Integral Quadratic Constraints for Nonnegative Uncertainties *

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Abstract: This paper presents some preliminary results on robust stability analysis of positive systems. A general analysis framework with Integral Quadratic Constraints (IQC) is refined for nonnegative uncertain systems. Based on this framework, a new approach for robust control design is proposed. The new framework not only enables a stability analysis of a positive systems, but also suggests new robust control design strategy for systems which has nothing to do with the positive systems at first glance. An example is shown in which such a control design method provides a sharper result than the control design using existing IQC techniques.

Keywords: Robust control, Positive systems, Integral quadratic constraints, H-infinity control, Structured singular value, Decentralized control, Convex optimization.

1. INTRODUCTION

The description of system uncertainty through integral quadratic constraints (IQC) provides a general framework for robust stability analysis. The notion of quadratic separation unifies two major ideas for establishing robust stability, namely the small gain theorem and the passivity theorem, as well as more general classes of sufficient conditions, such as the one provided by μ-analysis. The stability criteria of the IQC framework introduced by Megretski and Rantzer (1997) can also be viewed as a generalization of classical Popov/circle criteria.

This paper attempts to formulate a special case of IQC theory which is suitable for the analysis of a cone system which is, roughly speaking, a linear system whose output signal lives in a cone, as opposed to a vector space, as long as its initial condition and its input signal live in a cone space. As an important class of cone systems, we focus on positive systems (more precisely nonnegative system). We formulate a new IQC theorem for the stability analysis of the nonnegative uncertain system shown in Fig.1 where operators G and ∆ are such that the internal signals w and v are nonnegative as long as the injected signals e and f are nonnegative. In the sequel we refer to this newly developed framework as “nonnegative IQC” analysis whereas the usual framework is referred to as “standard IQC” analysis.

Many real chemical processes, economic systems and ecosystems are naturally modeled as nonnegative systems, since their state variables such as concentration, temperature, prices, and number of species are nonnegative quantities (Haddad et al. (2010)). However, few techniques have been developed to analyze such systems’ stability and/or performance with nonnegativity taken into account explicitly. One motivation for developing the nonnegative IQC framework is to provide a sharper analysis tool for such systems.

Another benefit of introducing the nonnegative IQC is in the control design process. The new framework is beneficial not only in the control design for positive systems as listed above, but also in the control design for systems which has nothing to do with the nonnegative systems at the first glance.

- As pointed out by Rami and Tadeo (2007), a control design for positive systems mentioned above without nonnegativity constraints taken into consideration may result in a closed loop system with infeasible dynamics. Sharing the same idea with Rami and Tadeo (2007), the nonnegative IQC approach provides the control design scheme with a guaranteed closed loop nonnegativity. Also by taking nonnegativity into account, a sharper robust control design may be available.
- The nonnegative IQC approach may be beneficial for the control design for general systems which has nothing to do with the nonnegative system by itself. For example, suppose now Fig.1 is a general feedback in which signals can take positive or negative values, but a nonlinear uncertainty ∆ satisfies more strict sector bound condition if injected signal v is restricted to the nonnegative region. In other words, ∆ is more “certain” on the nonnegative domain. Then one can focus on the set of controllers (if exists) that artificially makes the closed loop system Fig.1 nonnegative, so more strict sector bound condition on ∆ can be used. In Section 6, an example is shown in

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which such a strategy yields less conservative robust control design. 

The existence of an efficient numerical test for quadratic separation makes the IQC framework useful in practice. Namely, the so called Kalman-Yakubovich-Popov (KYP) lemma provides an equivalent linear matrix inequality (LMI) condition which can be evaluated efficiently using semidefinite programming algorithms. Moreover, LMI optimization techniques enable one to search for an adequate quadratic form numerically (Jonsson (2001)) in some important cases. For example, stability analysis in the presence of a bounded structured uncertainty can be performed by the search for the convex upper bound of the structured singular value \( \mu \) by diagonal scaling (Packard and Doyle (1993)). The same technique can be seen as the search for a suitable quadratic form in the IQC framework. In this paper, a similar numerical test for quadratic separation in the context of non-negative IQC analysis is presented. However the result is not as flexible as the one for standard IQC analysis. We will present a “non-negative version” of the bounded real lemma, whose extension to more general KYP lemma is the future work (The bounded real lemma is a special case of the KYP lemma with the quadratic form being restricted to the “small-gain” type). Nevertheless, we will see that the \( \mu \)-like analysis is still valid for the nonnegative structured uncertainties. A significance is that the “non-negative” bounded real lemma argues the existence of diagonal Lyapunov function whereas the classical bounded real lemma argues the existence of general quadratic Lyapunov function. The benefit of the diagonal Lyapunov function to the structured/decentralized control design is well known. Hence, the development of the non-negative IQC framework is also beneficial from the viewpoint of decentralized control.

