Performance Analysis of Random Dither Quantizers in Feedback Control Systems

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Abstract: This paper analyzes the performance of the random dither quantizers from the view of feedback control. In particular, we focus here on the uniform random dither quantizers, which use an additional random signal with a uniform distribution at each sampling period. First, two interesting properties are shown by numerical examples, which motivate us to analyze the performance of such quantizers in feedback control loop. Next, an upper bound of the performance is derived, which enables us to easily estimate the performance of the random dither quantizers. Finally, the relation between the performance and the sampling period is clarified.

1. INTRODUCTION

Quantized control is one of the most important topics in recent years. This is partly because various quantizing devices, such as communication networks and discrete-level actuators/sensors, are essential to make control systems flexible, intelligent, and robust. Furthermore, they are useful for reducing installation and maintenance costs. On the other hand, in considering the quantized control, it is necessary to handle discrete-valued signals as well as continuous-valued ones, which poses challenging control problems.

Thus, the topic has been actively studied from various viewpoints (see e.g., Nair et al. [2007], Goodwin et al. [2008]). The authors also have approached it from the dynamic quantizer design (Minami et al. [2007]). The dynamic quantizers considered there are in the form of a difference equation which maps continuous-valued signals into discrete-valued ones depending on the history of both signals. By using well-designed dynamic quantizers, one can easily construct high-performance control systems even when some signals in the control systems are restricted to be discrete-valued. However, as is proven in Minami et al. [2007], the performance of dynamic quantizers heavily depends on the dimension of their state variables. In other words, we often need more complex (or higher-dimensional) dynamic quantizers to achieve higher performance, as shown in Fig. 1 (a). This fact often becomes a crucial problem in the implementation stage.

Here, we are interested in a different type of quantizers, called the random dither quantizers. The quantizer transforms a given continuous-valued signal to a discrete-valued signal by using an artificially added random signal. The structure is very simple and the performance is comparable with the dynamic quantizers, as shown in Fig. 1 (b). This type of quantizer has been originally proposed by Roberts [1962], and widely used for the signal processing (see Gray and Stockham [1993], Wannamaker et al. [2000]), e.g., the digital image halftoning (Ulichney [1988]), the consensus algorithm (Aysal et al. [2008]), the digital watermarking (Chen and Wornell [2001]), and the smoothing of nonlinear systems (Iannelli et al. [2003]). However, to our best knowledge, theoretical results on the performance for control, which allow us to quantify the performance when it is used in feedback control, has never been obtained so far.

This paper thus analyzes the performance of the random dither quantizers from the viewpoint of feedback control. In particular, we focus here on the uniform random dither quantizer, which uses an additional random signal with a uniform distribution at each sampling period. The contributions of this paper are summarized as follows. First, through examples, we show two interesting properties of such quantizers in feedback control loop: (i) the random dither quantizers exhibit much better performance than the simple uniform (rounding-off type) quantizers, and (ii) the performance is improved by small sampling period. Next, based on stochastic properties of its quantization error, an upper bound of the performance is derived. This explains property (i) and enables us to easily estimate the performance of the random dither quantizers. Finally, the relation between the performance and the sampling period is clarified, which reveals a mechanism behind property (ii). Furthermore, by this result, the random signal can be designed so as to satisfy desired performance.
Fig. 2. Static nearest-neighbor quantizer toward $-\infty$ with quantization interval $d$

Fig. 3. Quantized feedback system (Minami et al. [2007]) given by a two-dimensional state equation. Fig. 7 illustrates the output signals of

Notation: Let $\mathbb{R}$ and $\mathbb{N}$ be the real number field and the set of positive integers, respectively. We use $I$ and 0 to express the identity matrix and the zero matrix of appropriate dimensions. For the real number $a$ and the positive real number $b$, let $\lfloor a\rfloor_b$ denote the largest element of the discrete set $\{0, \pm b, \pm 2b, \ldots\}$ not greater than $a$. For a probabilistic event $a$, let $P_r(a)$ denote its probability. For a random variable $x$, let $E(x)$ and $V(x)$ denote the expectation and variance of $x$, respectively. In addition, for the matrix $X$, let $\|X\|$ denote its induced 2-norm except as otherwise noted. Finally, let $q$ denote the static nearest-neighbor quantizer toward $-\infty$ with the quantization interval $d$ as shown in Fig. 2.

