose the use of the LPV systems framework. The approaches presented herein deal with the control of the joint positions and provide an attenuation of the deflection at the end-effector level.

The first step towards designing LPV control laws is the LPV modeling of the system. An appropriate LPV model of the system can be derived either by means of a direct identification of a parameter-dependent model from experimental data, or based on the use of the physical equations of motion. In the last case, the nonlinear dynamic model has to be reformulated into an LPV one. One can distinguish the linearization-based methods (such as the Jacobian linearization and the velocity-based linearization (Leith and Leithead, 1998)) from the change-of-variable methods such as the use of some state transformations and the nonlinear functions substitution (Toth, 2010). The last approach has been adopted here for the LPV modeling of a flexible-link manipulator.

The LPV modeling stage may lead to an LPV state-space representation with rational parametric dependence. This rational dependence causes some issues for the derivation of parameter-dependent controllers based on the resolution of a finite number of linear-matrix-inequality (LMI) conditions (Boyd et al., 1994). Furthermore, we consider that the state variables corresponding to the deflections are not measured, which is usually the case in practical applications. Therefore, full-order state feedback controllers are not feasible and a dynamic output feedback control scheme may be preferred. It is well known, in the control literature related to convex optimization, that such control problems naturally lead to bilinear-matrix-inequality (BMI) conditions that are nonconvex and generally difficult to solve.

In order to overcome the above mentioned control design issues and to synthesize LPV dynamic output-feedback controllers that ensure the internal stability and some
performance level over a wide operating range, we propose
the use of the two following distinct approaches:

(1) To derive an affine LPV descriptor model that is an
accurate approximation of the original rational LPV
model and design the controller based on available
results for this particular class of models (Masubuchi
et al. 2003).

(2) To use some augmented LMI conditions that account
for the particular structure of robots models in the
synthesis conditions. This method is an extension of
the synthesis conditions given in Halalchi et al. (2010)
for the control of a rigid robot manipulator. Herein,
flexible manipulators are considered, and an $H_{\infty}$
performance index is guaranteed by the synthesis.

Both methods finally allow to remove the rational depen-
dence of the LMI synthesis conditions and to reduce to
affine parameter-dependent LMIs that can be solved on
the vertices of the parametric space.

Our paper is organized as follows. Section 2 deals with the
nonlinear and the LPV modeling of flexible manipulators.
The proposed control strategies are presented in Section 3.
The forth section is devoted to the implementation of the
control laws and presents some simulation results. Section 5
concludes our paper.

Notations - If $A$ and $B$ are symmetric matrices $A > 0$
(resp. $\geq 0$) means that $A$ is positive definite (resp. positive
semi-definite), $A^T$ is the transpose of $A$, $\text{He}(A) = A + A^T$,
$\text{diag}(A, B) = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ and ($\ast$) represents the blocs
that can be obtained by symmetry. Matrix dimensions will be
omitted whenever possible.

2. MODELING OF FLEXIBLE MANIPULATORS

Robot manipulators can undergo two large classes of flex-
bility (De Luca and Book, 2008): joint flexibility or elas-
ticity that is concentrated at the joints of the manipulator
and may be caused by the use of compliant transmission
elements, and link flexibility that is distributed along the
mechanical structure and may be caused by the use of
lightweight materials or high bandwidth input torques.
Our paper focuses on the modeling and the control of
flexible-link manipulators which is the most challenging
case.

2.1 Dynamic modeling

Modeling of the bending deflection

A classical approach for the modeling of the bending deforma-
tion along a flexible link is the so-called assumed modes
technique (De Luca and Siciliano, 1991). The transverse
deflection in the horizontal plane varies with respect to
the position $x_k$ in the link $k$ and can be expressed as:

$$w_k(x_k, t) = \sum_{i=1}^{\infty} \phi_i(x_k) \delta_{ki}(t), \quad x_k \in [0, l_k]$$

where $l_k$ is the length of the link, $\phi_i(x_k)$ are shaping functions that are taken herein as eigenfunctions of the
deformation and $\delta_{ki}(t)$ is the time-varying coordinate
associated to the corresponding assumed mode. This infinite
dimensional model may be truncated up to a fixed order
$N$ giving a satisfactory approximation of the behaviour of
the system.