This paper is organized as follows. Section 2 reviews some standard results of the IQC analysis by Megretski and Rantzer (1997) and positive systems. The main theoretical contribution of this paper is presented in Section 3. Section 4 presents the bounded real lemma that makes the nonnegative IQC analysis computationally tractable. In the same section, an LMI test for robust stability in the presence of a structured nonnegative uncertainty is provided. A robust control synthesis based on the nonnegative IQC analysis is presented in Section 5. Finally Section 6 shows a robust control problem to which an application of the synthesis developed in this paper yields a better result than the standard IQC.

2. PRELIMINARIES

2.1 Notations

Let \( L_2^2[0, \infty) \) denote the space of functions \( f : [0, \infty) \rightarrow \mathbb{R}^n \) of finite energy

\[
\|f\|^2 = \int_0^\infty \|f(t)\|^2 dt < +\infty,
\]

(1)

and \( L_2^\infty[0, \infty) \) denote the space of functions \( f : [0, \infty) \rightarrow \mathbb{R}^n \) of bounded energy on any finite interval, i.e.,

\[
\int_0^T \|f(t)\|^2 dt < +\infty \quad \forall T > 0.
\]

Since the spaces above are the main object of interest, the following notations are introduced for simplicity:

\( \mathcal{H} := L_2[0, \infty), \mathcal{H}_e := L_2^\infty[0, \infty). \)

We also define the space \( \mathcal{H}^+ \) and \( \mathcal{H}^+_e \) by

\( \mathcal{H}^+ := \{ f \in \mathcal{H} : f(t) \geq 0 \text{ almost everywhere in } [0, \infty) \} \)

\( \mathcal{H}^+_e := \{ f \in \mathcal{H}_e : f(t) \geq 0 \text{ almost everywhere in } [0, \infty) \} \)

where “almost everywhere” is with respect to the Lebesgue measure on \([0, \infty)\). It is clear that \( \mathcal{H} \subset \mathcal{H}_e \) and \( \mathcal{H}^+ \subset \mathcal{H}^+_e \).

For \( u \in \mathcal{H}^+_e, \|u\| \) is understood as the value computed from (1). An operator \( G : \mathcal{H}_e \rightarrow \mathcal{H}_e \) (or \( G : \mathcal{H} \rightarrow \mathcal{H} \)) is said to be causal if

\[
P_T G P_T = P_T G \forall T \in [0, \infty)
\]

where \( P_T \) is the truncation operator. A causal operator \( G : \mathcal{H}_e \rightarrow \mathcal{H}_e \) is said to be bounded if the gain

\[
\|G\| = \sup_{u \in \mathcal{H}_e, u \neq 0} \frac{\|Gu\|}{\|u\|}
\]

is bounded. It is known that \( G \) is causal and bounded on \( \mathcal{H}_e \) if and only if \( G \) is causal and bounded on \( \mathcal{H} \). (A detailed discussion is given in Jonsson (2001).) If \( G : \mathcal{H} \rightarrow \mathcal{H} \) is an operator defined on a cone \( \mathcal{C} \subset \mathcal{H} \), \( \|G\| \) is understood as the value defined by

\[
\|G\| = \sup_{u \in \mathcal{C}, u \neq 0} \frac{\|Gu\|}{\|u\|}
\]

Let \( RH_\infty \) be the set of proper rational functions with real coefficients without poles in the closed right half plane. We also define the notation

\[
RH^+_\infty = \{ G \in RH_\infty | Gu \in \mathcal{H}^+ \forall u \in \mathcal{H}^+ \}
\]

The expression \( M \geq 0 \) means that a real matrix (vector) \( M \) is entry-wise non-negative. For a Hermitian matrix, \( P > 0 \) means that \( P \) is positive definite (positive semidefinite).

