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Checking Robust Practical Stability for Flatness Based Tracking Controllers Using Interval Methods

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Abstract: The robustness analysis of tracking controllers based on differential flatness is a field of research with many open problems. In earlier publications the authors have shown that interval methods can be used to determine maximal admissible uncertainties in the plant such that the controlled system can be guaranteed to remain within a specified neighbourhood of the desired trajectory. In this contribution we consider set point changes and show that it is possible to proof, using interval methods, whether, for given bounds of the uncertainties in the plant, the controller yields practical stability at the final desired equilibrium point.

Keywords: Differential Flatness, Tracking Control, Robustness, Interval Methods, Practical Stability

1. INTRODUCTION

Flatness based controller design (see Fliess et al. [1995, 1999]) is a powerful tool for motion planning and trajectory tracking for linear and nonlinear systems. Especially for nonlinear systems there is a wide acceptance of this approach, which has been applied successfully to numerous problems of industrial relevance (see e.g. Lévine [1999], Sira-Ramirez and Agrawal [2004]).

There is rather few literature which considers robustness of flatness based controllers. In Hagenmeyer and Delaleau [2003] a so-called feedforwared linearizing controller has been proposed which is supposed to have improved robustness properties. For this controller the asymptotic stability of the trajectory can be guaranteed when additional restrictions on the velocity of the trajectory are satisfied. In Cazaurang et al. [2000], additionally to a nominal feedback linearization, a robust linear controller based on H_∞/μ-synthesis methods is applied.

It has been shown in (Antritter et al. [2007], Kletting et al. [2007, 2006]) that interval methods are a suitable tool for analyzing the robustness properties of tracking controllers over a bounded period of time. Using these methods, the maximum admissible range of parameter uncertainties in the plant and in the initial state can be determined such that the deviation from a desired trajectory does not violate pre-specified bounds. The approach imposes no restrictions on the velocity of the trajectory and even sensor errors can be taken into account. If the trajectory is a transition between equilibrium points of the system the deviation from the final set point at the end of the trajectory or at a specified point of time can be computed. The approach even allows to determine optimal control parameters for robust transitions.

In this paper we consider tracking controllers which achieve a set point transition and we show that the developed interval tools are also suitable to proof that the tracking controller achieves practical stability (see, e.g., Lakshmikantham et al. [1990], LaSalle and Lefschetz [1961]) at the final equilibrium point, i.e. that the system remains in a bounded neighbourhood of the desired equilibrium for all $t > T$ with T some point of time after the specified transition time. For given bounds of the uncertain parameters and of the initial states, the approach allows to compute bounds for the neighbourhood of the final set point in which the system evolves.

We will discuss the application to a magnetic levitation system (see Levine et al. [1996]). This is a structurally rather simple single input differentially flat system and hence simplifies the discussion. For the example it can be shown that the approach is feasible. Together with the the tools provided in Antritter et al. [2007], Kletting et al. [2007, 2006] this yields a powerful approach for the robustness analysis of flatness based tracking controllers. The investigation of the example shows, however, that the methods have to be improved to get a tight enclosure of the neighbourhood of the final equilibrium point in which the system evolves.

The verified intergration of the controlled system, which is the basis of the approach, is done using a taylor model based solver as implemented in COSY-VI (see Berz and Makino [1998]).

This paper is organized as follows. In Section 2.1, we shortly recall the flatness based design of tracking controllers for nonlinear single input systems, taking explicitely into account uncertain parameters in the plant. In Section 3 we introduce the magnetic levitation system and design a suitable trajectory together with a stabilizing tracking controller. Section 4 describes briefly the taylor model based verified integration of nonlinear uncertain systems. This provides the basis for the robustness analysis in Section 5, where the practical stability at the final equilibrium point is shown. Clearly, this paper presents a very first approach to proofing practical stability using interval methods. Thus, in Section 6, we discuss the achieved results and give an outlook on possible improvements.

2. FLATNESS BASED TRACKING CONTROLLER DESIGN

2.1 Differential Flatness

Flatness based controller design has been introduced e.g. in Fliess et al. [1995] (differential algebraic setting) and Fliess et al. [1999] (differential geometric setting). Various aspects of flatness are illustrated e.g. in Sira-Ramirez and Agrawal [2004]. In this contribution the following relations for nonlinear single input systems are used, where explicitely the dependence of the relations on the parameters are stated: For a flat system

$$\dot{x} = f(p, x, u) \tag{1}$$

with $x \in \mathbb{R}^n$, $u \in \mathbb{R}$ and the parameter vector $p \in \mathbb{R}^{n_p}$ the flatness property implies the existence of a flat output $y_f \in \mathbb{R}$, such that