2. PROBLEM FORMULATION

2.1 Motivating examples

Let us illustrate several interesting properties of the random dither quantizers. Consider the system shown in Fig. 3, where $P$ and $K$ are the discrete-time plant and controller given by the continuous-time ones

$$
P : \begin{cases} 
\dot{x}(t) = \begin{bmatrix} 0 & 4 \\
-3 & 2 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\
1 \end{bmatrix} v(t), \\
z(t) = \begin{bmatrix} 1 \\
0 \end{bmatrix} x(t), \\
y(t) = x(t),
\end{cases}$$

$$
K : u(t) = -[0.3 \ 4] y(t) + r(t)
$$

and the zero-order hold with the sampling period 0.1[s]. The initial state of $P$ is $x(0) = [-1, 1]^T$. The reference input is supposed to be $r(k) \equiv 0$ for every $k \in \{0\} \cup \mathbb{N}$ and the input of $P$ is limited to be a value on the discrete set $\{0, \pm 2, \pm 4, \ldots\}$. The system $Q$ is a quantizer which plays a role of the interface between $P$ and $K$.

First, suppose that $Q$ is the simple uniform quantizer $v(k) = q(u(k))$ where $q$ is defined in section 1. Fig. 4 shows the time responses of $v$ and $z$ of the system in Fig. 3. In addition, the second graph shows the output of the unquantized system in Fig. 5 (in the same situation) by the thick line for comparison. It turns out that the output is quite different from that of the unquantized system.

We next consider the quantizer $v(k) = q(u(k) + w(k))$, called the random dither quantizer, where $w(k) \in \mathbb{R}$ is an i.i.d. random variable with the uniform probability distribution on $[-1, 1)$. Fig. 6 shows the time responses of $v$ and $z$. Similar to the first example, the thick line in the second graph indicates the output of the unquantized system in Fig. 5. We see that the random dither quantizer gives a similar output to that of the unquantized system. Figs. 4 and 6 show that the random dither quantizer could achieve much better performance than the simple uniform quantizer.

Also, the advantage of random dither quantizers can be seen by comparing with the optimal dynamic quantizer (Minami et al. [2007]) given by a two-dimensional state equation. Fig. 7 illustrates the output signals of
the quantized system with the optimal quantizer and the continuous-input system. This and Fig. 6 demonstrate that even though the random dither quantizer has a much simpler structure, it has similar performance to that of the optimal dynamic quantizer.

Now, let us consider the case where the sampling period is set to 0.01[s] (it is 0.1[s] in the above example) and the other conditions are the same as before. Then we have the responses of \(v\) and \(z\) shown in Fig. 8, where the output response of the unquantized system (with the smaller sampling period) is displayed by the thick line. Figs. 6 and 8 tell us that the performance of the random dither quantizer would be improved by smaller sampling period.

The above examples motivate us (a) to evaluate the performance of the random dither quantizer and (b) to clarify the relation between the performance and the sampling period.

### 2.2 Problem formulation

Consider the system \(\Sigma_Q\) in Fig. 9 (a). The system \(G\) is a discrete-time linear system given by

\[
\begin{align*}
  x(k+1) &= Ax(k) + B_1 r(k) + B_2 v(k), \\
  z(k) &= C_1 x(k) + D_1 r(k), \\
  u(k) &= C_2 x(k) + D_2 r(k)
\end{align*}
\]

(1)

where \(k \in \{0\} \cup \mathbb{N}\) is the discrete time, \(x(k) \in \mathbb{R}^n\) is the state, \(r(k) \in \mathbb{R}\) and \(v(k) \in \mathbb{R}\) are the inputs, \(z(k) \in \mathbb{R}\) and \(u(k) \in \mathbb{R}\) are the outputs, \(A \in \mathbb{R}^{n \times n}\), \(B_1 \in \mathbb{R}^{n \times 1}\),

\[
B_2 \in \mathbb{R}^{n \times 1}, \quad C_1 \in \mathbb{R}^{1 \times n} \quad \text{and} \quad C_2 \in \mathbb{R}^{1 \times n}
\]

and \(D_1 \in \mathbb{R}\) and \(D_2 \in \mathbb{R}\) are constant matrices, and \(R_1 \in \mathbb{R}\) and \(R_2 \in \mathbb{R}\) are constant numbers. The initial value is given as \(x(0) = x_0\) by \(x_0 \in \mathbb{R}^n\). The quantizer \(Q\) is in the form of

\[
v(k) = q(u(k) + w(k))
\]

(2)

where \(w(k) \in \mathbb{R}\) is an i.i.d. random variable with the uniform probability distribution on \([-d/2, d/2]\), as shown in Fig. 10. Note that \(w(k_1)\) and \(w(k_2)\) are independent each other for \(k_1 \neq k_2\). The signal \(w\) is called the random dither signal and the quantizer is called the random dither quantizer.