2.2 IQC

Define a quadratic form \( \sigma_\Pi : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{R} \) by

\[
\sigma_\Pi(v, w) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{v}(j\omega) \Pi(j\omega) \hat{w}(j\omega) d\omega
\]

where \( \Pi \) is a self adjoint operator of the form

\[
\Pi(j\omega) = \Psi^*_H(j\omega) M_H \Psi_H(j\omega)
\]

\(
\Psi_H(j\omega) = C_H(j\omega I - A_H)^{-1} [B_{P_H} B_{P_H}] + [D_{P_H} D_{P_H}].
\]

In the time domain, it is equivalently

\[
\sigma_\Pi(v, w) = \int_0^\infty y_H^T(t) M_H y_H(t) dt
\]

where

\[
x = A_H x_H + B_{P_H} v + B_{P_H} w, x_H(0) = 0
\]

\[
y_H = C_H x_H + D_{P_H} v + D_{P_H} w.
\]

An operator \( \Delta : \mathcal{H} \rightarrow \mathcal{H} \) is said to satisfy an IQC defined by \( \Pi \) if

\[
\sigma_\Pi(v, \Delta(v)) \geq 0 \quad \forall v \in \mathcal{H}.
\]

(2)

Furthermore, if \( G : \mathcal{H} \rightarrow \mathcal{H} \) satisfies

\[
\exists \epsilon > 0 : \sigma_\Pi(Gv, w) \leq -\epsilon \|v\|^2 \quad \forall v \in \mathcal{H}
\]

then two graphs \( \{ (v, w) \in \mathcal{H} \times \mathcal{H} | w = Gv \} \) and \( \{ (v, w) \in \mathcal{H} \times \mathcal{H} | w = \Delta(v) \} \) are said to be quadratically separated by \( \sigma_\Pi \).

The following theorem establishes stability of the feedback interconnection of Fig.1 based on the quadratic separation.
Theorem 1. (Megretski and Rantzer (1997)) Let $G \in RH_\infty$ and $\Delta$ be a bounded causal operator on $\mathcal{H}$. Assume

i) for every $\tau \in [0,1]$, the interconnection of $G$ and $\tau \Delta$ is well-posed

ii) for every $\tau \in [0,1]$, $\tau \Delta$ satisfies the IQC $\sigma_\Pi(v, \tau \Delta(v)) \geq 0 \forall v \in \mathcal{H}$

iii) there exists $\epsilon > 0$ such that

$$\sigma_\Pi(Gw, w) \leq -\epsilon \|w\|^2 \forall w \in \mathcal{H}.$$ 

Then the feedback interconnection of $G$ and $\Delta$ is stable.

For precise definitions of well-posedness and stability, we refer to the original paper by Megretski and Rantzer (1997). Condition (iii) in the above theorem can be equivalently written as a frequency domain inequality (FDI)

$$\left[ \frac{G(j\omega)}{I} \right]^* \Pi(j\omega) \left[ \frac{G(j\omega)}{I} \right] \leq -\epsilon I \forall \omega \in \mathbb{R}. \quad (4)$$

Notice that if $\Delta$ is a linear operator, the quadratic separation (2) and (3) implies condition i) (see Dullerud and Paganini (2005)). However in general, (2) and (3) do not necessarily imply well-posedness, which needs to be checked separately. Moreover, in order to guarantee stability, quadratic separation and the well-posedness is required not only for $\Delta$ but also for $\tau \Delta$ for all $\tau \in [0,1]$.

