Delay-Dependent Robust $H_{\infty}$ Output Tracking Control for Uncertain Networked Control Systems

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Abstract: In this paper, the $H_{\infty}$ output tracking problem for networked control systems (NCSs) liable to model uncertainties and with delays varying within intervals is investigated. A new method is proposed to the robust $H_{\infty}$ output tracking analysis and control design for NCSs. The results presented here are an improvement over previous ones, for the analysis incorporates state-of-the-art stability techniques for systems with time-varying delays such as convex optimization technique, piecewise analysis method, and a new delay fractioning approach with the introduction of a novel auxiliary delayed state. The analysis is enriched with a numerical example that illustrates the advantages of our criteria which outperform previous criteria in the literature. Moreover, an illustrative example shows the effectiveness of the proposed $H_{\infty}$ output tracking control design.

1. INTRODUCTION

Networked Control Systems (NCSs) refer to a class of control systems whose elements (plant, controller, actuators and sensors) are linked together through a multipurpose shared communication network and the information is exchanged in the form of data packets [Zhang et al., 2001]. This class of systems have many advantages compared to the traditional local control architecture, including lower costs, simple installation and maintenance, and reduced weight [Zhang et al., 2010]. Consequently, NCS modeling, stability, and stabilization problems have emerged as a topic of significant interest to the control community, which is highlighted by several surveys on the subject [see, e.g., Yang, 2006, Hespanha et al., 2007, Antsaklis and Baillieul, 2007]. However, although a fundamental issue in control theory, tracking performance analysis and control design has received little attention in the NCS literature. In this paper, we study the $H_{\infty}$ output tracking problem for uncertain NCSs with time-varying delay.

It is well recognized that the tracking problem is more general and challenging than stability and stabilization problems [Gao and Chen, 2008]. The main objective of tracking control is to synthesize feedback controllers to make the output of a given plant asymptotically tracks a desired reference whereas ensuring disturbances attenuation properties. The importance of tracking arises from the extensive applications in robot control [Braganza et al., 2008], flight control [Liao et al., 2002], and so on. For other applications and important results on output tracking see Chen and Su [2002], Shieh et al. [2003], and the references there in.

The existing results on output tracking rarely focus on control design for NCSs subjected to time delays and packet droputs. Among the recent works concerning the tracking problem for NCSs, the following should be mentioned for their contribution to the analysis. Lopez et al. [2008] provide conditions for network-based tracking of non-delayed systems with reference to the mutual information rate between the feedback and the reference signal. Wang and Yang [2009] study the $H_{\infty}$ approach towards the output tracking problem in terms of the communication protocol and the network medium access. Li et al. [2008] investigate the tracking problem for switched linear systems with delayed states, however the effects of the network-induced delay are not taken into account in the feedback-loop. In [van de Wouw et al., 2007, 2009], the approximate tracking problem for NCSs is solved under the framework of sampled-data systems; however the delay is highly dependent on the time-varying sampling interval. The work from Gao and Chen [2008] addresses to the robust $H_{\infty}$ output tracking problem with respect to NCSs with time-varying delay and constant sampling period. Nevertheless, none of the previous results on output tracking control have taken into consideration the recent advances from the time-delayed systems’ stability analysis.

During the last decade, various methods have been taken for deriving stability conditions for delayed systems and for NCSs using different Lyapunov–Krasovskii functionals (LKF), however few have been extended to deal with the $H_{\infty}$ output tracking problem. Particularly, the employment of Jensen’s inequality instead of using the cross-terms bounding is a well-established approach that leads to less conservative results. However, this still is a conservative analysis, for the time-varying delay is bounded when considering terms containing not only the delay bounds, but also the delay itself. Instead of bounding the time-varying...
delay, the convex optimization technique incorporated with the Jensen’s inequality proved to be effective in [Park and Ko, 2007]. Recently, new Lyapunov functional candidates inspired on Gouaisbaut and Peaucelle [2006] have enriched the stability analysis by extending the piecewise analysis method from Gouaisbaut and Peaucelle [2006] to systems with time-varying delays, [see, e.g., Fridman et al., 2009, Yue et al., 2009, Figueredo et al., 2010, Oribuela et al., 2010]. Particularly, Fridman et al. [2009], Figueredo et al. [2010] also explore the information about the delay derivative’s lower bound through the employment of delay-interval-dependent terms in the LKF.