Nonlinear dynamic model

We consider that the vector of generalized coordinates
$q(t)$ consists of the joint positions that are measur-
able by the encoders $\theta_k(t)$, $k = 1..n$ and the de-
formation amplitudes associated with each link $\delta_{ni}(t):$
$q(t) = [\theta_1(t), ..., \theta_n(t); \delta_{11}(t), ..., \delta_{NN}(t)]^T$.
The use of the kinematics of the robot allows to obtain the
kinetic and the potential energies $T$ and $U$. The dynamic
model is then established using Euler-Lagrange equations:

$$\frac{d}{dt} \left( \begin{array}{c} \partial L \\ \partial \dot{q} \end{array} \right) - \frac{\partial L}{\partial q} = F$$

where $L = T - U$ is the Lagrangian of the system and $F$ is
the vector of external and friction torques.

Considering the vectors of joint positions $\theta(t) = [\theta_1(t), \ldots, \theta_n(t)]^T$ and deflection variables $\delta(t) = [\delta_{11}(t), \ldots, \delta_{NN}(t)]^T$, the dynamic model can be put into
the following generalized second-order form:

$$\begin{bmatrix} M_{\theta\theta}(q) & M_{\theta\delta}(q) \\ M_{\delta\theta}(q) & M_{\delta\delta} \end{bmatrix} \begin{bmatrix} \ddot{\theta} \\ \ddot{\delta} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & D_\delta \end{bmatrix} \begin{bmatrix} \dot{\theta} \\ \dot{\delta} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & K_\delta \end{bmatrix} \dot{\delta} + \begin{bmatrix} c_\theta(q, \dot{q}) + g_\theta(q) \\ c_\delta(q, \dot{q}) + g_\delta(q) \end{bmatrix} = \tau$$

$M_{xy}$ with $x, y \in \{\theta, \delta\}$ are the submatrices of the
rigid/flexible partition of the inertia matrix $M(q)$. $D_\delta$ and
$K_\delta$ are the damping and the stiffness matrices. $c_\theta$ and $c_\delta$
are the vectors of Coriolis and centripetal torques, and $g_\theta$ and $g_\delta$ the vectors of gravitational torques. $\tau$ is the vector
of control torques provided by the motors.

A meaningful state vector of the system is
$x(t) = [q^T(t), \dot{q}^T(t)]^T$. Provided that $M(q)$ is always invertible,
and under the assumption of small deformations, the non-
linear dynamic model is of the form: $\dot{x} = A(q, \dot{q})x + B(q)\tau$.

2.2 LPV modeling

A strictly-proper linear parameter varying (LPV) model
can be described by the following state-space equations:

$$\begin{cases}
\dot{x}(t) = A(\rho(t))x(t) + B(\rho(t))u(t) \\
y(t) = C(\rho(t))x(t)
\end{cases}$$

where the state matrices $A(\rho(t))$, $B(\rho(t))$, $C(\rho(t))$ depend
on a vector of parameters $\rho(t) = [\rho_1(t), \ldots, \rho_p(t)]^T
\rho(t)$ whose trajectories are unknown a priori, but whose values
may be measured in real time. Furthermore, the parameters
and their time derivatives are supposed to be bounded.

These bounds generate the admissible sets $S_\rho$ and $S_{\dot{\rho}}$.
When the parameters are independent, the admissible
sets are hyperrectangles delimited by the minimum
and the maximum values of each parameter. The vertices
sets defined this way are denoted $S_{\rho}^v$ and $S_{\dot{\rho}}^v$. LPV systems
can be classified according to the parametric dependence of
their state matrices. Among the commonly used classes we
can mention: affine LPV systems, polytopic LPV systems, rational LPV systems and LPV systems in an LFT (Linear Fractional Transformation) form.

The approximation or transformation of a nonlinear plant model into an LPV one can be performed by several methods (Rugh and Shamma, 2000). The interested reader can find in Toth (2010) a recent overview on the LPV modeling of physical systems. In this paper, we have adopted a change-of-variable approach based on function substitution for the LPV modeling of a 2-links flexible robot manipulator.