The system in Fig. 9 (a) is a generalized version of the quantized system in Fig. 3. Indeed, the part indicated by the dashed line frame in Fig. 3 corresponds to the system \(G\).

### 2.3 Performance index

In order to define a performance index for \(Q\), we consider the unquantized system \(\Sigma_I\) shown in Fig. 9 (b) as an ideal system. To distinguish the outputs of the systems \(\Sigma_Q\) and \(\Sigma_I\), let \(z_Q(k, x_0, R)\) denote the output \(z\) of the system \(\Sigma_Q\) at the \(k\)-th time for \(x(0) = x_0\) and \((r(0), r(1), \ldots) = R\) and let \(z_I(k, x_0, R)\) be the output for \(\Sigma_I\). Then, the performance of \(Q\) is evaluated by

\[
J := \sup_{k \in \{0\} \cup \mathbb{N}} \sup_{(x_0, R) \in \mathbb{R}^{n \times \ell_w}} E\left(\left(z_Q(k, x_0, R) - z_I(k, x_0, R)\right)^2\right)
\]

(3)

The performance index \(J\) corresponds to the worst-case output difference between the systems \(\Sigma_Q\) and \(\Sigma_I\). If \(J\) is
small enough, the output of $\Sigma Q$ becomes similar to that of $\Sigma I$. Note here that $Q$ is evaluated in an expectation sense because $\Sigma Q$ contains the random variable $w(k)$.

The purpose of this paper is two-fold: One is to determine the value of $J$, and the other is to clarify the relation between the performance of the random dither quantizers and the sampling period, based on which determines the discrete-time system (12) from the original continuous-time ones.

3. PERFORMANCE ANALYSIS OF RANDOM DITHER QUANTIZERS

3.1 Stochastic property of quantization error

From the definition of the random dither quantizer, we note that its output is a random variable. Therefore, the quantization error $\eta(k) := v(k) - u(k)$ is also a random variable. Furthermore, by using the symbol $\eta$, the system $\Sigma Q$ is equivalently transformed into the system in Fig. 11 and the output is specified by $\eta$. So, in this section, we clarify the stochastic property of the quantization error, as a preliminary.

Let us derive the probability distribution of $\eta$. By definition, we have $-d/2 \leq w(k) < d/2$, and so

$$u(k) - \frac{d}{2} \leq u(k) + w(k) < u(k) + \frac{d}{2}. \quad (4)$$

Therefore, it follows from (2) that $v(k)$ takes one of the values $\lfloor u(k) \rfloor_d$ and $\lfloor u(k) \rfloor_d + d$. This implies that if $u(k)$ is fixed, $\eta(k)$ is a random variable with a binomial distribution. Furthermore, by considering Fig. 10 and the equivalence relation

$$\eta(k) = \lfloor u(k) \rfloor_d - u(k) \iff u(k) + w(k) < \lfloor u(k) \rfloor_d + \frac{d}{2},$$

$$\eta(k) = \lfloor u(k) \rfloor_d + d - u(k) \iff u(k) + w(k) \geq \lfloor u(k) \rfloor_d + \frac{d}{2},$$
	he probability distribution of $\eta(k)$ can be expressed as

$$Pr(\eta(k) = \lfloor u(k) \rfloor_d - u(k)) = Pr\left(\frac{-d}{2} \leq w(k) < \lfloor u(k) \rfloor_d + \frac{d}{2}\right)$$

$$= \left(\frac{\lfloor u(k) \rfloor_d - u(k) + \frac{d}{2}}{d}\right) \frac{1}{d}$$

$$= \frac{d - u(k) + \lfloor u(k) \rfloor_d}{d}, \quad (5)$$

$$Pr(\eta(k) = \lfloor u(k) \rfloor_d + d - u(k)) = Pr\left(\frac{d}{2} \leq w(k) < \frac{d}{2}\right)$$

$$= \left(\frac{d}{2} - \frac{\lfloor u(k) \rfloor_d - u(k) + \frac{d}{2}}{d}\right) \frac{1}{d}$$

$$= \frac{u(k) - \lfloor u(k) \rfloor_d}{d}. \quad (6)$$

From (5) and (6), the following lemma is obtained.