In this context, the present paper brings an important contribution to the $H_{\infty}$ output tracking analysis and control design for NCSs with time-varying delay. By developing a novel Lyapunov functional and applying state-of-the-art stabilized techniques for systems with time-varying delays, we are able to derive conditions under which the prescribed $H_{\infty}$ output tracking performance for uncertain NCS is achieved. It should be mentioned that although dealing only with the $H_{\infty}$ output tracking problem, the proposed method if adapted to the NCS stability problem also yields less conservative results than previous criteria in the literature. Furthermore, the criterion is extended to deal with the $H_{\infty}$ output tracking control design problem. The analysis is enriched with a numerical example that illustrates the advantages of our criteria which outperform previous criteria in the literature. Finally, an illustrative example shows the effectiveness of the proposed $H_{\infty}$ output tracking controller design. Theoretical proofs will be omitted due to lack of space, but can be found in Figueredo [2011].

2. PROBLEM FORMULATION AND PRELIMINARIES

We shall consider a closed-loop NCS consisting of an LTI plant and a reference model connected to a controller module through a shared network. All the network communications is performed by the Sender and the Receiver elements, which are responsible for transmitting and acquiring data packets through the network, respectively.

At instants $n\ell_k$, the sensor module sends measurements from the plant and from the reference model over the network, where $h$ is the sampling period and $n \in \mathbb{N}^*$. The Controller and Actuator modules are event-driven and start to process a new packet immediately after its arrival. The integers $\ell_k$, $k \in \mathbb{N}^*$, denote the $k$th sample number which is carried by the $k$th received network packet at the actuator’s input. Single packet transmission is assumed, i.e., all data is assembled together into one network packet and transmitted at the same time.

Remark 1. If \{$\ell_1, \ell_2, \ldots, \ell_n, \ldots$\}, then no packet dropout occurred in the transmission. However, if the $p$-th sample was lost, then $\exists q \in \mathbb{N}^*$, such that $\ell_q=p$. Packet disordering occurs when a packet reaches its destination later than its successors, i.e., $\exists p, q \in \mathbb{N}^*$, $p > q$, such that $\ell_q < \ell_p$. In this case, the old data are discarded.

Furthermore, the following delays are considered:

- $\tau_{sc}^g$: sensor to controller delay for the $k$-th packet;
- $\tau_c^k$: controller to actuator delay for the $k$-th packet;
- $\tau_s^k$: total delay (sensor to actuator) for the $k$-th packet;

We assume the existence of constants $\tau_{min}$ and $\tau_{max}$,

$$\begin{align*}
\tau_{min} & \leq \tau_k, \\
\tau_{max} & \geq \tau_k,
\end{align*}$$

where $\tau_{max}$ denotes the upper bound of the total network-induced delay, involving both transmission delays and packet dropouts. The term $\tau_{min}$ denotes the lower bound and has an analogous definition.

The plant’s process has a state space model of the form

$$\begin{align*}
\dot{x}_p(t) = (A_p + \Delta A_p)x_p(t) + (B_p + \Delta B_p)u_p(t) + B_w w(t), \\
y_p(t) = (C_p + \Delta C_p)x_p(t) + (D_p + \Delta D_p)u_p(t),
\end{align*}$$

where $x_p(t) \in \mathbb{R}^{n_p}$ is the plant’s state vector, $u_p(t) \in \mathbb{R}^{r_p}$ and $y_p(t) \in \mathbb{R}^{r_y}$ are the plant’s input and output vectors, respectively, and $w(t) \in \mathbb{R}^{r_w}$ is the exogenous disturbance signal which is assumed to belong to $L_2[0, \infty)$. The matrices $A_p, B_p, C_p, D_p$ are real constant matrices with appropriate dimensions. The uncertainties $\Delta A_p, \Delta B_p, \Delta C_p$, and $\Delta D_p$ are time-varying matrices, which are defined as follows:

$$\begin{align*}
\Delta A_p & = H_1 (\Delta t) [\Xi_A \Xi_B], \\
\Delta C_p & = H_2 (\Delta t) [\Xi_C \Xi_D],
\end{align*}$$

where $H_1, H_2, \Xi_A, \Xi_B, \Xi_C, \Xi_D$ are constant matrices with appropriate dimensions and $(\Delta t)$ represents an unknown time-varying matrix, which is Lebesgue measurable in $t$ and satisfies $\Delta^2 (\Delta t) \leq I$.

We consider the reference signal, $y_r(t) \in \mathbb{R}^{r_y}$, to be the output of the given linear system:

$$\begin{align*}
\dot{x}_r(t) &= A_r x_r(t) + r(t), \\
y_r(t) &= C_r x_r(t),
\end{align*}$$

where $x_r(t)$, $r(t) \in \mathbb{R}^{r_y}$ are the reference’s state vector and the energy bounded reference input, respectively, $A_r$ is a Hurwitz matrix, and $C_r$ is an appropriately dimensioned constant matrix.

Considering the communication delay from sensor to controller, and a state feedback control law with gains $K_1$ and $K_2$, the plant’s input $u_p(t)$ can be described as

$$u_p(t) = K_1 x_p(t) + K_2 x_r(t),$$

$t \in \ell_k h + \tau_k, \ell_k+1 h + \tau_{k+1}, \forall k \in \mathbb{N}^*$, where $\tau_k = \tau_{sc}^k + \tau_c^k + \tau_s^k$.

Therefore, from (1)-(6), defining $\tau(t) \in \mathbb{R}^{r_y}$, such that $\tau(t):=[x_p^T(t), x_r^T(t)]$, $\tau^T(t):=[w^T(t), r^T(t)]$, and considering

$$A=\begin{bmatrix} A_p & 0 \\ 0 & A_r \end{bmatrix}, B=\begin{bmatrix} B_p \\ 0 \end{bmatrix}, A_w=\begin{bmatrix} B_w & 0 \end{bmatrix}, C=[C_p \ C_r], D=D_p,$$

we obtain the augmented closed-loop system for $t \in \ell_k h + \tau_k, \ell_k+1 h + \tau_{k+1}, \forall k \in \mathbb{N}^*$,

$$\begin{align*}
(\overline{\tau})(t) &= (\overline{\tilde{A}}+\Delta \overline{\tilde{A}}) \overline{\tau}(t) + \overline{\tilde{B}} \overline{\tau}(t-d(t)) + \overline{A}_w \overline{\tau}(t), \\
e(t) &= (\overline{\tilde{C}}+\Delta \overline{\tilde{C}}) \overline{\tau}(t) + (\overline{\tilde{D}}+\Delta \overline{\tilde{D}}) \overline{\tau}(t-d(t)),
\end{align*}$$

where

$$\begin{align*}
\overline{\tau}(t) &= (t) + \overline{\tau}(t), \\
e(t) &= y_r(t) - y_r(t),
\end{align*}$$

and $\rho(t)$ is a given function which describes the state’s initial condition. The time-varying matrices are defined as follows:

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where $0 \leq \tau_{min} \leq \tau_{max}$ are constants. It’s notable that $d(t)$ is piecewise linear with derivative $d(t) = 1$ for $t \neq \ell_k h + \tau_k$. Therefore, the time-varying delay $d(t)$ is discontinuous at the interrupt points $t = \ell_k h + \tau_k$, $\forall k \in \mathbb{N}^*$. 

**Tracking problem:** We desire the plant’s output $y_d(t)$ to asymptotically track a given reference signal $y_r(t)$. Our purpose is therefore to design a robust state-feedback controller $K$ such that the output tracking performance $\gamma$ is ensured in the $H_\infty$ sense. 