2.3 Case study

The case study discussed in our paper is the model of the FLEXARM robot taken from De Luca et al. (1990).

Nonlinear model

The considered robot is a planar manipulator with a flexible forearm. Its nonlinear dynamic model is given by:

\[ M(q) \ddot{q} + c(q, \dot{q}) + Kq + D \dot{q} = G \tau \]  

where \( q(t) = \theta_1(t), \theta_2(t), \theta_1(t), \theta_2(t) \), \( \theta_k : k = 1, 2 \) are the joint angular positions and \( \delta_j : j = 1, 2 \), are the amplitudes of the two flexible modes considered for the forearm (the second link). \( M(q) \) is the vector of Coriolis and centrifugal torques, \( K \) and \( D \) are the stiffness and the damping matrices respectively, \( \tau \) is the vector of control torques provided by the motors and \( G \) is an input matrix that is different from identity because of the chosen structure for the inertia matrix. The gravitational torques can be neglected due to the horizontal workspace of the robot.

The matrices of the dynamic model (5) have the following expressions:

\[
M(q) = \begin{bmatrix}
m_{11} & m_{12} & m_{13} & m_{14} \\
m_{21} & m_{22} & 0 & 0 \\
m_{31} & 0 & 1 & 0 \\
m_{41} & 0 & 0 & 1 \\
\end{bmatrix}
\]  

with:

\[
m_{11} = J_{11} + J_{21} + 2h_1 \cos \theta_2 - 2(h_1 \dot{\theta}_1 + h_2 \dot{\theta}_2) \sin \theta_2 \\
m_{12} = J_{21} + h_3 \cos \theta_2 - (h_1 \dot{\theta}_1 + h_2 \dot{\theta}_2) \sin \theta_2 \\
m_{13} = h_1 \cos \theta_2 \\
m_{14} = h_2 \cos \theta_2 \\
m_{22} = J_{21} \\
\]

The entries of the vector \( c(q, \dot{q}) \) are given by:

\[
c_1 = -2(\dot{\theta}_1 \dot{\theta}_2 + \dot{\theta}_2^2)(h_3 \sin \theta_2 + (h_1 \dot{\theta}_1 + h_2 \dot{\theta}_2) \cos \theta_2) \\
c_2 = \dot{\theta}_1^2 h_3 \sin \theta_2 + (h_1 \dot{\theta}_1 + h_2 \dot{\theta}_2) \cos \theta_2 \\
c_3 = \dot{\theta}_2^2 h_1 \sin \theta_2 \\
c_4 = \dot{\theta}_2^2 h_2 \sin \theta_2
\]

The input matrix is expressed as:

\[
G = \begin{bmatrix}
1 \\
G_\delta
\end{bmatrix}
\]  

with \( G_\delta = \begin{bmatrix} 0 & \phi_{10} \\
0 & \phi_{20}
\end{bmatrix} \)

The damping and stiffness matrices \( D \) and \( K \) have diagonal structures:

\[
D = \text{diag}(0, 0, d_1, d_2) \quad \text{and} \quad K = \text{diag}(0, 0, k_1, k_2)
\]

Moreover, the deflection of the end-effector of the robot is given by:

\[
y_{up}(t) = (\phi_{10} x_2 - \phi_{10}' x_1) + (\phi_{20} x_2 - \phi_{20}' x_1)
\]

LPV model

An LPV model of system (5) has been obtained by neglecting the terms involving the deformation variables \( \delta_j \) in the inertia matrix, and reformulating some nonlinear terms as varying parameters:

\[
\rho_1 = \cos(\theta_2) \quad \rho_2 = \dot{\theta}_2 \sin(\theta_2) \quad \rho_3 = \dot{\theta}_2 \sin(\theta_2) \\
\rho_4 = \dot{\theta}_2^2 \cos(\theta_2) \quad \rho_5 = \dot{\theta}_2 \cos(\theta_2) \quad \rho_6 = \dot{\theta}_2^2 \cos(\theta_2)
\]