Lemma 1. For the random dither quantizer $Q$ defined in Section 2.2, suppose that the quantization interval $d$ is given. Then,

(i) $E(\eta(k)) = 0$,

(ii) $V(\eta(k)) \leq \frac{d^2}{4}$,

(iii) $E(\eta(k_1)\eta(k_2)) = \begin{cases} 0 & (k_1 \neq k_2), \\ V(\eta(k_1)) & (k_1 = k_2) \end{cases}$

hold for every $u(k), u(k_1), u(k_2) \in \mathbb{R}$.

(Proof) By the definition of the expectation, we have

$$E(\eta(k)) = \left(\lfloor u(k) \rfloor_d - u(k)\right)\frac{d - u(k) + \lfloor u(k) \rfloor_d}{d}$$

$$+ \left(\lfloor u(k) \rfloor_d + d - u(k)\right)\frac{u(k) - \lfloor u(k) \rfloor_d}{d}$$

$$= 0,$$

which proves (i). Next, from (i) and the definition of the variance, it follows that

$$V(\eta(k)) = \left(\lfloor u(k) \rfloor_d - u(k)\right)^2\frac{d - u(k) + \lfloor u(k) \rfloor_d}{d}$$

$$+ \left(\lfloor u(k) \rfloor_d + d - u(k)\right)^2\frac{u(k) - \lfloor u(k) \rfloor_d}{d}$$

$$= (d - u(k) + \lfloor u(k) \rfloor_d)(u(k) - \lfloor u(k) \rfloor_d)$$

$$= -\left(\lfloor u(k) \rfloor_d - u(k)\right)^2\frac{d^2}{4}$$

$$\leq \frac{d^2}{4}.$$

Hence, (ii) is obtained. Finally, (iii) is given by straightforward calculation with (5) and (6). \qed

Lemma 1 provides several stochastic properties of the quantization error of the random quantizer. This will play an important role to analyze the performance in the system $\Sigma Q$.

3.2 Performance analysis

Next, we evaluate $J$ with Lemma 1. By introducing the variable $\eta$ to (1), the system $\Sigma Q$ in Fig. 9 (a) is expressed as

$$\begin{cases} x(k+1) = \tilde{A}x(k) + \tilde{B}r(k) + B_2\eta(k), \\ z(k) = C_1x(k) + D_1r(k), \\ u(k) = C_2x(k) + D_2r(k), \end{cases} \quad (7)$$

where $\tilde{A} := A + B_2C_2$ and $\tilde{B} := B_1 + B_2D_2$. Then, the output of the system $\Sigma Q$ at the $k+1$-th time is given by Fig. 11. Feedback system with input noise.
\[ z_Q(k+1, x_0, R) = C_1 \tilde{A}^{k+1} x_0 + \sum_{i=0}^{k} \left( C_1 \tilde{A}^{k-i} B_2 \eta(i) \right) \]
\[ + \sum_{i=0}^{k} \left( C_1 \tilde{A}^{k-i} \tilde{B} r(i) \right) + D_1 r(k+1). \]  

(8)

On the other hand, since \( \Sigma_T \) is equivalent to the system in Fig. 11 with \( \eta(k) = 0 \), the output of the system \( \Sigma_T \) is

\[ z_I(k+1, x_0, R) = C_1 \tilde{A}^{k+1} x_0 \]
\[ + \sum_{i=0}^{k} \left( C_1 \tilde{A}^{k-i} \tilde{B} r(i) \right) + D_1 r(k+1). \]  

(9)

Thus, the output difference between the two systems (in the expectation sense) is estimated as

\[
E \left( (z_Q(k+1, x_0, R) - z_I(k+1, x_0, R))^2 \right)
= E \left( \left( \sum_{i=0}^{k} C_1 \tilde{A}^{k-i} B_2 \eta(i) \right)^2 \right)
\leq E \left( \sum_{i=0}^{k} \left( C_1 \tilde{A}^{k-i} B_2 \eta(i) \right)^2 \right)
= \sum_{j=0}^{k} (C_1 \tilde{A}^j B_2)^2 e(k-j)^2
= \sum_{j=0}^{k} (C_1 \tilde{A}^j B_2)^2 d^2 \frac{j^2}{4}.
\]  

(10)

The first equation is obtained from (8) and (9). The second relation is given by Lemma 1 (iii) and the triangle inequality. The third one is the variable transformation as \( j = k - i \). The last one is obtained from Lemma 1 (ii) and (iii).

