Power System State Estimation Using Contraction Mapping and Singular Value Decomposition

Arina Aboonabi ∗ Mehrdad Saif∗

∗ Simon Fraser University, 8888 University Dr, Burnaby, BC, Canada
(e-mail: aab27, msaif @ sfu.ca).

1. INTRODUCTION

State estimation is the process of assigning values to unknown states of a system based on a set of measurements. Among this set, there might be some data, so-called bad data, which adversely affect the estimation process. In order to obtain a precise estimation, the effect of this data should be filtered.

State estimation had been introduced to power system in 1960s and since then a board class of techniques have been developed to address the problem of state estimation. The most well known technique is Least Square (LS) estimation method, (Wood et al. (1996)). To find an optimal solution, the underlying assumption in the LS estimation is that the error corrupting the system measurements must be Gaussian distributed, (Abur et al. (2004)). However, if only a part of network is affected by noise, or if the error has another probability distribution, or an unknown pattern, the LS method leads serious challenges as discussed in (Christensen et al. (1991) and Monticelli (2000)).

In (Christensen et al. (2007)), it was shown that the use of (LAV) Least Absolute Value estimator results in more accurate estimation for linear systems. Same results have then been verified in (Aboonabi et al (2010)) for nonlinear power systems by local linearization approach. However there is still a challenging issue that for large-scale system, a considerable subset of measurement data results in ill-condition system matrices and are not suitable for estimation process. This issue is addressed by introducing a modified LAV estimation technique. The set of nonlinear equations from the system is again linearized by calculating Taylor series expansion and the Jacobian matrix. The System matrices are then modified using Singular Value decomposition (SVD) in order to overcome the observability and ill conditioning problems. In the next step, the contraction mapping rule is applied to eliminate redundant measurements and obtain accurate equations which are essential for estimation. The estimated values are finally computed through the iterative process of solving selected linear equations.

The remaining parts of this paper is organized as follows: In section 2 the mathematical formulation for the nonlinear form and linearized form of the system are discussed. Characteristics of bad data and estimation criteria are also defined in this section. In section 3, the major concepts of contraction mapping and SVD are explained and an algorithm is developed to find the optimal solution for the problem at hand. In section 4, the effectiveness of the proposed technique is applied to the IEEE 118-buses power system with two different sets of bad data and the results are evaluated and compared with the existing approach.

2. PROBLEM FORMULATION

2.1 Mathematical Formulation of Nonlinear Systems

Consider a system with m simultaneous nonlinear state-space equations

\[
\begin{bmatrix}
    z_1 \\
    z_2 \\
    \vdots \\
    z_m
\end{bmatrix} =
\begin{bmatrix}
    h_1(x_1, \ldots, x_n) \\
    h_2(x_1, \ldots, x_n) \\
    \vdots \\
    h_m(x_1, \ldots, x_n)
\end{bmatrix} +
\begin{bmatrix}
    v_1 \\
    v_2 \\
    \vdots \\
    v_m
\end{bmatrix}
\]

(1)

where \( z_k, k = 1, \ldots, m \), represents the measurements obtained from the system. \( x_k, k = 1, \ldots, n \), denotes the state variables. \( h_k(x), k = 1, \ldots, m \), is a vector of nonlinear rational polynomial functions that relates the states to the measurements. \( v_k, k = 1, \ldots, m \), denotes the error in measurements.

The compact form of (1) is given by

\[
z = h(x) + v
\]

(2)

where, \( z = [z_1 \ z_2 \ \cdots \ z_m]^T \), \( x = [x_1 \ x_2 \ \cdots \ x_n]^T \), \( h(x) = [h_1(x) \ h_2(x) \ \cdots \ h_m(x)]^T \) and \( v = [v_1 \ v_2 \ \cdots \ v_m]^T \).

Measurement error \( v \) is zero for accurate measurements. For erroneous measurements, \( v \) is typically assumed to be a random number with Gaussian distribution and Probability Density Function (PDF) (3), (Wood et al. (1996)). However, experimental evidence verifies that it can have another distribution function such as Raleigh distribution (4).

\[
PDF(v) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-(v - \mu)^2}{2\sigma^2}\right)
\]

(3)
\[
PDF(v) = -\frac{v}{\sigma^2} \exp\left(-\frac{v^2}{2\sigma^2}\right)
\] (4)

In (3) and (4), \(\mu\) and \(\sigma\) present mean and variance accordingly.