Note that the varying parameters are functions of the measured state variables only. If we consider the output as the joint angular positions, the obtained LPV model takes the following state-space form:

\[
\begin{aligned}
\dot{x}(t) &= A(p(t))x(t) + B(p(t))u(t) \\
y(t) &= Cx(t)
\end{aligned}
\]

where \( A(p(t)) = M_0^{-1}(p_1)A_1(p) \), \( B(p(t)) = M_0^{-1}(p_1)B_1(p) \) and \( C = [I_2 0_{2\times6}] \), with \( M_0(p_1) = \text{diag}(I_4, M(p_1)) \),

\[
B_1(p) = \begin{bmatrix}
0_{4\times2} & I_2 \\
G & A_1(p) = \begin{bmatrix} 0_4 & I_4 \\
0_{4\times2} & S(p) \end{bmatrix}
\end{bmatrix}
\]

\[
S^T(p) = \begin{bmatrix}
-h_1(2p_5 + p_4) h_1 p_6 - k_1 & 0 \\
-h_2(2p_5 + p_4) h_2 p_6 & -k_2 \\
-h_2(2p_5 + p_4) h_2 p_6 & -f_1 \\
-h_2(2p_5 + p_4) h_2 p_6 & -f_2
\end{bmatrix}
\]

Due to the inversion of the inertia matrix \( M(p_1) \) in the state matrices, the LPV system (11) is rational with a particular structure. This structure can be accounted for in the synthesis of LPV controllers.

3. LPV CONTROL STRATEGIES

In order to address the analysis and synthesis problems for rational LPV systems several approaches have been discussed in the literature such as: parameter griding (Aparician and Adams 1998), minimal convex polytope finding (Anstett et al. 2009), LFT representation (Scherer 2001), and affine LPV descriptor representation (Masubuchi et al. 2003). In the following, the latter method will be used, as well as a second method based on dilated LMI conditions and accounting for the system’s particular structure. Let us recall that the dynamic output feedback LPV control problem is to synthesize a dynamic controller \( K(s, p) \) of state matrices \( (A_K(p), B_K(p), C_K(p), 0) \) that stabilizes the closed-loop system and guarantees some performance index. The closed-loop system is described by:

\[
\begin{aligned}
\dot{x}_{CL}(t) &= A_{CL}(p)x_{CL}(t) \\
y(t) &= C_{CL}x_{CL}(t)
\end{aligned}
\]

where \( x_{CL} = [x^T \ x_{K}^T]^T \) is the closed-loop state vector, \( x_K \) is the controller state vector, \( A_{CL} \) and \( C_{CL} \) are partitioned...
according to the states of the system and the controller:

\[ A_{\text{Cl}}(\rho) = \begin{bmatrix}
A(\rho) & B(\rho)C_K(\rho) \\
B_K(\rho)C(\rho) & A_K(\rho)
\end{bmatrix} \quad \text{and} \quad C_{\text{Cl}} = [C \quad 0]. \]

### 3.1 Equivalent affine LPV descriptor representation

It has been stated in Masubuchi et al. (2003) that any rational LPV system can be transformed into a descriptor form with affine state matrices. This result has been used in Halalchi et al. (2010a) for the LPV control of a rigid robot. Herein, the aim is to use similar methods for the control of flexible robots, while accounting for the Coriolis and centripetal terms in the synthesis model.

#### Modeling

A rational LPV system is described by classical parameter-dependent state-space equations:

\[
\begin{aligned}
\dot{x}(t) &= A(\rho)x(t) + B(\rho)u(t) \\
y(t) &= C(\rho)x(t)
\end{aligned}
\]  

(13)

in which the state matrices \(A(\rho), B(\rho)\) and \(C(\rho)\) are rational functions of the parameter vector \(\rho(t)\). The transformation of system (13) into an affine descriptor system can be performed either by making some changes of variables involving the parameter-dependent entries of the state matrices (ad-hoc methods), or systematically by introducing an intermediate step which is the LFR (Linear Fractional Representation) modeling of the rational LPV system. The LFR model itself can be obtained by manual calculations or by using available numerical software (Magni, 2006).