Nevertheless this requirement is mild, since in many cases it is possible to reformulate the problem statements so the uncertainty description meets this requirement. Note that not all Hermitian operators $\Pi$ can be used in the IQC analysis since the assumptions of Theorem 1 implicitly rule out a certain class of Hermitian operators. For example, consider a single-input single-output system with a Hermitian operator $\Pi = \begin{bmatrix} -4 & 5 \\ 5 & -6 \end{bmatrix}$, and suppose $\Delta(v) = \delta v$ where $\delta$ is a real constant. Then assumption ii) requires that

$$\begin{bmatrix} 1 \\ \tau \delta \end{bmatrix}^T \Pi \begin{bmatrix} 1 \\ \tau \delta \end{bmatrix} \geq 0, \text{ i.e., } 2 \leq \tau \delta \leq 1$$

for all $\tau \in [0,1]$. However there does not exist such a real constant $\delta$. We say that a Hermitian operator $\Pi$ is admissible for standard IQC analysis if there exist nonzero bounded causal operators $G$ and $\Delta$ on $\mathcal{H}$ satisfying assumptions i), ii) and iii) in Theorem 1. In general, an operator $\Pi(j\omega)$ can be a function of frequency. However for the sake of simplicity, we focus on the static IQCs in which $\Pi$ is a Hermitian matrix.

### 2.3 Positive systems

Consider the following LTI system

$$\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t), \; x(0) = x_0 \quad (5a) \\
y(t) &= Cx(t) + Du(t). \quad (5b)
\end{align*}$$

Definition 2. (Kaczorek (2002)) The system (5) is called externally positive if for all $u(t) \geq 0, \; t \geq 0$ and zero initial conditions $x_0 = 0$, the output satisfies $y(t) \geq 0$ for $t \geq 0$. The system (5) is called internally positive if for every $x_0 \geq 0$ and all $u(t) \geq 0, \; t \geq 0$, the state and output satisfy $x(t) \geq 0$ and $y(t) \geq 0$ for $t \geq 0$.

It is known that (5) is internally positive if and only if $A$ is a Metzler matrix (all off-diagonal entries are non-negative) and $B, C, D \geq 0$ (Kaczorek (2002)). It is also easy to check that $D \geq 0$ is necessary if (5) is externally positive.

Theorem 3. (Barker, Berman, Plemmons) (see Shorten et al. (2009)) There exists a diagonal solution $P \geq 0$ to the Lyapunov equation $A^T P + PA < 0$ if and only if $A X$ has at least one negative diagonal entry for all nonzero $X \geq 0$.

In this paper we introduce notions of positively well-posedness and positive stability defined below.

Definition 4. The interconnection of operators $G : \mathcal{H}_+^2 \to \mathcal{H}_+^2$ and $\Delta : \mathcal{H}_+^2 \to \mathcal{H}_+^2$ in Fig.1 is said to be positively well-posed if the map $g : (e, w) \mapsto (e, f)$ has a causal inverse $g^{-1} : \mathcal{H}_+^2 \times \mathcal{H}_+^2 \to \mathcal{H}_+^2$. In addition, the interconnection is said to be positively stable if there exists a constant $c > 0$ such that

$$\int_0^T (|v|^2 + |w|^2) \, dt \leq c \int_0^T (|f|^2 + |e|^2) \, dt$$

for any $T \geq 0$ and for all $f, e \in \mathcal{H}_+^2$.

Intuitively, the interconnection of Fig.1 is positively well-posed if there exists a nonnegative unique solution $(e, w)$, $(e, f)$ in $\mathcal{H}_+^2 \times \mathcal{H}_+^2$ when nonnegative signals $(e, f) \in \mathcal{H}_+^2 \times \mathcal{H}_+^2$ are injected. Now let us consider the implication of the positively well-posedness when both operators $G$ and $\Delta$ are in $RH_\infty$. The feedback interconnection of Fig.1 is said to be well-posed in the classical sense if there exists a causal inverse to operator $(I - G\Delta) : \mathcal{H}_c \to \mathcal{H}_c$. If $G$ and $\Delta$ are LTI system with $G(s) = C_1(sI - A_1)^{-1}B_1 + D_1, \Delta(s) = C_2(sI - A_2)^{-1}B_2 + D_2$, then the interconnection is well-posed if and only if $(I - D_2D_1)$ is non-singular. In that case, the transfer function $g^{-1} : (e, f) \mapsto (e, w)$ has the realization

$$g^{-1}(s) = C(sI - A)^{-1}B + D \quad (6)$$

where

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad C = \begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix}, \quad D = \begin{bmatrix} D_1 \\ D_2 \end{bmatrix}.$$
3. MAIN RESULT

The following is our central result for nonnegative IQC analysis. Theorem 5 can be seen as a refinement of Theorem 1 for nonnegative systems.