$$y_f = h_f(p, x) \tag{2}$$

$$x = \psi_x(p, y_f, \dot{y}_f, \ldots, y_f^{(n-1)}) \tag{3}$$

$$u = \psi_u(p, y_f, \dot{y}_f, \ldots, y_f^{(n)}) \tag{4}$$

holds, with h_f, ψ_u, ψ_x smooth functions of their arguments. Introducing the new coordinates

$$\zeta = (\zeta_1, \ldots, \zeta_n) = (y_f, \dot{y}_f, \ldots, y_f^{(n-1)})\ , \tag{5}$$

the flat system (1) can be transformed via the well defined diffeomorphism

$$\zeta = \Phi(p, x) \tag{6}$$

into controller normal form

$$\begin{aligned} \dot{\zeta}_i &= \zeta_{i+1}, \qquad i = 1, 2, \ldots n-1 \\ \dot{\zeta}_n &= \alpha(p, \zeta, u)\ . \end{aligned} \tag{7}$$

Setting $v = y_f^{(n)}$ yields

$$u = \psi_u(p, \zeta, v) \tag{8}$$

in view of (4) and (5). In Hagenmeyer [2003] it has been shown that

$$\alpha(p, \zeta, \psi_u(p, \zeta, v)) = v \tag{9}$$

holds and thus by application of the feedback law (8), system (1) is diffeomorphic to the Brunovský normal form

$$\begin{aligned} \dot{\zeta}_i &= \zeta_{i+1}, \qquad i = 1, 2, \ldots n-1 \\ \dot{\zeta}_n &= v \end{aligned} \tag{10}$$

with new input v.

2.2 Flatness Based Feedforward Controller

Due to the derived relations in Section 2.1 a (sufficiently smooth) reference trajectory $y_{f,d} : [t_0, t_0 + T] \to \mathbb{R}$ for the flat output y_f can be assigned almost arbitrarily (excluding singularities of the differential parameterization (3)–(4)). If the reference trajectory $y_{f,d}$ satisfies the boundary conditions

$$x(t_0) = \psi_x(p_0, y_{f,d}(t_0), \dot{y}_{f,d}(t_0), \ldots, y_{f,d}^{(n-1)}(t_0)) \tag{11}$$

then a corresponding feedforward controller that provides $y_f(t) = y_{f,d}(t)$ for $t \in [t_0, t_0 + T]$ is given by

$$u_d(t) = \psi_u(p_0, y_{f,d}(t), \dot{y}_{f,d}(t), \ldots, y_{f,d}^{(n)}(t)) \tag{12}$$

For (11) and (12) it has been assumed that the parameters of the plant (1) match a nominal parameter vector p_0.

2.3 Flatness Based Tracking Controller Design

To stabilize the tracking of a given reference trajectory $y_{f,d}$ for the flat output, the tracking error e is introduced as

$$e = y_f - y_{f,d} = \zeta_1 - \zeta_{1,d} \tag{13}$$

In view of (10) it follwos that

$$e^{(i)} = \zeta_{i+1} - \zeta_{i+1,d}, \quad i = 0, 1, \ldots, n-1 \tag{14}$$

Thus, when setting the new input v in (10) to

$$v = \dot{\zeta}_{n,d} - \sum_{i=0}^{n-1} \lambda_i(\zeta_{i+1} - \zeta_{i+1,d}) = \dot{\zeta}_{n,d} - \sum_{i=0}^{n-1} \lambda_i e^{(i)} \tag{15}$$

the tracking error obeys the differential equation

$$0 = e^{(n)} + \sum_{i=0}^{n-1} \lambda_i e^{(i)} \tag{16}$$

which can be achieved to be stable by suitable choice of the λ_i. Substituting (15) into the differential parameterization (4) of the input yields in view of (5) the feedback law

$$u = \psi_u(p, y_f, \dot{y}_f, \ldots, y_f^{(n-1)}, y_{f,d}, \dot{y}_{f,d}, \ldots, y_{f,d}^{(n)})\ . \tag{17}$$

Using the diffeomorphism (6), the feedbacklaw (17) can be implemented as

$$u = \psi_u'(p_0, x, y_{f,d}, \dot{y}_{f,d}, \ldots, y_{f,d}^{(n)}) = \psi_u''(p_0, x, t)\ . \tag{18}$$

where again the plant parameters p are assumed to be equal to the nominal parameter vector p_0. As a consequence, for the feedback controller (18), the controlled system can be summarized as

$$\dot{x} = f(p, x, \psi_u''(p_0, x, t)) = f_{cl}(p, x, t) \tag{19}$$

where $p \neq p_0$ can occur due to not exactly known parameters. To improve the robustness of the tracking controller an integral error feedback is often introduced, i.e. the error feedback (15) is extended according to

$$\begin{aligned} \dot{e}_I &= \zeta_1 - \zeta_{1,d} \\ v &= \dot{\zeta}_{n,d} - \sum_{i=0}^{n-1} \lambda_i e^{(i)} - \lambda_{-1} e_I \end{aligned} \tag{20}$$

this feedback can clearly be implemented as a state feedback of the kind

$$\begin{aligned} \dot{e}_I &= h_f(p, x) - y_{f,d}(t) \\ u &= \psi_{u,I}''(p_0, x, e_I, t)\ . \end{aligned} \tag{21}$$