**Definition 1:** In this paper true states are refereed to states which are calculated by solving the mathematical equations of system for any given operating condition in the absence of noise.

The goal of estimation is to estimate states from a given set of erroneous measurements, with a satisfying level of estimation error.

The performance matrix is defined as the Least Absolute Value estimation error given by (5):

\[
J(x) = \sum_{k=1}^{m} |e_k| = \sum_{k=1}^{m} |z_k - h_k(x)|
\] (5)

which is the total sum of absolute errors at the end of estimation process. Clearly, \(J(x)\) is zero for true states but when there are erroneous measurements, the value of \(J(x)\) would not normally go to zero, (Wood et al. (1996)).

### 2.2 Linearized System

By using Taylor series expansion technique and eliminating the derivatives of second order and higher, function \(h(x)\) is approximated around its equilibrium point, \(x_0\), by:

\[
h(x) = h(x_0) + A \Delta x
\] (6)

where \(A = H(x_0) = \frac{\partial h(x)}{\partial x}\) is the \(m \times n\) Jacobian matrix of \(h(x)\) at \(x_0\) and \(\Delta x = x - x_0\).

By substituting (6) into (2) and defining \(\Delta z = z - v - h(x_0)\), the equation (6) can be written as

\[
\Delta z = A \Delta x
\] (7)

### 3. STATE ESTIMATION BY USING CONTRACTION MAPPING AND SVD

#### 3.1 Contraction Mapping Theorem

Contraction mapping was first used in (Christensen et al. (2007)) for Least Absolute Value estimation in linear systems. The authors also introduced a nonlinear state estimator (Aboonabi et al. (2010)), using the following contraction mapping concepts.

**Assumption 1:** A complete normed vector space is considered for this study. Therefore, every cauchy sequence will have its limit within the space and if any transformation within the space is a contraction mapping, its fixed point will exist in the space.

A contraction mapping on a metric space \((M, d)\) is a function \(f\) from \(M\) to itself, with the property that there is a real number \(0 \leq \alpha < 1\) such that for all \(x, y \in M\),

\[
d(f(x), f(y)) < \alpha d(x, y)
\] (8)

where \(d(x, y)\) is the distance norm given by:

\[
d(x, y) = |x - y|
\] (9)

Thus, a contraction mapping brings every two points \(x\) and \(y\) in \(M\) closer together.

For the system given by (7), it was shown that (7) is a contraction mapping if

\[
\sum_{i} |a_{ij}| < 1
\] (10)

where \(a_{ij}\) are the elements of \(A\), the Jacobian matrix, i.e.,

\[
A = [a_{ij}], \quad \text{(Christensen et al. (2007))}
\]

If (10) is satisfied, the estimation error converges to zero by the method of successive approximation, starting from any arbitrary initial vector in the subspace. In addition, the best approximation of network and minimum error of estimation is achieved for a set of classified data with smallest contraction coefficient.

**Definition 2:** A sparse matrix is a matrix populated primarily with zeros.

The application of above concepts was tested on an IEEE 5 bus system which results in a satisfactory estimation results, Aboonabi et al (2010), however when the dimension of system increases the algorithm based on contraction mapping fails to succeed due to the sparsity problems in the system matrices.

In large-scale power networks, each bus is related to a limited number of buses within the system. Regarding the huge size of the network, this physical structure gives a sparse nature to the related Jacobian matrix of the system, since partial derivatives for a large number of states will become zero, Nor et al (2009).

Finding the smallest contraction coefficient in our linearized model could be challenging due to the mentioned sparsity of Jacobian matrix. If the number of zeros in the column gets greater than the number of states , contraction mapping could not differ between the zero elements. As a result the estimator will gain a random feature in selecting the equations, which is not desired. To overcome this problem and precise the method, SVD technique has been applied to modify system matrices such that the contraction mapping can be applied.

#### 3.2 Modified Contraction Mapping using SVD

SVD is a factorization process in which a matrix is represented as the product of three matrices as follows:

\[
A = U S V^T
\] (11)

For a \(m \times n\) matrix \(A\), \(S\) is a diagonal \(m \times n\) matrix where elements on the diagonal are non negative. Subsequently, \(U\) and \(V\) are respectively \(m \times m\) and \(n \times n\), unitary matrices and their columns contains left and right singular vectors of diagonal elements of \(S\).

Since the SVD technique deals with matrices and equations which are singular or very close to singularity (Madtharad et al (2003)), the technique can be used to resolve the issue of zero elements in the Jacobian matrix as discussed below.