In (10), the upper bound of \( E((z_Q(k+1, x_0, R) - z_Q(k+1, x_0, R))^2) \) is monotonic increasing. Therefore, considering the case \( k \to \infty \) results in the following theorem.

**Theorem 2.** For the system \( \Sigma_Q \), suppose that the system \( G \) is given and let \( Q \) be the random dither quantizer defined in (2). Then,

\[ J \leq \sum_{j=0}^{\infty} (C_1 \tilde{A}^j B_2)^2 d^2 \frac{j^2}{4}. \]  

(11)

Theorem 2 presents an upper bound of the performance index \( J \) in (3). In (11), \( C_1 \tilde{A}^j B_2 \) \((j = 0, 1, \ldots)\) are the impulse response matrices of the system in Fig. 11 (from \( \eta \) to \( z \)) and \( d^2/4 \) is the square of the amplitude of the dither signal \( w \), which characterize the upper bound.

The following example shows the validity of this theorem.

**Example 3.** Consider again the example in Fig. 6. The performance of the quantizer is quantified as \( J \leq 3.23 \times 10^{-2} \) by Theorem 2. On the other hand, the maximum value of \( E((z_Q(k, x_0, R) - z_I(k, x_0, R))^2) \) at \( 0 \leq k \leq 50 \) (based on 1000 trials) is \( 2.24 \times 10^{-2} \). By noting that the former is the value for the worst \( (x_0, R) \) and the latter is one for the specified \( x_0 \) and \( R \), i.e., \( x_0 = [-1 \ 1]^T \) and \( R = (0, 0, \ldots) \), it turns out that the upper bound in Theorem 2 gives a good estimation.

4. RELATION BETWEEN THE PERFORMANCE AND THE SAMPLING PERIOD

Consider the system \( \Sigma_Q \). For the given sampling period \( h \in \mathbb{R} \) and the continuous-time parameters \( A_c \in \mathbb{R}^{n \times n}, B_{1c} \in \mathbb{R}^{n \times 1}, B_{2c} \in \mathbb{R}^{1 \times 1}, C_{1c} \in \mathbb{R}^{1 \times n}, C_{2c} \in \mathbb{R}^{1 \times n}, D_{1c} \in \mathbb{R}, D_{2c} \in \mathbb{R}, \) let \( J_h \) denote the performance index \( J \) of (3) when

\[
\begin{align*}
    k &:= \left\lfloor \frac{t}{h} \right\rfloor, \\
    \tilde{A} &:= e^{\tilde{A} h}, \\
    B_1 := \int_0^h e^{A_c \tau} d\tau B_{1c}, & B_2 := \int_0^h e^{A_c \tau} d\tau B_{2c}, \\
    C_1 := C_{1c}, & C_2 := C_{2c}, \\
    D_1 := D_{1c}, & D_2 := D_{2c}
\end{align*}
\]  

(12)

where \( \tilde{A}_c := A_c + B_{2c} C_{2c} \). \( t \) is the time variable in the continuous-time domain, and \( \tilde{A} \) is assumed to be Hurwitz. Roughly speaking, when \( h_1 > h_2 \), \( J_{h_2} \) is a more precise index than \( J_{h_1} \). In fact, we have \( J_{h_1} \leq J_{h_2} \) for \( h_2 = h_1/N \) \((N \in \mathbb{N})\). In the following part, it will be shown that \( J_h \) goes to zero as \( h \) gets smaller despite the fact that \( J_h \) becomes more precise as \( h \to 0 \).