**Definition 1.** For a prescribed scalar $\gamma > 0$, the $H_\infty$ output tracking performance is achieved with an $H_\infty$ norm bound $\gamma$, if the following hold

1. The augmented closed-loop NCS (7) with $\mathcal{P}(t) \equiv 0$ is robustly asymptotically stable;
2. Regardless of the values of $u(t)$ and $r(t)$, the disturbance effect on the tracking error $e(t)$ is attenuated below a desired level in the sense of $H_\infty$ performance with index $\gamma$. Therefore, $\|e\|_2 < \gamma \|w\|_2$ must hold for all nonzero $w \in L_2(0, \infty)$ under zero initial condition. 

3. $H_\infty$ OUTPUT TRACKING CONTROL DESIGN

This section presents the main results of this paper. Firstly, we shall, similarly to [Fridman et al., 2009, Figueroa et al., 2010], divide the delay range $[\tau_{min}, \tau_{max}]$. Here we will consider two equally spaced subintervals: $[\tau_1, \tau_2]$ and $[\tau_2, \tau_3]$, where $\tau_1 = \tau_{min}$, $\tau_3 = \tau_{max}$, and $\tau_2 = \frac{1}{2} (\tau_{max} + \tau_{min})$. Therefore, the augmented uncertain delayed system (7) can be rewritten as

$$
\begin{align*}
\bar{\mathcal{P}}(t) = & \left( A + \Delta A \right) \bar{x}(t) + \chi_{[\tau_1, \tau_2]}(d(t)) (B + \Delta B) \bar{K} \bar{x}(t-d(t)) + \bar{A} \bar{w}(t) + (1 - \chi_{[\tau_1, \tau_2]}(d(t))) (D + \Delta D) \bar{x}(t-d(t)) \\
& + \left( C + \Delta C \right) \bar{x}(t) + \chi_{[\tau_1, \tau_2]}(d(t)) (B + \Delta B) \bar{K} \bar{x}(t-d(t)) + (1 - \chi_{[\tau_1, \tau_2]}(d(t))) (D + \Delta D) \bar{K} \bar{x}(t-d(t)),
\end{align*}
$$

(10)

where $\chi_{[\tau_1, \tau_2]} = 1$, if $s \in [\tau_1, \tau_2]$, and $\chi_{[\tau_1, \tau_2]} = 0$, otherwise. 

3.1 $H_\infty$ output tracking performance Analysis

In this subsection, we derive conditions under which the augmented closed-loop NCS achieves $H_\infty$ output tracking performance $\gamma$, namely, the augmented-closed-loop system is stable and satisfies the performance conditions $\|e\|_2 < \gamma \|w\|_2$. The proposed stability analysis is based on the Lyapunov-Krasovskii functional candidate

$$
V(t) = \sum_{i=1}^{5} V_i(t),
$$

(11)

where

$$
\begin{align*}
V_1(t) &= x^T(t) P x(t), \\
V_2(t) &= \int_{\ell_{n-1} h}^{t} \left[ x^T(s) \right]^{T} \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix} \left[ \begin{bmatrix} x(s) \\ x(s-\tau_2+1) \end{bmatrix} \right] ds, \\
V_3(t) &= \int_{t-\tau_2}^{t} \left[ x^T(s-\tau_2+1) \right]^{T} \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \left[ \begin{bmatrix} x(s) \\ x(s-\tau_2+1) \end{bmatrix} \right] ds,
\end{align*}
$$

$$
\begin{align*}
V_4(t) &= \left( \frac{1}{2} \right) \int_{-\tau_1}^{t} \dot{x}^T(s) Z_1 \dot{x}(s) ds dt, \\
V_5(t) &= \left( \frac{1}{2} \right) \int_{-\tau_1}^{t} \dot{x}^T(s) Z_2 \dot{x}(s) ds dt,
\end{align*}
$$