When SVD factorization is applied to the matrix \(A\), matrix \(B\) can be calculated which is a non sparse \(m \times n\) matrix (12).

\[
B = U S
\] (12)

Therefore, the linearized system equation (7) can be written as (13),

\[
\Delta z = B V^T \Delta x
\] (13)

The contraction mapping criteria can now be applied to non sparse \(B\) matrix in (13) to choose the desired set of \(n\) equations.

After choosing the equations the states can be estimated using equation (14).

\[
\Delta x = (V S^T S V^T)^{-1} V B^T \Delta z_{selected}
\] (14)

where \(B_{selected}\) represents an \(n \times n\) matrix related to selected system equations and \(\Delta z_{selected}\) is an \(n \times 1\) row-vector corresponding to the related measurements.
3.3 The Estimation Algorithm

In this section an algorithm is proposed regarding the modified contraction mapping concept. The algorithm is developed for power system state estimation and it eliminates most of the bad data points within the set.

(1) Consider the system equation (2) with nonlinear \( h(x) \).

(2) Read the power system data and all the available measurements that include redundant measurements.

(3) Build the matrices in (1) based on the given data.

(4) Assign initial values to the state variables. The flat start, which sets all the phase angles equal to zero and all the voltage magnitudes equal to one, is considered.

(5) Find the Jacobian matrix and construct the linearized model around the state vector (7).

(6) Use SVD factorization (11) to find matrices \( U \), \( S \) and \( V \) of the constructed Jacobian matrix accordingly.

(7) Compute \( B \) matrix and compute the contraction coefficients, \( \alpha_i = \sum_{i=1}^{n} |b_{ij}| \), for the \( n \) smallest elements of each column in the \( B \) matrix.

(8) Compare \( \alpha_i \) s for all the columns: Pick up the set of \( n \) equations related to the smallest \( \alpha_i \) within the columns which satisfies the contraction mapping condition and also makes the system observable.

(9) Find \( \Delta x \) by using (14).

(10) Update the value of \( x \) by adding the estimated value of \( \Delta x \) to the previous value of \( x \).

(11) If the calculated \( \Delta x \) is larger than the convergence criteria or if the maximum number of iteration is not reached, repeat steps 4 to 9.

(12) Calculate the error using (5).

4. CASE STUDY

4.1 Power system Formulation

In power systems the measurements obtained from the system, \( Z \), are usually active power, \( P \), and reactive power, \( Q \), at buses and branches in the network. The unknown states, \( x_i \) s, are also the voltage magnitude, \( V_i \), and the phase angle, \( \theta_i \), at network buses except for the slack bus which is the reference bus in the network.

Table 1. The 3-bus power system’s parameters

| \( P_i \) | Active power of the \( i \)-th bus |
| \( Q_i \) | Reactive power of the \( i \)-th bus |
| \( V_i \) | Voltage magnitude of the \( i \)-th bus |
| \( \theta_i \) | Phase angle of the \( i \)-th bus |
| \( P_{ij} \) | Active power flow between bus \( i \) and bus \( j \) |
| \( Q_{ij} \) | Reactive power flow between bus \( i \) and bus \( j \) |
| \( \theta_{ij} \) | \( \theta_i - \theta_j \) |
| \( g_{ij} \) | The real part of line admittance |
| \( g_{ij} \) | The real part of shunt admittance |
| \( b_{ij} \) | The imaginary part of line admittance |
| \( b_{ij} \) | The imaginary part of shunt admittance |

By defining the system variables as shown in Table 1, The power flow equations of system can be written as follows, (Abur et al. (2004)),

\[
P_i = V_i \sum_{j \in N_i} V_j (g_{ij} \cos \theta_{ij} + b_{ij} \sin \theta_{ij})
\]  

(15)

\[
Q_i = V_i \sum_{j \in N_i} V_j (g_{ij} \sin \theta_{ij} + b_{ij} \cos \theta_{ij})
\]  

(16)

where \( N_j \) are all the buses connected to bus \( i \).

The true states are obtained by solving power flow equations (15) and (16), together by using Newton Raphson method (Glover et al (2008)) for instance.