**Theorem 5.** Let $G \in RH_{\infty}^+$, $\Delta : H^+ \to H^+$ be bounded causal operators. Assume

i) for every $\tau \in [0,1]$, the interconnection of $G$ and $\tau \Delta$ is positively well-posed,

ii) for every $\tau \in [0,1]$, $\tau \Delta$ satisfies the IQC defined by $\sigma_{\Pi}(v, \tau \Delta(v)) \geq 0 \forall v \in H^+$,

iii) there exists $\epsilon > 0$ such that $\sigma_{\Pi}(Gw, w) < -\epsilon \|w\|^2 \forall w \in H^+$.

Then the feedback interconnection of $G$ and $\Delta$ is positively stable.

**Remark 6.** If the Hermitian operator $\Pi$ is of the bounded real type $\Pi = \begin{bmatrix} I & \mathbf{0} \\ 0 & -\mathbf{I} \end{bmatrix}$, it is shown in Tanaka and Langbort (2010) that the condition iii) in above theorem is equivalent to the $H_\infty$ norm condition $\|G\|_{\infty} < \gamma$ and the FDI (4).

We say that a Hermitian operator $\Pi$ is **admissible for nonnegative IQC analysis** if there exist nonzero bounded causal operators $G$ and $\Delta$ on $H^+$ satisfying assumptions i), ii) and iii) in Theorem 5. Notice that the set of admissible $\Pi$ in Theorem 5 is smaller than the set of admissible $\Pi$ for standard analysis in Theorem 1. For example, the choice $\Pi = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$ gives the passivity theorem in Theorem 1.

However, condition iii) in Theorem 5 cannot be satisfied by any nonnegative operator $G$ with this choice of $\Pi$, since nonnegativity of $G$ implies $\sigma_{\Pi}(Gw, w) = 2\langle w, Gw \rangle \geq 0$ for all $w \in H^+$. In fact, knowing that the positive feedback Fig.1 is comprised of non-negative operators, it is impossible to establish the stability through the passivity argument.

**Proof.** (of Theorem 5) We need to prove that for all $f, e \in H^+$, the unique solution of

$$\begin{align*}
\begin{cases}
v = Gw + f \\
w = \Delta(v) + e
\end{cases}
\end{align*}$$

satisfies $v, w \in H^+$. That is, there exists a bounded causal inverse $(I - \Delta)^{-1} : H^+ \to H^+$.

First we will show that there is a positive constant $\kappa$ such that

$$\|v\| \leq \kappa \|v - \tau \Delta(v)\| \forall v \in H^+, \forall \tau \in [0,1]. \quad (7)$$

To prove (7), notice that for any $v \in H^+$ and $w \in H^+$ defined by $w = \tau \Delta(v)$,

$$\begin{align*}
\sigma_{\Pi}(v, w) - \sigma_{\Pi}(Gw, w) &= \left\langle \begin{bmatrix} Gw \\ w \end{bmatrix}, \Pi \begin{bmatrix} v - Gw \\ 0 \end{bmatrix} \right\rangle + \\
&+ \left\langle \begin{bmatrix} v - Gw \\ 0 \end{bmatrix}, \Pi \begin{bmatrix} Gw \\ w \end{bmatrix} \right\rangle + \left\langle \begin{bmatrix} v - Gw \\ 0 \end{bmatrix}, \Pi \begin{bmatrix} v - Gw \\ 0 \end{bmatrix} \right\rangle \\
&\leq 2\|\Pi\| \left\| \begin{bmatrix} Gw \\ w \end{bmatrix} \right\| \left\| \begin{bmatrix} v - Gw \\ 0 \end{bmatrix} \right\| + \|\Pi\| \left\| \begin{bmatrix} v - Gw \\ 0 \end{bmatrix} \right\|^2 \\
&= 2\|\Pi\| \left\| \begin{bmatrix} Gw \\ w \end{bmatrix} \right\| \left\| \begin{bmatrix} v - Gw \\ 0 \end{bmatrix} \right\| + \|\Pi\| \left\| \begin{bmatrix} v - Gw \\ 0 \end{bmatrix} \right\|^2.
\end{align*}$$