In this case the closed loop system has the form

$$\begin{bmatrix} \dot{x} \\ \dot{e}_I \end{bmatrix} = \begin{bmatrix} f(p, x, \psi_{u,I}''(p_0, x, t)) \\ h_f(p, x) - y_{f,d}(t) \end{bmatrix} = f_{cl,I}(p, x, e_I, t) \tag{22}$$

This controller achieves the linear tracking error dynamics

$$0 = e_I^{(n+1)} + \sum_{i=0}^{n} \lambda_{i-1} e_I^{(i)} \tag{23}$$

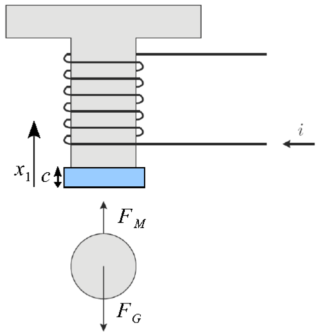

Fig. 1. Magnetic levitation system.

2.4 Practical Stability for Flatness Based Tracking Controllers and Interval Bounded Uncertainties

Given a point of time t_i and the interval vectors $[x_i] = [\underline{x}_i, \overline{x}_i]$, $[p] = [\underline{p}, \overline{p}]$, $[x_f] = [\underline{x}_f, \overline{x}_f]$ we will say in the following that the tracking controller (18) yields robust practical stability with respect to $([x_i], [p], [x_f])$ for the closed loop system (19), if

$$x(t, t_i, x_i, p) \in [x_f] \quad \forall\, t > t_i \tag{24}$$

for all $x(t_i) \in [x_i]$ and for all $p \in [p]$. Here, $x(t, t_i, x_i, p)$ means the flow of (19) at time t resulting from the initial condition $x(t_i) = x_i$. This is a suitable adaption of practical stability as defined in Lakshmikantham et al. [1990] or LaSalle and Lefschetz [1961]. The adaption of the concept to the controller (21) is straight forward.

3. MAGNETIC LEVITATION SYSTEM

A simplified model of a magnetic levitation system (see Figure 1) is given by Levine et al. [1996]

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= \frac{k}{m}\frac{u^2}{(c-x_1)^2} - g. \end{aligned} \tag{25}$$

The system is already given in the nonlinear controller normal form (7) and thus a flat output of (25) is given by

$$y_f = x_1. \tag{26}$$

The relations (3)–(4) can directly be derived from (25)

$$(x_1, x_2) = (y_f, \dot{y}_f), \tag{27}$$

$$u = (c - y_f)\sqrt{\frac{m}{k}(\ddot{y}_f + g)}. \tag{28}$$

The nominal parameters of system (25) have been taken to be $k_0 = 58.041 \frac{\mathrm{kg\,cm^3}}{\mathrm{s^2\,A^2}}$, $g_0 = 981\,\frac{\mathrm{cm}}{\mathrm{s^2}}$, $m_0 = 0.0844\,\mathrm{kg}$, $c_0 = 0.11\,\mathrm{cm}$. In the following we will assume that only the parameter k is uncertain, i.e. $m = m_0$, $g = g_0$ and $c = c_0$. Therefore we will drop the index for the parameters m, g and c.

Based on the results in Section 2.1 a tracking controller for system (25) is given by

$$\begin{aligned} u_{fb} = {} & (c - x_1)\cdot \\ & \sqrt{\frac{m}{k_0}(\ddot{y}_{f,d} - \lambda_1(x_2 - \dot{y}_{f,d}) - \lambda_0(x_1 - y_{f,d}) + g)}. \end{aligned} \tag{29}$$

We extend this controller with an integral error feedback

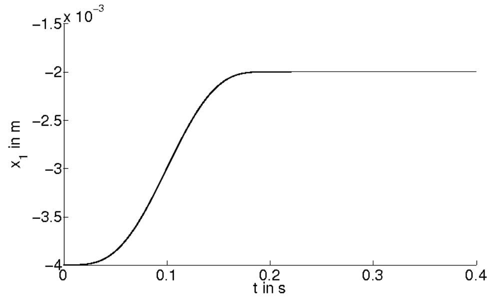

Fig. 2. Reference trajectory for the position x_1.