Branch measurements are then calculated based on the exact solution and by using (17) and (18)

\[
P_{ij} = V_i^2 (g_{si} + g_{ij}) - V_i V_j (g_{ij} \cos \theta_{ij} + b_{ij} \sin \theta_{ij})
\]  

(17)

\[
Q_{ij} = -V_i^2 (b_{si} + b_{ij}) - V_i V_j (g_{ij} \sin \theta_{ij} + b_{ij} \cos \theta_{ij})
\]  

(18)

For a system of full measurements the total number of measurements equals to \( 2 \) (number of branches) \( + 2 \) (number of buses -1). The number of unknown states is \( 2 \) (number of buses -1). The ratio of measurements to unknown states is called Redundancy Ratio and in most cases the ratio is between two and three which results in an overdetermined system of nonlinear equations, (Luenberger (1997)).

These measurements along with the original bus measurements constructs the set of \( m \) true measurements of active and reactive power. The system equations in the form of (6) is then given by :

\[
\begin{bmatrix}
P(V_\theta) \\
Q(V_\theta)
\end{bmatrix}_{m \times 1} = \begin{bmatrix}
\frac{\partial P}{\partial V} & \frac{\partial P}{\partial \theta} \\
\frac{\partial Q}{\partial V} & \frac{\partial Q}{\partial \theta}
\end{bmatrix}_{m \times n} \begin{bmatrix}
\Delta V \\
\Delta \theta
\end{bmatrix}_{n \times 1} + \begin{bmatrix}
P(V_\theta, \theta_0) \\
Q(V_\theta, \theta_0)
\end{bmatrix}_{n \times 1}
\]  

(19)

4.2 Illustrative Example and Simulation Results

The proposed modified LAV estimation technique is now applied to the IEEE network with 118 buses. Figure 1 shows the complexity of the network structure. Due to limitation in space, interested readers are directed to (Christie (1993)) for system data.

In the first step, true states are calculated using the Newton Raphson method for the problem at hand (Glover et al (2008)). The original measurement set is then constructed based on the specification tables and actual values, such that all the bus and branch measurements are considered.

To simulate the effect of bad data on a number of buses in the network, \( m_1 \) \( (m_1 < m \) \), \( m \) equals to number of all measurements) elements in the set are corrupted with noise. For corruption a random generator algorithm has been used to produce random errors. The errors were created as so to be the representative of values drawn from a set of numbers having a Gaussian PDF (3) for the first experiment and Rayleigh PDF (4) for the second experiment. These generated values have been added to \( m_1 \) measurements in the original set to construct a new set of measurements with erroneous data. This set represents a situation where only some of the measurements in the network contain bad data.

Since the measurement error is not consistent in all measurements, weighted least squares method is not applicable. The
Fig. 1. Schematics of IEEE 118bus
typical LS estimation technique leads also to significant estimation errors in buses that their measurements contains noise.

In this research the redundancy ratio is 2.59 and \( m_1 \) is considered approximately one forth of the measurement set, or measurements related to the first 30 buses of system. The redundant measurement set and the system physical specification data are considered for estimation. At the first step, system equations are linearized around the flat start, and the initial matrix format of system is created. Then SVD factorization is applied to Jacobian matrix to modify it for contraction mapping selection. In the next step, the processed matrices are examined to find the observable set with the smallest contraction mapping coefficient. The values of states will be updated after solving the equations related to the selected set. The procedure will be iterated until it satisfies the estimation convergence criteria.

The result of state estimation is presented in figure (2) for voltage magnitudes and phase angles. The LS technique and the purposed algorithm both have been applied to this system regarding the procedure. The measurement set for this experiment is corrupted with the Gaussian noise.

Figure (3) also shows the estimation results for both methods with Rayleigh noise on the first 30 buses.

Experimental evidence proves the proposed estimation technique results in satisfactory level of performance for various distributions of error such as Gaussian and Rayleigh.

5. CONCLUSION

In this paper a new algorithm has been introduced for power system state estimation using contraction mapping modified by singular value decomposition. The method is developed for large-scale networks where the sparse nature of system matrix may result in singularity and ill conditioning problems. The performance of the proposed technique has been evaluated on the IEEE 118-bus power system in the presence of bad data and the simulation results are compared to the Least Square (LS) estimator. In contrast to LS estimator, the new method appropriately classifies system data and removes the effects of erroneous sections in a robust estimation process.

6. ACKNOWLEDGMENTS

The authors would like to express their thanks to Seyed Mohammad Mahdi Alavi for proof reading the paper.
Fig. 2. The simulation results for 118 bus system and Gaussian Noise

Fig. 3. The simulation results for 118 bus system and Rayleigh Noise
REFERENCES