An LFR form of the rational LPV system (13) is:

\[
\begin{bmatrix}
\dot{x}(t) \\
y(t) \\
z(t)
\end{bmatrix} =
\begin{bmatrix}
N_{11} & N_{12} & N_{13} \\
N_{21} & N_{22} & N_{23} \\
N_{31} & N_{32} & N_{33}
\end{bmatrix}
\begin{bmatrix}
x(t) \\
u(t) \\
v(t)
\end{bmatrix} +
\begin{bmatrix}
0 \\
M_A \\
M_B
\end{bmatrix}
\begin{bmatrix}
\dot{\rho}(t) \\
\dot{\rho}(t) \\
\dot{\rho}(t)
\end{bmatrix}
\]

and \(v = (\Delta(\rho)z)\).

where \(x(t) \in \mathbb{R}^r\) is the state, \(z(t) \in \mathbb{R}^nu\) and \(v(t) \in \mathbb{R}^nv\) are the input and output signals of the uncertainty block \(\Delta(\rho)\) that is affine in \(\rho\) and that represents the only parameter-dependent matrix of the LFR. \(N_{ij}\), \(i, j = 1, 3\) are the submatrices resulting from the partition of the interconnection matrix \(N\).

Defining an augmented state vector as \(x(t) = [x(t)^T \, v(t)^T]^T\) allows to obtain the affine LPV descriptor system:

\[
\begin{aligned}
\dot{E}_{\text{x}}(t) &= A(\rho)x(t) + B(\rho)u(t) \\
y(t) &= Cx(t)
\end{aligned}
\]

(15)

with \(E = \text{diag}(I_r, 0_{n_v})\), \(A(\rho) = \begin{bmatrix} N_{11} \quad N_{13} \\ \Delta(\rho)N_{31} \quad \Delta(\rho)N_{33} - I_{n_v} \end{bmatrix}\), \(B(\rho) = \begin{bmatrix} N_{12} \\ \Delta(\rho)N_{32} \end{bmatrix}\) and \(C = \begin{bmatrix} C_r \quad 0_{n_v \times n_v} \end{bmatrix}\).

The alternative choice \(x(t) = [x(t)^T \, v(t)^T]^T\) can be made in order to obtain a constant \(B\) matrix.

#### \(H_\infty\) control

Let us define the following augmented descriptor system with a performance channel \(w(t) \rightarrow z(t)\), where \(w(t)\) is an external input and \(z(t)\) is the controlled output:

\[
\begin{aligned}
E_1\dot{x}(t) &= A(\rho)x(t) + B_1w(t) + B_2u(t) \\
y(t) &= C_1x(t) \\
z(t) &= C_2x(t)
\end{aligned}
\]

(16)

The LMI synthesis conditions used in this paper have been given in Masubuchi et al. (2004). Constant Lyapunov matrices \(X\) and \(Y\) are considered herein.

**Theorem 1.** The closed-loop system is stable and has an \(H_\infty\) performance index less than a positive scalar \(\gamma\) if the following LMIs have a solution \(p = \{X, Y, F, G, H\}\), \(\forall \rho \in S_p^\rho\):

\[
\begin{bmatrix}
YE_1^T & E_1 \\
E_2^T & E_1^T X
\end{bmatrix} \geq 0
\]

(17)

and

\[
\begin{bmatrix}
M_A + M_A^T & M_B & M_C^T \\
M_B^T & -\gamma I & 0 \\
M_C & 0 & -\gamma I
\end{bmatrix} < 0
\]

(18)

where \(M_A = \begin{bmatrix} A_1(\rho) & B_2F \end{bmatrix}
\begin{bmatrix} A_1(\rho) \\ H^T \\ X^T A_1(\rho) + G^T C_2 \end{bmatrix}\), \(M_B = \begin{bmatrix} B_1 \\ X^T B_1 \end{bmatrix}\) and \(M_C = [C_1^T \, C_1\]

The output feedback LPV controller can be obtained directly in a regular state-space form: \(K(s; \rho) = \{A_K(\rho), B_K(\rho), C_K(\rho), 0\}\) and can be implemented in this form on the real system (see Masubuchi et al. (2004) for details).