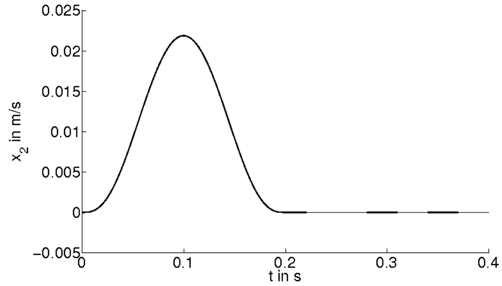

Fig. 3. Reference trajectory for the velocity x_2.

(see (21)):

$$\begin{aligned} \dot{e}_I &= x_1 - y_{f,d}, \\ u_{fb,I} &= (c - x_1)\cdot \\ & \Bigg(\frac{m}{k_0}(\ddot{y}_{f,d} - \lambda_1(x_2 - \dot{y}_{f,d}) - \lambda_0(x_1 - y_{f,d}) \\ & \qquad - \lambda_{-1}e_I) + g)\Bigg)^{-\frac{1}{2}}. \end{aligned} \tag{30}$$

This controller achieves the linear time invariant tracking error dynamics

$$e_I^{(3)} + \lambda_1 \ddot{e}_I + \lambda_0 \dot{e}_I + \lambda_{-1} e_I = 0. \tag{31}$$

For the load of the levitation system a set point change is considered. A trajectory has been planned such that the following boundary conditions are satisfied

$$\begin{aligned} (x_1(0\,\mathrm{s}), x_2(0\,\mathrm{s})) &= (-0.4\,\mathrm{cm}, 0\,\frac{\mathrm{cm}}{\mathrm{s}}), \\ (x_1(0.2\,\mathrm{s}), x_2(0.2\,\mathrm{s})) &= (-0.2\,\mathrm{cm}, 0\,\frac{\mathrm{cm}}{\mathrm{s}}). \end{aligned} \tag{32}$$

In view of the differential parameterization (27) this yields the following boundary conditions for a corresponding trajectory $y_{f,d}$ for the flat output

$$\begin{aligned} y_{f,d}(0\,\mathrm{s}) &= -0.4\,\mathrm{cm}, & \dot{y}_{f,d}(0\,\mathrm{s}) &= 0\,\frac{\mathrm{cm}}{\mathrm{s}}, \\ y_{f,d}(0.2\,\mathrm{s}) &= -0.2\,\mathrm{cm}, & \dot{y}_{f,d}(0.2\,\mathrm{s}) &= 0\,\frac{\mathrm{cm}}{\mathrm{s}}. \end{aligned} \tag{33}$$

Since the final point is a set point, it is natural to extend the trajectory with a constant trajectory for the flat output, i.e. $y_{f,d} \equiv -0.2\,\mathrm{cm}$ for all $t > 0.2\,\mathrm{s}$. In order to obtain also a continuous control signal u_d at $t = 0\,\mathrm{s}$ and $t = 0.2\,\mathrm{s}$, the additional boundary conditions $\ddot{y}_{f,d}(0\,\mathrm{s}) = \ddot{y}_{f,d}(0\,\mathrm{s}) = 0\,\frac{\mathrm{cm}}{\mathrm{s^2}}$ have also been incorporated. This can be achieved by assigning for $y_{f,d}$ a fifth order polynomial in t. The resulting trajectory for y_f can be seen in Figures 2–3. The right hand side of the closed loop system for the tracking controller with integral feedback, has the form

$$\begin{bmatrix} \dot{x} \\ \dot{e}_I \end{bmatrix} = \begin{bmatrix} f(k, x, e_I, u_{fb,I}(k_0, x, e_I, t)) \\ x_1 - y_{f,d}(t) \end{bmatrix} = f_{cl,I,tv}(k, x, e_I, t) \tag{34}$$

for $t \in [0\,\mathrm{s}, 0.2\,\mathrm{s}]$. Note that for the considered reference trajetory the closed loop system becomes time invariant for $t > 0.2\,\mathrm{s}$:

$$\begin{bmatrix} \dot{x} \\ \dot{e}_I \end{bmatrix} = \begin{bmatrix} f(k, x, u_{fb,I}(p_0, x)) \\ x_1 + 0.4\,\mathrm{s} \end{bmatrix} = f_{cl,I,ti}(k, x, e_I)\,. \tag{35}$$

Using interval methods we will determine in Section 5, for a given uncertainty of the parameter $[k] = [\underline{k}, \overline{k}]$ (with $k_0 \in [k]$) and a given uncertainty of the state, whether the closed loop system (35) is practically stable, i.e. whether the system remains in a neighbourhood (described by an interval box $[x_f]$) of the final set point.