Since $\sigma_{\Pi}(Gw, w) \leq -\epsilon \|w\|^2$ and $G$ is bounded, there exists $\epsilon_1 > 0$ such that $\sigma_{\Pi}(Gw, w) \leq -\epsilon_1 \left\| \begin{bmatrix} Gw \\ w \end{bmatrix} \right\|^2$. Since $\sigma_{\Pi}(v, w) \geq 0$ we have

$$\epsilon_1 \left\| \begin{bmatrix} Gw \\ w \end{bmatrix} \right\|^2 \leq \sigma_{\Pi}(v, w) - \sigma_{\Pi}(Gw, w)$$

and thus

$$\epsilon_1 \left\| \begin{bmatrix} Gw \\ w \end{bmatrix} \right\|^2 \leq 2\|\Pi\| \left\| \begin{bmatrix} Gw \\ w \end{bmatrix} \right\| \left\| v - Gw \right\| + \|\Pi\| \left\| v - Gw \right\|^2. \quad (8)$$

Notice that

$$0 \leq \frac{\epsilon_1}{2} \left\| \begin{bmatrix} Gw \\ w \end{bmatrix} \right\|^2 - 2\|\Pi\| \left\| \begin{bmatrix} Gw \\ w \end{bmatrix} \right\| \left\| v - Gw \right\| + \|\Pi\| \left\| v - Gw \right\|^2$$

and by adding (8) and (9) we obtain

$$\epsilon_1 \left\| \begin{bmatrix} Gw \\ w \end{bmatrix} \right\|^2 \leq \left( \frac{2\|\Pi\|^2}{\epsilon_1} + \|\Pi\| \right) \left\| v - Gw \right\|^2. \quad (9)$$

Thus

$$\left\| \begin{bmatrix} Gw \\ w \end{bmatrix} \right\| \leq C \left\| v - Gw \right\|. \quad (10)$$

By the triangle inequality

$$\left\| v \right\| \leq \left\| v - Gw \right\| + \left\| Gw \right\| \leq \left\| v - Gw \right\| + \left\| \begin{bmatrix} Gw \\ w \end{bmatrix} \right\|$$

and

$$\left\| v \right\| \leq (1 + C) \left\| v - Gw \right\| = \kappa \left\| v - \tau \Delta(v) \right\|$$

and we arrived at (7). Next we show that $\left( I - \Delta(v) \right)^{-1}$ is bounded on $H^+$ for all $\tau \in [0,1]$. By the positive well-posedness assumption, the inverse is well-defined on $H_\tau^+$. The proof is by induction. Assume that for $\tau_0 \in [0,1]$ is bounded for some $\tau_0 \in [0,1], \forall u \in H_\tau^+$, define $u_T = (I - \Delta(v))^{-1}P_T(I - \tau_0 G\Delta(v))u \in H^+$. Then for every $T > 0$,

$$\|P_Tu_T\| = \|P_Tv_T\| \leq \|u_T\|$$

$$\leq \kappa \|I - \tau_0 G\Delta(v)u\|$$

$$= \kappa \|P_T(I - \tau_0 G\Delta(v)u)\|$$

$$\leq \kappa \|P_T(I - \Delta(v)u + (\tau - \tau_0) P_T G\Delta(v)u)\|$$

$$\leq \kappa \|P_T(I - \tau_0 G\Delta(v)u) + \kappa |\tau - \tau_0|\|G\Delta(v)\|P_Tu\|.$$  \(10\)

If $|\tau - \tau_0| < (\kappa)\|G\Delta(v)\|$, then

$$\delta = 1 - \kappa |\tau - \tau_0|\|G\Delta(v)\| > 0.$$

and (10) implies

$$\|P_Tu\| \leq (\kappa/\delta) \|P_T(I - \tau G\Delta(v)u)\|.$$  \(11\)

For every $\tau \in H_\tau^+$, substituting $u = (I - \tau G\Delta(v))^{-1}w \in H_\tau^+$ in (11),

$$\|P_T(I - \tau G\Delta(v))^{-1}w \leq (\kappa/\delta) \|P_Tw\| \forall T > 0, w \in H_\tau^+$$

Thus we have shown that $I - \tau G\Delta(v)$ is bounded on $H^+$ which is equivalent to the boundedness of $H^+$. Notice that the boundedness of $(I - \tau G\Delta(v))^{-1}$ is trivial when $\tau = 0$. Starting from $\tau_0 = 0$ and repeating above operations finite times, the boundedness of $(I - \tau G\Delta(v))^{-1}$ is obtained for all $\tau \in [0,1]$. Hence the claim is established.