#### 3.2 Dilated LMI conditions and structure constraints

Dilated LMI conditions have been introduced in Halalchi et al. (2010) for the stabilization and the asymptotic joint tracking of a rigid manipulator. In the following, these conditions are extended to \(H_\infty\) performance and exploited for the control of flexible manipulators. An augmented representation of (12) with the performance channel \(w(t) \rightarrow z(t)\) is the following:

\[
\begin{aligned}
\dot{x}_c(t) &= A_{\text{Cl}}(\rho)x_c(t) + B_{w}w(t) \\
y(t) &= C_{z}x_c(t) + D_{w}w(t)
\end{aligned}
\]

(19)

where \(w(t) \in \mathbb{R}^{nu}\) is an external input and \(z(t) \in \mathbb{R}^{n_z}\) is the controlled output. Matrices \(B_{w}\) and \(C_{z}\) are partitioned according to the states of the system and the controller:

\[
B_{w} = \begin{bmatrix} B_{1w} \\ 0 \end{bmatrix}\quad \text{and} \quad C_{z} = \begin{bmatrix} C_{1z} \quad 0 \end{bmatrix}.
\]

The following \(H_\infty\) analysis conditions, given in Bara and Daafouz (2001) are based on the dilated LMI conditions that have been proposed in Apkarian et al. (2001).

**Theorem 2.** The closed-loop system (19) is stable and has an \(H_\infty\) performance index less than a positive scalar \(\gamma\) if: \(\exists P(\rho) = P^T(\rho)\) (Lyapunov matrix) and \(\exists V(\rho)\) (slack variable) such that, \(\forall (\rho, \phi) \in S^p \times S^p\):

\[
\begin{bmatrix}
-\text{He}(V(\rho)) \\
A_{\text{Cl}}(\rho)V(\rho) + P(\rho) - P(\rho) + \bar{P}(\rho) \quad \gamma \bar{I} \quad \gamma \bar{I} \quad \gamma \bar{I} \\
C_{v}V(\rho) \quad 0 \quad \bar{V}(\rho) \\
0 \quad B_{w}^T \quad D_{w}^T \quad -\bar{I} \quad \bar{I} \quad \bar{I} \\
V(\rho) \quad 0 \quad 0 \quad 0 \quad -P(\rho)
\end{bmatrix} < 0
\]

(20)
If they are applied to the closed-loop system in (19), these analysis conditions lead to the following controller synthesis conditions.

**Theorem 3.** The closed-loop system (19) is stable and has an $H_{\infty}$ index less than a positive scalar $\gamma$ if the following LMI has a solution $\Phi = [\hat{V}_1, \hat{V}_2]$, $\hat{A}_K, \hat{B}_K, \hat{C}_K, \hat{U}(\rho)$, $\hat{X}(\rho)$, $\forall \rho \in S_{\rho}$:

$$
\begin{bmatrix}
-\hat{A}_K(M_1(\rho)V_1) + \hat{B}_K(M_1(\rho)W_11) + \hat{C}_KU(\rho) - \hat{X}(\rho) & \hat{A}_K(M_1(\rho)V_1) + \hat{B}_K(M_1(\rho)W_11) + \hat{C}_KU(\rho) - \hat{X}(\rho) \\
\alpha_1 & \alpha_2 + \hat{X}(\rho) - \hat{X}(\rho) & \alpha_3 \\
C_{12}V_11 & C_{12}(I + \hat{V}_22) & 0 & \gamma I & 0 \\
0 & 0 & \beta \hat{D}_v & \gamma I & 0 \\
M_1(\rho)\hat{V}_11 & M_1(\rho) & W_{\alpha 1} & \hat{V}(\rho) & W_{\alpha 1}M_1(\rho)
\end{bmatrix} < 0
$$

(21)

with: $\alpha_1 = A_1(\rho)\hat{V}_11 + B_1\hat{C}_K\hat{M}_1(\rho)$, $\alpha_2 = A_1(\rho)$, $\alpha_3 = A_\hat{K}M_1(\rho)$, $\alpha_4 = \hat{W}_{11}A(\rho) + \hat{B}_K\hat{C}$, $\beta = [\hat{B}_1^T \hat{B}_1^T \hat{W}_{11}]$, and $\hat{B}_{1w} = M_1^{-1}(\rho)\hat{B}_{1w}$.