(8)

and given controller gain $\bar{K}$, the achieved $H_\infty$ output tracking performance $\gamma$ if there exist scalars $\epsilon_1 > 0$ and $\epsilon_2 > 0$, $\gamma \leq 1, 2, 3, 4$, and matrices $P, Z_j, j \in \{1, 2, 3, 4\}, N$ and $M$, with appropriate dimensions satisfying (12), and free-weighting matrices $\Sigma_1 \in \mathbb{R}^{\tau_1 \times \tau_1}, \Sigma_2 \in \mathbb{R}^{\tau_1 \times \tau_2}, \Sigma_3 \in \mathbb{R}^{\tau_2 \times \tau_2}$, and $\Sigma_4 \in \mathbb{R}^{\tau_2 \times \tau_2}$, such that the following LMIs hold:

$$
\begin{align*}
\Omega_{121} &= \left[ \frac{\psi_1^{(1)}}{\psi_2^{(1)}} \right] \begin{bmatrix} (\tau_2 - \tau_1) S_1 \Gamma_m & \gamma_1 \bar{A}_w \Gamma_m Z_k & \gamma_1 \Gamma_m \end{bmatrix} \\
& \quad + (S_1 G_1 + G_1^T S_2) \begin{bmatrix} (\tau_2 - \tau_1)^2 Z_3 & 0 & 0 \\ 0 & -\gamma^2 I & 0 \\ 0 & 0 & -\gamma^2 I \end{bmatrix} \\
& \quad + \begin{bmatrix} I & \Gamma_m \end{bmatrix}, \quad \epsilon_m \leq 0 \\
\Omega_{221} &= \left[ \frac{\psi_1^{(2)}}{\psi_2^{(2)}} \right] \begin{bmatrix} (\tau_2 - \tau_1) S_2 \Gamma_m & \gamma_2 \bar{A}_w \Gamma_m Z_k & \gamma_2 \Gamma_m \end{bmatrix} \\
& \quad + (S_2 G_2 + G_2^T S_5) \begin{bmatrix} (\tau_2 - \tau_1)^2 Z_4 & 0 & 0 \\ 0 & -\gamma^2 I & 0 \\ 0 & 0 & -\gamma^2 I \end{bmatrix} \\
& \quad + \begin{bmatrix} I & \Gamma_m \end{bmatrix}, \quad \epsilon_m \leq 0
\end{align*}
$$

(13)

where

$$
\begin{align*}
& \psi_1^{(1)} = \bar{A} \bar{w}(t) - \bar{A} \bar{w}(t-d(t)) + \bar{A} \bar{w}(t) - \bar{A} \bar{w}(t-d(t)), \\
& \psi_2^{(1)} = \bar{A} \bar{w}(t) - \bar{A} \bar{w}(t-d(t)) + \bar{A} \bar{w}(t) - \bar{A} \bar{w}(t-d(t)), \\
& \gamma_1 \bar{A}_w = \begin{bmatrix} 0 & I \\ 0 & -I \end{bmatrix}, \\
& \gamma_2 \bar{A}_w = \begin{bmatrix} 0 & I \\ 0 & -I \end{bmatrix}, \\
& \bar{A} \bar{w}(t) = \begin{bmatrix} A \bar{w}(t) \\ B \bar{w}(t) \end{bmatrix}, \\
& \bar{A} \bar{w}(t-d(t)) = \begin{bmatrix} A \bar{w}(t-d(t)) \\ B \bar{w}(t-d(t)) \end{bmatrix}.
\end{align*}
$$

with $m \in \{1, 2\}$, and

$$
\begin{align*}
& G_1 = \begin{bmatrix} 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \\ 0 & 0 & 0 & I \\ 0 & 0 & 0 & I \end{bmatrix}, \\
& G_2 = \begin{bmatrix} 0 & 0 & 0 & I \\ 0 & -I & 0 & 0 \\ 0 & 0 & -I & 0 \\ 0 & 0 & 0 & I \end{bmatrix}, \\
& \Gamma_m = \begin{bmatrix} \Sigma_1 \Sigma_2 \Sigma_2 \Sigma_1 \end{bmatrix}, \\
& \Gamma_m = \begin{bmatrix} \Sigma_1 \Sigma_2 \Sigma_2 \Sigma_1 \end{bmatrix},
\end{align*}
$$