4. VERIFIED INTEGRATION BASED ON TAYLOR MODELS

The controlled systems (34) and (35) respectively can be described by a set of time varying nonlinear ordinary differential equations

$$\dot{x}(t) = f_x(x(t), p(t), t), \tag{36}$$

where $x \in \mathbb{R}^{n_x}$ is the state vector (including eventually the controller state for the integral error feedback) and $p \in \mathbb{R}^{n_p}$ the parameter vector. The parameter vector p and the initial conditions $x(0)$ are assumed to be uncertain with $p \in [\underline{p}; \overline{p}]$ and $x(0) \in [\underline{x}(0); \overline{x}(0)]$. If the parameters may vary over time within their bounds and if upper and lower bounds of the variation rate are known then

$$\dot{p}(t) = \Delta p \quad \text{with} \quad \Delta p \in [\Delta \underline{p}; \Delta \overline{p}] \tag{37}$$

holds. The state vector can be extended by the parameter vector according to

$$\dot{z}(t) = f(z(t)) \tag{38}$$

with

$$z(t) = [x(t)^T, p(t)^T]^T, \quad f = \begin{bmatrix} f_x(x(t), p(t)) \\ \Delta p \end{bmatrix} \tag{39}$$

where $f : D \mapsto \mathbb{R}^n$, $D \subset \mathbb{R}^n = \mathbb{R}^{n_x} \times \mathbb{R}^{n_p}$. Uncertain parameters which are time-invariant are described by $\Delta p = 0$.

For the robustness analysis a verified integration of the system model has to be performed. In this paper a Taylor model based integrator as implemented in COSY VI (see Berz and Makino [1998]) is used. Verified integration techniques like VNODE (see Nedialkov and Jackson [2001]) are based on talyor series expansion in time. COSY-VI performs in addition to the expansion in time also an expansion in the initial state vector, which is in the following denoted by $\mathfrak{z}$. The domain interval vector for $\mathfrak{z}$ is given by $[\mathfrak{z}]$. The expansion point for the expansion in the initial state vector $\mathfrak{z}$ is given by $\hat{\mathfrak{z}}$ with $\hat{\mathfrak{z}} \in [\mathfrak{z}]$. The expansion point for the expansion in time is t_k. The flow of the differential equation in a given time interval $[t_k; t_{k+1}]$ is enclosed by a n-dimensional Taylor model

$$T_\rho(\mathfrak{z} - \hat{\mathfrak{z}}, t - t_k) := P_\rho(\mathfrak{z} - \hat{\mathfrak{z}}, t - t_k) + I_{\rho,k+1}, \tag{40}$$

with $\mathfrak{z} \in [\mathfrak{z}]$ and $t \in [t_k; t_k + 1]$. $P_\rho(\mathfrak{z} - \hat{\mathfrak{z}}, t - t_k)$ is the multivariate polynomial part of order ρ and $I_{\rho,k+1}$ the remainder interval vector. Components i of $T_\rho(\mathfrak{z} - \hat{\mathfrak{z}}, t - t_k)$ are denoted by $T_{\rho,i}(\mathfrak{z} - \hat{\mathfrak{z}}, t - t_k)$. The Taylor model at $t = t_{k+1}$ is

$$T_{\rho,k+1}(\mathfrak{z} - \hat{\mathfrak{z}}) := P_{\rho,k+1}(\mathfrak{z} - \hat{\mathfrak{z}}) + I_{\rho,k+1} \ . \tag{41}$$

Components i of $T_{\rho,k+1}(\mathfrak{z} - \hat{\mathfrak{z}})$ are given by $T_{\rho,i,k+1}(\mathfrak{z} - \hat{\mathfrak{z}})$.

Verified integration methods which use a single interval vector or a single parallelepiped for the state enclosure may suffer from large overestimation, especially in case of nonlinear systems. The flow representation by Taylor models makes it possible to obtain tight enclosures of non-convex sets and leads to a reduction of overestimation. For the integration the differential equation (38) is rewritten to a fixed point equation

$$\mathcal{O}(z)(t) := z(t_k) + \int_{t_k}^{t} f(z(t'), t')dt' \ . \tag{42}$$

Applying the Operator $\mathcal{O}$ to a Taylor model for the integration in the time-interval $[t_k; t_{k+1}]$ yields

$$\begin{aligned} &\mathcal{O}(P_\rho(\mathfrak{z} - \hat{\mathfrak{z}}, t - t_k) + I_{\rho,k+1}) \\ &\quad = z(t_k) + \int_{t_k}^{t} f(P_\rho(\mathfrak{z} - \hat{\mathfrak{z}}, t' - t_k) + I_{\rho,k+1})dt' \ , \end{aligned}$$

where $z(t_k)$ is represented by its corresponding Taylor model enclosure at $t = t_k$.