The state matrices of the controller are determined by:

$$
B_K(\rho) = W_{21}^{-T}(\rho)\hat{B}_K; C_K(\rho) = \hat{C}_KV_21^{-1}(\rho);
A_K(\rho) = W_{21}^{-T}(\rho)[\hat{A}_K - W_{21}A_1(\rho)V_11M_1^{-1}(\rho) - W_{21}^T\hat{B}_K(\rho)C\hat{V}_11M_1^{-1}(\rho) - W_{21}^T\hat{B}_1\hat{C}_K(\rho)\hat{V}_21M_1^{-1}(\rho)].
$$

**Proof.** Obtained by left multiplying inequality (20) by $T(\rho)$ and right multiplying by its transpose, where:

$$
T(\rho) = \begin{bmatrix}
M_2(\rho)\Pi_{W}^T & 0 & 0 & 0 & 0 & 0 \\
0 & M_2(\rho)\Pi_{W}^T & 0 & 0 & 0 & 0 \\
0 & 0 & I_{n_v} + n_w & 0 & 0 & 0 \\
0 & 0 & 0 & M_2(\rho)\Pi_{W}^T & 0 & 0
\end{bmatrix}
$$

(22)

with $\Pi_{W} = \begin{bmatrix} V_{11} & I \end{bmatrix}$, $\Pi_{W} = \begin{bmatrix} W_{11} & 0 \end{bmatrix}$, $V = \begin{bmatrix} V_{11} & V_{12} \end{bmatrix}$, and $W = V^{-1} = \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix}$.

(23)

(24)

$M_2(\rho) = \text{diag}(M_1(\rho), I_4)$ and matrices in (24) result from a partitioning of $V$ and $W$ according to the system and the controller dimensions. The relations $WH_1 = H_2W$ and $VH_1 = H_2V$ are also exploited.

**Remark 1.** Matrices $W_{21}(\rho)$ and $V_{21}(\rho)$ used in the controller are determined the following way. Actually $\hat{U}(\rho) = (W_{21}^TV_{11} + W_{21}^TV_21M_1(\rho), W_{11} = M_1(\rho)\hat{W}_{11}$ and $\hat{V}_{11} = V_11M_1(\rho)$. Therefore, $W_{21}^TV_21 = M_1^{-1}(\rho)\hat{U}(\rho) - W_1^T\hat{M}_1(\rho)\hat{V}_{11}$. Thus $W_{21}(\rho)$ and $V_{21}(\rho)$ can be obtained from a SVD decomposition plus a Cholesky factorization.

## 4. IMPLEMENTATION

### 4.1 Control scheme

Let us consider the two-bloc $H_{\infty}$ control scheme presented in Figure 1. The closed-loop augmented system can be described by (19) where the external input $w(t) = [w_1^T(t) w_2^T(t)]^T$ is decomposed into two parts:

$$
\begin{align*}
w_1(t) &= z_1(t) - y_{in}(t) \\
w_2(t) &= z_2(t) - y_{in}(t)
\end{align*}
$$

(25)

Fig. 1. $H_{\infty}$ control scheme

The trajectory reference signal $w_1(t) = r(t)$ and an additive input disturbance $w_2(t)$. The controlled output $z(t) = [e(t)^T y_{in}(t)]^T$ contains the tracking error signals $e(t) = r(t) - y(t)$ and the end-effector deflection $y_{in}(t)$ that is supposed to be unmeasured because it is a function of the state variables involving deformation. However, the output matrix $C_{in}$ is known from (9). The part $w_3(t) = z_1(t)$ of the performance channel allows to impose constraints on the modulus margin, the precision and the bandwidth, whereas the action of the part $w_3(t) = z_2(t)$ is described as follows: the choice of $z_2(t) = y_{in}(t)$ allows to minimize the end-effector deflection and $w_2(t)$ allows to compensate the input disturbances such as Coulomb friction and gravitational torques.