(12)

where $\Delta_k = \begin{bmatrix} \epsilon_1 \hat{H}_1 & 0 \epsilon_1 \hat{G}_1 \epsilon_2 \hat{G}_1 \epsilon_2 \hat{G}_1 \end{bmatrix}$, $\epsilon_1 = \text{diag} \{\epsilon_{11}, \epsilon_{12}, \epsilon_{11}, \epsilon_{12}\}$, $\epsilon_2 = \text{diag} \{\epsilon_{11}, \epsilon_{12}, \epsilon_{11}, \epsilon_{12}\}$, $\epsilon_{11} = \text{diag} \{\epsilon_{11}, \epsilon_{12}, \epsilon_{11}, \epsilon_{12}\}$.
Example 1. Consider the following augmented system (7) with no uncertainties:

\[
\begin{bmatrix}
\Psi_{11} & 0 & P & Z_1 + M_1 & 0 & 0 & 0
\\
* & 0 & 0 & 0 & 0 & 0 & 0
\\
* & * & \Psi_{33} & 0 & 0 & 0 & 0
\\
* & * & * & \Psi_{44} & Z_2 - M_1 & 0 & 0
\\
* & * & * & * & \Psi_{55} & N_1 & 0
\\
* & * & * & * & * & N_2 - N_1 & -Z_4 - N_1 & Z_4
\\
* & * & * & * & * & * & * & -N_2 & -Z_2
\end{bmatrix}
\]

and

\[
\begin{bmatrix}
\Psi_{11} = M_1 - Z_1,
\\
\Psi_{33} = \left(\frac{\tau_1}{\tau_2}\right)^2 (Z_1 + Z_2) + (\tau_2 - \tau_1)^2 Z_3 + (\tau_3 - \tau_2)^2 Z_4,
\\
\Psi_{44} = M_2 - M_1 - Z_1 - Z_2,
\\
\Psi_{55} = N_1 - M_2 - Z_2.
\end{bmatrix}
\] (14)

Remark 2. Theorem 1 presents conditions which guarantee the \( H_\infty \) output tracking performance \( \gamma \). To establish this result, we have applied state-of-the-art stability techniques for systems with time-varying delays. In the derivative of the Lyapunov functionals \( V_4(t) \) and \( V_5(t) \), the cross-term bounding is avoided by the employment of Jensen’s inequality which, when incorporated with the recent convex optimization technique, introduced in [Park and Ko, 2007], yields even better results. The convex analysis is used to reduce the conservativeness related to the bounding of the time-varying delay \( d(t) \). Moreover, the analysis is further improved through the employment of the piecewise analysis method which divides the delay-range into two equally spaced subintervals, as stressed in (10). This method enables us to investigate and to establish different LMIs conditions for each subinterval, reducing considerably the conservativeness which arises from the LMI analysis of the interval \([\tau_{\text{min}}, \tau_{\text{max}}]\). The improvements are consequence of the different expressions obtained in the derivative of the Lyapunov functional when \( d(t) \leq \tau_2 \) and when \( d(t) > \tau_2 \). Notwithstanding, another contribution to the analysis is obtained with the employment of a new delay-fractioning approach. The benefits from this analysis arise from the introduction of the auxiliary delayed state \( x(t - \hat{\omega}_m) \) which allows further exploitation of the delay’s lower bound value, \( \tau_{\text{min}} \). Therefore, less conservative conditions are obtained, especially when \( \tau_{\text{min}} \to \tau_{\text{max}} \).

3.2 \( H_\infty \) output tracking controller design

Based on Theorem 1, we now propose a novel criterion which if satisfied solves the problem of robust \( H_\infty \) output tracking controller design for uncertain NCSs with time-varying delay.