$$T_{\rho,k} = P_{\rho,k}(\mathfrak{z} - \hat{\mathfrak{z}}) + I_{\rho,k} \ . \tag{43}$$

This leads to

$$\begin{aligned} &\mathcal{O}(P_\rho(\mathfrak{z} - \hat{\mathfrak{z}}, t - t_k) + I_{\rho,k+1}) \\ &= P_{\rho,k}(\mathfrak{z} - \hat{\mathfrak{z}}) + I_{\rho,k} + \int_{t_k}^{t} f(P_\rho(\mathfrak{z} - \hat{\mathfrak{z}}, t - t_k) + I_{\rho,k+1})dt' \ . \end{aligned} \tag{44}$$

The goal for the integration from t_k to t_{k+1} is to determine a Taylor model $T_\rho(\mathfrak{z} - \hat{\mathfrak{z}}, t - t_k)$ such that

$$\mathcal{O}(P_\rho(\mathfrak{z} - \hat{\mathfrak{z}}, t - t_k) + I_{\rho,k+1}) \subset P_\rho(\mathfrak{z} - \hat{\mathfrak{z}}, t - t_k) + I_{\rho,k+1} \tag{45}$$

$\forall \mathfrak{z} \in [\mathfrak{z}]$ and $\forall t \in [t_k; t_{k+1}]$. The polynomial part and the interval remainder are determined in two separate steps. A detailed description of these steps is given in Berz and Makino [1998] and Kletting [2009].

The expansion in initial states reduces the overestimation which may occur during the integration process. However, the interval remainder part reamains as a source for overestimation. In order to limit the long-term growth of the remainder error and to further reduce overestimation the following strategies can be applied:

- Shrink Wrapping: Here, the interval remainder is absorbed in the polynomial part (Makino and Berz [2005b]).
- Preconditioning: The Solution of ODE is studied in a different coordinate system in order to minimize long-term error growth (Makino and Berz [2005a]).
- The domain interval vector $[\mathfrak{z}]$ can be split into subboxes and the enclosure of $z(t)$ is given by a list of Taylor models (Kletting [2009]).

For numerical and implementation reasons it is advantageous to have the unit box $[-1; 1]^n$ as a domain interval vector in each integration step Makino and Berz [2005b]. Thus the initial enclosure $[z(0)]$ of the extended state vector $z(t)$ is expressed as a Taylor model according to

$$[z(0)] = T(\mathfrak{z}) = c + D\,\mathfrak{z} \text{ with } \quad \mathfrak{z}_i \in [-1; 1] \ , i = 1 \ldots, n \ , \tag{46}$$

where c is the midpoint of $[\mathfrak{z}]$ and D is a diagonal matrix with $d_{i,i} = \mathrm{rad}([\mathfrak{z}_i]) = |\overline{\mathfrak{z}}_i - \underline{\mathfrak{z}}_i|$.

5. CHECKING PRACTICAL STABILITY

For simplicity of the presentation we assume that only the parameter k and the initial state $x_1(0)$ are uncertain.

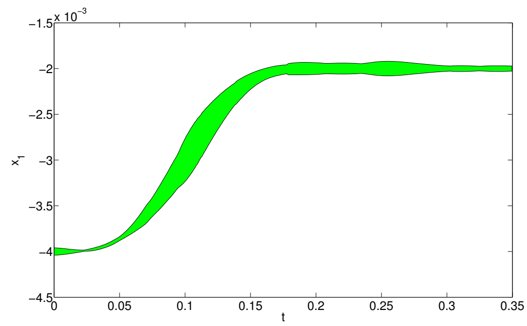

Fig. 4. Verified Enclosure of the load position x_1.

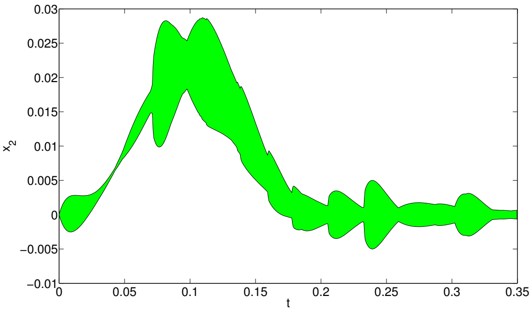

Fig. 5. Verified Enclosure of the load velocity x_2.