Recall that the control objective is to ensure the internal stability of the system and to minimize the $H_{\infty}$ performance (actually the $L_2$-induced gain) of the closed-loop transfer from $w(t)$ to $z(t)$. The performance requirements can be adapted among frequencies by using appropriate weighting functions (Skogestad and Postlethwaite 2005).

Time simulations have been carried out in order to evaluate the proposed control methods. Let us define the following limits that generate the admissible set $S_\rho$: $|p_1| < 1$ (by definition of the cosine function), $|p_2|, |p_3| < 0.3$ rad/s, and $|p_4|, |p_5|, |p_6| < 0.045 (rad/s)^2$. While both control methods ensure the stability of the closed-loop system over the whole admissible set, the performance levels presented in the paper are guaranteed for $|p_1| < 0.5$. The LMI conditions have been solved on the vertices set $S_\rho$ using the solver SeDuMi (Sturm 1999) with the YALMIP interface (Loberg 2004). The joint reference signals $\theta_1^*(t)$ and $\theta_2^*(t)$ are selected so as to assess the accuracy of the tracking and the decoupling of the two joints over the whole operating range. Smooth reference signals are involved in order to cope with the limitations on the parameters and to keep a small excitation of the flexible modes.

### 4.2 Simulation results

The previously described control scheme, associated with the methods described in Section 3, is used in simulation for the control of the FLEXARM robot.

The descriptor representation method presented in Subsection 3.1 (referred to as the first method in the figures) has been applied to one block augmented scheme in which only the performance channel $w_1(t) \rightarrow z_1(t)$ is considered, whereas the dilated LMI method with slack variables presented in Subsection 3.2 (referred to as the second method in the figures) is used to control the overall block diagram in Figure 1. When considering unit weighting functions, the obtained optimal performance indices are $\gamma_1 = 2.37$ and $\gamma_2 = 1.80$. These performance indices allow to guar-
antee a modulus margin greater than $\frac{1}{\gamma_1} = 0.422$ and $\frac{1}{\gamma_2} = 0.556$ respectively. The frequency responses of the realized transfers, i.e. the output-sensitivity function $S_y(s)$ and the sensitivity $S_{w_2y_1}(s)$ of the tip-deflection $y_{tip}$ to the variations of the input disturbance $w_2(t)$, are compared to the frequency template given by $\gamma_{dB}$ in this case. The comparison has been performed for 4 particular values of the parameter vector $\rho$ that have been selected among the 64 vertices of the admissible set: $\rho(1) = [p_1 \ldots p_6]^T$, $\rho(2) = [\overline{p}_1 \overline{p}_1 \ldots \overline{p}_6]^T$, $\rho(3) = [\overline{p}_1 p_1 \ldots p_6]^T$ and $\rho(4) = [\overline{p}_1 \overline{p}_1 \ldots \overline{p}_6]^T$.

The results are shown in Figures 2 and 5. The obtained bandwidths are of about 10 rad/s.

The subfigures (a) and (b) of Figures 3 and 6 show the tracking of the reference joint trajectories and the corresponding tracking errors for both control methods. The subfigures (a) to (f) of Figure 4 and 7 represent the realized trajectories for all six varying parameters. The presented figures show an accurate tracking of reference joint trajectories as well as a significant decoupling of the two joints. The imposed performance constraints are satisfied for the considered operating points.

**Fig. 3. Tracking of the reference trajectories (first method)**

**Fig. 4. Evolution of the varying parameters (first method)**

**Fig. 5. Frequency transfers (second method)**

**Fig. 6. Tracking of the reference trajectories (second method)**
5. CONCLUSION

This paper deals with the LPV modeling and control of flexible manipulators. A general methodology that addresses these problems has been proposed. It is based on an appropriate reformulation of the original nonlinear model following a quasi-LPV approach. Based on the obtained LPV model, synthesis conditions of output feedback LPV controllers have been used that guarantee the stability and the $H_\infty$ performance over a wide operating range. The nonlinear model of a 2-links flexible manipulator has been successfully used in simulation in order to validate the proposed control algorithms. In a future work, we aim at improving the performances achieved by the designed controllers and to tackle the more difficult problem of direct trajectory tracking of the end-effector of the robot.

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