Theorem 2. For given scalars \( \gamma > 0, \tau_{\text{min}}, \) and \( \tau_{\text{max}} \), such that \( 0 \leq \tau_{\text{min}} \leq \tau_{\text{max}} \), there exist a state-feedback gain \( \hat{K} \) such that the augmented closed-loop NCS (7) with time-varying delay \( d(t) \) satisfying (9), and uncertainties described by (8) achieves the \( H_\infty \) output tracking performance \( \gamma \), if there exist scalars \( \epsilon_{11} > 0 \) and \( \epsilon_{12} > 0 \),

\[
\begin{array}{l}
\text{Table 1. Min. value of } \gamma \text{ for } \tau_{\text{max}}=0.430 (\text{Ex. 1}) \\
\hline
\text{Method} & \tau_{\text{min}} & 0.05 & 0.1 & 0.15 & 0.2 \\
\hline
\text{Gao and Chen (2008)} & 3.9018 & 3.1017 & 2.5700 & 2.1922 & 1.9103 \\
\hline
\text{Theorem 1} & 1.6283 & 1.5795 & 1.5296 & 1.4783 & 1.4783 \\
\text{obtained improvement:} & (58.3\%) & (49.1\%) & (40.5\%) & (32.5\%) & (22.6\%) \\
\hline
\end{array}
\]

\( i = \{1, 2, 3, 4\} \), and matrices \( P, Z_i, j \in \{1, 2, 3, 4\}, N \) and \( M \), with appropriate dimensions satisfying (12), free-weighting matrices \( S_1 \in \mathbb{R}^{r_x \times r_x}, S_2 \in \mathbb{R}^{r_y \times r_y}, Y \in \mathbb{R}^{r_y \times r_y}, \) and a definite positive matrix \( X \in \mathbb{R}^{r_x \times r_x}, \) such that (13) hold, with

\[
\begin{bmatrix}
\Psi(1) & \Psi(2)
\end{bmatrix}
\]

and

\[
\begin{align*}
\Psi(1) &= \\
\Psi(2) &=
\end{align*}
\]

4. EXAMPLES

In this section, we first consider the example presented in [Gao and Chen, 2008] to illustrate the advantages of the proposed \( H_\infty \) performance criterion which yields less conservative results when compared with state-of-the-art criteria. Also, we shall illustrate the effectiveness of Theorem 2 for the design of feedback controllers to make the system’s output tracks a desired reference whereas ensuring disturbances attenuation properties. The second example concerns the \( H_\infty \) output tracking control design problem for a satellite system. The effectiveness of the analysis is also illustrated in a simulation scenario.

Example 1. Consider the following augmented system (7) with no uncertainties:

\[
\begin{align*}
\text{Table 1. Min. value of } \gamma \text{ for } \tau_{\text{max}}=0.430 (\text{Ex. 1}) \\
\hline
\text{Method} & \tau_{\text{min}} & 0.05 & 0.1 & 0.15 & 0.2 \\
\hline
\text{Gao and Chen (2008)} & 3.9018 & 3.1017 & 2.5700 & 2.1922 & 1.9103 \\
\hline
\text{Theorem 1} & 1.6283 & 1.5795 & 1.5296 & 1.4783 & 1.4783 \\
\text{obtained improvement:} & (58.3\%) & (49.1\%) & (40.5\%) & (32.5\%) & (22.6\%) \\
\hline
\end{align*}
\]
Assuming $\tau_{\text{max}}=0.430$, the minimum guaranteed $H_{\infty}$ output tracking performances $\gamma$ for various values of $\tau_{\text{min}}$ are listed in Table 1. The obtained results clearly represent an important improvement over those from Gao and Chen [2008]. The improvements over the results from Gao and Chen [2008] are as high as 58% for $\tau_{\text{min}}=0$, which means that Theorem 1 assures that the actual tracking error for the described NCS is lesser than 42% of the expected value obtained in [Gao and Chen, 2008]. Nevertheless, further improvements for the system’s performance may be achieved by designing a feedback controller which takes the network characteristics in consideration. In this context, by solving the LMIs presented in Theorem 2 with

$$\begin{bmatrix} \bar{A} & 0 & 0 \\ -1 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0.2 \\ 0.1 \end{bmatrix}, \quad \begin{bmatrix} 0.1 \\ 0 \\ 0 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 0 & -0.5 \end{bmatrix}, \quad \begin{bmatrix} 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \end{bmatrix},$$

$$\bar{C}=[1 \ 0 \ -0.5], \quad \bar{D}=0.5, \quad \bar{K}=\bar{K}_{\text{previous}}=[-1 \ 1 \ 1].$$