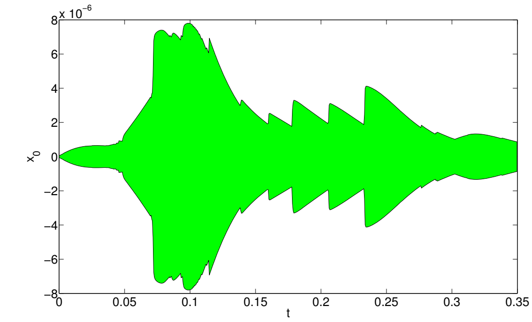

Fig. 6. Integral error e_I.

We assume that the uncertain parameter can be bounded by $k \in [0.99, 1.01] \cdot 58.041 \frac{\mathrm{kg\,cm}^3}{\mathrm{s}^2\,\mathrm{A}^2}$. Additionally, the initial state is assumed to be bounded by the interval $x_1(0) \in [0.99, 1.01] \cdot (-0.4)$ cm. For these parameters the enclosure of the flow of the controlled system (34)/(35) has been computed at discrete time steps using taylor models. The computed enclosures are shown in Figures 4–6, when the roots of (31) are placed at -70 (the periodic pattern in x_2 and e_I is due to the integration algorithm).

To investigate practical stability at the final equilibrium point the following steps have been performed: Firstly, the Talyor model $T_1(\mathfrak{z})$ at $t_1 = 0.3\,\mathrm{s} > 0.2\,\mathrm{s}$ has been stored during the integration that yielded Figures 4–6. This taylor model has been approximated by a single interval box to simplify the computations. The interval enclosure $[s_1]$ of T_1 is described by

$$\begin{aligned} [x_1] &= [-0.202782, -0.197218] \cdot 10^{-2}\,\mathrm{cm}\ , \\ [x_2] &= [-0.120365, \quad 0.120367] \cdot 10^{-2}\,\frac{\mathrm{cm}}{\mathrm{s}}\ , \\ [e_I] &= [-0.101926, \quad 0.101641] \cdot 10^{-5}\,\mathrm{cm}\cdot\mathrm{s}\ . \end{aligned} \tag{47}$$

The enclosure $[s_1]$ has been obtained by evaluating the Taylor model T_1 over the whole domain interval vector $[\mathfrak{z}]$ using interval arithmetic. Note that the interval enclosure does not introduce much additional overestimation compared to T_1 since the error remainder part is dominating in comparison to the independent variables $\mathfrak{z}_i$. As a next step, the interval inclusion $[s_1]$ has then been rewritten as a taylor model $\tilde{T}_1(\mathfrak{z})$ of the form (46).

Then, a verified integration of (35) with initial conditions according to $\tilde{T}_1(\mathfrak{z})$ with $N-1$ integration steps has been performed. At $t_N = 0.35\,\mathrm{s}$, the corresponding taylor model is denoted by $T_N(\mathfrak{z})$. The interval enclosure $[s_N]$ of the taylor model $T_N(\mathfrak{z})$ is given by

$$\begin{aligned} [x_1] &= [-0.202774, -0.197245] \cdot 10^{-2}\,\mathrm{cm}\ , \\ [x_2] &= [-0.630385, \quad 0.637955] \cdot 10^{-3}\,\frac{\mathrm{cm}}{\mathrm{s}}\ , \\ [e_I] &= [-0.867383, \quad 0.862579] \cdot 10^{-6}\,\mathrm{cm}\cdot\mathrm{s}\ . \end{aligned} \tag{48}$$

It can be easily verfied that $[s_N]$ lies completely inside the interval enclosure $[s_1]$ of T_1 (see (47)). Thus, the subset of the state space described by $T_N(\mathfrak{z})$ is completely contained in $\tilde{T}_1$. A two-dimensional interpretation of these relations is shown in Fig. 7.

Note that $[s_1]$ (see (47)) is not a positively invariant set, since the inclusion test described above has been applied in each integration step t_k, $1 < k \leq N$ and $t_N = 0.35$ is the first point of time were the inclusion could be established. A projection of the relations on a single variable of the state space is shown in Figure 8. An enclosing interval box $[x_f]$ such that for all $t_1 \leq t \leq t_N$, $(x_1(t), x_2(t), e_I(t)) \in [x_f]$ can be computed by evaluating the $N-1$ taylor models T_i $i = 1, 2, \ldots, N-1$ over the corresponding time intervals $[t_i, t_{i+1}]$, $i = 1, 2, \ldots, N-1$. This yields the sets $[s_{i,i+1}]$, $i = 1, 2, \ldots, N-1$. Then, an estimate of $[x_f]$ is obtained to $[x_f] \subset \bigcup_{i=1}^{N} [s_{i,i+1}]$. These computations yield the following estimate for $[x_f]$:

$$\begin{aligned} [x_1] &= [-0.404001, -0.193345] \cdot 10^{-2}\,\mathrm{cm}\ , \\ [x_2] &= [-0.459013, \quad 0.459326] \cdot 10^{-2}\,\frac{\mathrm{cm}}{\mathrm{s}}\ , \\ [e_I] &= [-0.711424, \quad 0.711134] \cdot 10^{-5}\,\mathrm{cm}\cdot\mathrm{s}\ . \end{aligned} \tag{49}$$

This naive evaluation is, unfortunately, very conservative. We still state it here to be able to complete the proof of practical stability.