From (11) and (12),

\[
\begin{align*}
    J_h &\leq \sum_{j=0}^{\infty} (C_1 \tilde{A}^j B_2)^2 d^2 \frac{j^2}{4} \\
    &\leq \sum_{j=0}^{\infty} \|C_1\|^2 \|\tilde{A}^j\|^2 \|B_2\|^2 d^2 \frac{j^2}{4} \\
    &\leq \|C_{1c}\|^2 \left( \sum_{j=0}^{\infty} \|e^{\tilde{A}_c j}\|^2 \right) \left( \int_0^h e^{A_c \tau} d\tau B_{2c} \right)^2 d^2 \frac{j^2}{4} \|B_{2c}\|^2 \|e^{\tilde{A}_c h}\|_\star \leq 1.
\end{align*}
\]  

(13)

holds. Since \( \tilde{A}_c \) is stable, there exists a norm \( \|e^{\tilde{A}_c h}\|_\star \), such that

\[ \|e^{\tilde{A}_c h}\|_\star \leq 1. \]  

(14)

Then, for \( \lambda := \|e^{\tilde{A}_c h}\|_\star \),

\[ \|e^{\tilde{A}_c h}\|_\star \leq \lambda h \]  

(15)

holds. Therefore, from the norm equivalence,

\[ \|e^{\tilde{A}_c h}\| \leq a \|e^{\tilde{A}_c h}\|_\star \leq a \lambda h \]  

(16)

for some constant number \( a > 0 \), and further

\[ \sum_{j=0}^{\infty} \|e^{\tilde{A}_c h}\|^2 \leq \sum_{j=0}^{\infty} a^2 \lambda^2 h^2 = \frac{a^2}{1 - \lambda^2 h^2}. \]  

(17)

On the other hand, the norm of \( B_2 \) is estimated as

\[
\begin{align*}
    \left\| \int_0^h e^{A_c \tau} d\tau B_{2c} \right\| &\leq \left\| \left( h + \frac{1}{2} A_c h^2 + \frac{1}{3!} A_c^2 h^3 + \cdots \right) B_{2c} \right\| \\
    &\leq \left( h + \frac{1}{2} \|A_c\| h^2 + \frac{1}{3!} \|A_c\|^2 h^3 + \cdots \right) \|B_{2c}\| \\
    &= \|B_{2c}\| \|e^{\|A_c\| h} - 1\|.
\end{align*}
\]  

Equations (13), (17) and (18) provide the following result.
Theorem 4. For the system \( \Sigma Q \), suppose that the system \( G \) is given and let \( Q \) be the random dither quantizer defined in (2). Then,

\[
\begin{align*}
(i) & \quad J_h \leq \frac{a^2 \|C_1\|^2 \|B_{2h}\|^2 (e^{\|A_c\|/h} - 1)^2 \|A_c\|^2}{1 - \lambda^{2h}} \frac{d^2}{4}, \\
(ii) & \quad \lim_{h \to 0} J_h = 0
\end{align*}
\]

hold, where \( a > 0 \) is some constant number satisfying (16).

(Proof) (i) It is a straightforward consequence of (13), (17) and (18).

(ii) Taylor’s theorem gives

\[
(e^{\|A_c\|/h} - 1)^2 = \left( \frac{1}{1!} \|A_c\| + \frac{1}{2!} \|A_c\|^2 h^2 + \cdots \right)^2
\]

\[
= h^2 \left( \frac{1}{1!} \|A_c\| + \frac{1}{2!} \|A_c\|^2 h^2 + \cdots \right)^2,
\]

\[
1 - \lambda^{2h} = - \left( \frac{1}{1!}(2 \log \lambda) h + \frac{1}{2!}(2 \log \lambda)^2 h^2 + \cdots \right)
\]

\[
= -h \left( \frac{1}{1!}(2 \log \lambda) + \frac{1}{2!}(2 \log \lambda)^2 h + \cdots \right). \tag{22}
\]

Applying (21) and (22) to the term \((e^{\|A_c\|/h} - 1)^2/(1 - \lambda^{2h})\) in (19) provides

\[
\lim_{h \to 0} \frac{(e^{\|A_c\|/h} - 1)^2}{1 - \lambda^{2h}} = 0. \tag{23}
\]

This proves (ii). \( \Box \)

The relation between the performance of random dither quantizers and the sampling period is given by Theorem 4. With this result, the controller or the sampling period can be determined as a random dither quantizer has the desired performance. In particular, if sampling period can be set small enough, the output of the system \( \Sigma Q \) becomes quite similar to that of the system \( \Sigma Y \).

5. CONCLUSION

In this paper, the performance of the uniform-type random dither quantizers has been analyzed from the view point of control. In particular, an upper bound of the performance has been presented and the relation between the performance and the sampling period has been shown. These results will be useful for the analysis and design of quantized control systems with random dither quantizers.

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