Example 2. We shall now consider an example of a satellite system that appears in [Gao and Chen, 2008]. The system consists of two rigid bodies joined together by a flexible link which is modeled as a spring with constant torque $k=0.09$ $Nm$ and viscous damping $f=0.04$ $Ns/m$. Considering the two bodies, we denote the yaw angles by $\theta_1$ and $\theta_2$, and the inertia moments by $J_1=1$ $Kgm^2$ and $J_2=1$ $Kgm^2$. The control torque is $u(t)$, and the torque disturbance is $w(t)$. Taking the angular position $\theta_2$ as the system’s output, we have the following state-space representation

$$\begin{bmatrix} \bar{\theta}_1(t) \\ \bar{\theta}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -0.09 & 0.09 & -0.04 & 0.04 \\ 0.09 & -0.09 & 0.04 & -0.04 \end{bmatrix} \begin{bmatrix} \bar{\theta}_1(t) \\ \bar{\theta}_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} w(t),$$

$$y(t)=\begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{\theta}_1(t) \\ \bar{\theta}_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} w(t).$$

The reference model given by

$$\dot{x}_r(t) = -x_r(t) + r(t), \quad \text{and} \quad y_r(t) = 0.5x_r(t).$$

In this example, we simulate the following scenario: considering the sampling interval, which may or may not be time-varying, packet dropouts, and network-induced delays, we have an upper bound $\tau_{\text{max}}=0.030$ for the time-varying delay. The lower bound of the delay is defined as $\tau_{\text{min}}=0.005$. By solving the LMIs presented in Theorem 2, we obtain the minimum guaranteed $H_{\infty}$ output tracking performance $\gamma=0.9915$ with the following controller gain

$$\bar{K}_{\text{new}}=[-72.82 \ -55.22 \ -33.74 \ -9586.62 \ 23080.56].$$

The obtained result are considerably less conservative than previous results. The improvements over the results from Gao and Chen [2008] ($\gamma=0.1267$) are as high as 27%.

For simulation purpose, we assume $w(t)=1.5\sin(5t)$, and $r(t)=10\sin(0.5t)$. The system’s initial condition is assumed to be $[-0.5 \ 1.3 \ 0.3 \ -0.3]$, and the initial condition of the reference model is assumed 0.5. The network-induced delay is obtained using the Matlab TCP/UDP/IP Toolbox, and, later, incremented with an uniform distributed random delay to achieve the prescribed delay’s bounds.

The system’s output obtained with the stabilizing controller $\bar{K}_{\text{new}}$, the system’s output obtained with the stabilizing controller proposed in [Gao and Chen, 2008], and the reference’s output are presented in Figure 1. From Figure 1, it is clear that the proposed method provides better results, and that it is effective in solving the output tracking problem whereas ensuring the disturbance attenuation properties.

The tracking error obtained with the controller $\bar{K}_{\text{new}}$ is compared with the results from...
the controller proposed in [Gao and Chen, 2008], and the results are shown in Figure 2. The mean error obtained with the stabilizing controller $\ell_{new}$ is reduced up to 25% when compared with the mean error obtained with Gao and Chen’s method.

5. CONCLUSION

This work’s main result concerns the establishment of a novel method for the $H_\infty$ output tracking analysis and control design for NCSs liable to model uncertainties and with network-induced delay varying within bounded intervals. The conservativeness of previous methods is considerably reduced by the employment of state-of-the-art stability techniques for systems with time-varying delays, and the introduction of new delay-interval-dependent LKF’s terms and a novel auxiliary delayed state. It should be mentioned that although dealing only with the $H_\infty$ output tracking problem, the proposed method if adapted to the NCS stability problem also yields less conservative results than previous criteria in the literature. The analysis is ratified with a numerical example that illustrates the advantages of our criteria, and with an illustrative example that shows the effectiveness of the proposed $H_\infty$ output tracking controller design.

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