Clearly, it is possible to bound $T_N(\mathfrak{z})$ at $t = 0.35\,\mathrm{s}$ by the bigger set (47). Since for $t > 0.2\,\mathrm{s}$ the closed loop system is time invariant, it can be concluded that a state enclosure at $t = 0.4\,\mathrm{s}$ is again given by $[s_N]$ (see (48)). The same holds for $t = 0.45\,\mathrm{s}$ and so on. This is illustrated by Figure 9. As a consequence, we have that for the initial condition $x(t_1) \in [s_1]$ and for all $k \in [\underline{k}, \overline{k}]$, $(x_1(t), x_2(t), e_I(t)) \in [x_f]$ for all $t > t_1$. Thus, the tracking controller yields practical stability with respect to $([s_1], [k], [x_f])$.

Let us finally remark that if the taylor model T_1 does not have a largely dominating interval remainder then a replacement by a single box is not suitable. In this case an inverse mapping of the set described by T_2 with the inverse Taylor model of T_1 can be done. The inverse Taylor model of T_1 maps T_1 into the domain interval vector $[\mathfrak{z}]$. If the application of the inverse of T_1 to the Taylor model T_2 leads to a box which included in $[\mathfrak{z}]$, and the desired inclusion is proven. The determination of inverse Taylor models is described in Hoefkens [2001].

6. CONCLUSIONS

In this contribution it has been shown that interval methods can be used to check practical stability of flatness based tracking controllers for given uncertainties in the

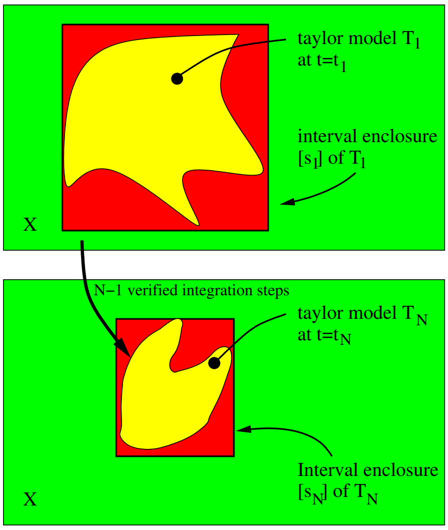

Fig. 7. Two dimensional explication of determination of positively invariant set.

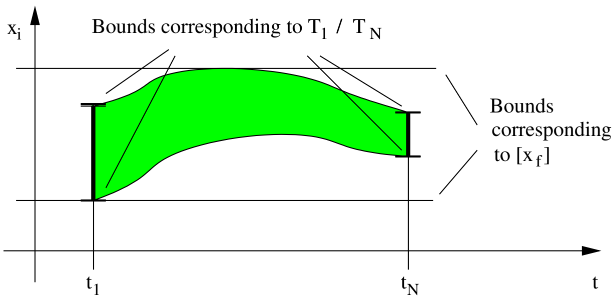

Fig. 8. One dimensional explanation of the determination of the bounds of the set $[x_f]$

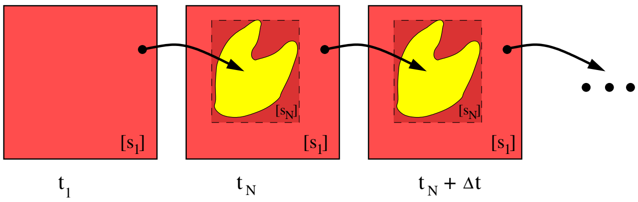

Fig. 9. Repetitive argument for bounding the flow at $t = t_1 + k \cdot \Delta t$ with $k = 1, 2, \ldots, \Delta t = t_N - t_1$.

plant and of the initial states. Unfortunately, the enclosure of the neighbourhood of the final equilibrium point in which the system evolves is far from being tight even for the rather small uncertainties which have been considered. Further work will consider improved methods for obtaining tighter inclusions of $[x_f]$. This might involve splitting strategies and monotonicity tests. Overall the approach seems to be promising and will yield a powerful approach in combination with the tools based on interval methods which have been developed in earlier work.

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