A small gain theorem for positive linear operators is also available. The proof of the following lemma is provided in Appendix.

**Lemma 7.** If a linear operator $Q : H^+ \to H^+$ satisfies $|Q| < 1$, then $(I - Q)^{-1} : H^+ \to H^+$ exists.
Strictly speaking, it is improper to call $Q$ linear, since it is defined only on the cone space $\mathcal{H}^+$. Nevertheless the meaning would be clear.

4. ANALYSIS

In this section, we seek to establish a counterpart to the KYP lemma for positive operators which converts the condition iii) in Theorem 5 to an equivalent LMI condition. Unfortunately, our result is not as flexible as the standard case, because equivalent LMI tests are available only for a limited class of Hermitian matrices $\Pi$. However it is possible to obtain the less general bounded real lemma by using the BBP result (Theorem 3) and the small gain theorem (Tanaka and Langbort (2010)). Namely, if

$$\Pi = \begin{bmatrix} I & 0 \\ 0 & -\gamma^2 I \end{bmatrix},$$

then there exists an efficient numerical test for the condition (iii) in Theorem 5. Without loss of generality, we set $\gamma = 1$.

Theorem 8. (Tanaka and Langbort (2010)) Let $G(s) = C(sI - A)^{-1}B + D \in RH^+_\infty$ be internally positive. If \(\Pi = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}\), then the following are equivalent.

(A) There exists a diagonal matrix $P > 0$ such that

$$\begin{bmatrix} A^TP + PA PB^T \\ B^TP \end{bmatrix} + \begin{bmatrix} C D \\ 0 I \end{bmatrix}^T \Pi \begin{bmatrix} C D \\ 0 I \end{bmatrix} < 0.$$

(B) There exists a diagonal matrix $P > 0$ such that

$$\begin{bmatrix} A^TP + PA PB^T \\ B^TP \end{bmatrix} + \begin{bmatrix} C D \\ 0 I \end{bmatrix}^T \Pi \begin{bmatrix} C D \\ 0 I \end{bmatrix} < 0.$$

The above result can be used for a robust stability analysis. The following result is the positive system version of scaled small gain test.

Theorem 9. Let $G(s) = C(sI - A)^{-1}B + D \in RH^+_\infty$ and $\Delta$ be a block diagonal operator $\Delta = \text{diag}(\Delta_1, \ldots, \Delta_d)$ such that

$$\Delta_k : \mathcal{H}^+ \rightarrow \mathcal{H}^+, \quad \|\Delta_k\| \leq 1, \quad k = 1, \ldots, d.$$

If $\Delta$ is restricted to the class of linear uncertainties, the assumption of positive well-posedness is redundant, since it follows from the LMI condition. Notice that LMI (12) means that $\|\Theta^2 G \Theta^{-\frac{1}{2}}\Delta\| < 1$. Since a linear operator $Q = \Theta^2 G \Theta^{-\frac{1}{2}} \Delta$ satisfies $\|Q\| < 1$, the inverse $(I - Q)^{-1} : \mathcal{H}^+ \rightarrow \mathcal{H}^+$ exists by Lemma 7. Nevertheless, it is important to keep the form of above statement so that the nonlinearity of $\Delta$ can be addressed, since as we will see in the example in Section 6, an advantage of the nonlinear IQC analysis over the standard IQC analysis is clear when $\Delta$ is nonlinear.

Proof. Let $\Pi = \begin{bmatrix} \Theta & 0 \\ 0 & -\Theta \end{bmatrix}$. Then, since $\|\Delta\| \leq 1$, $\tau \Delta$ satisfies

$$\sigma_\Pi(v, \tau \Delta(v)) \geq 0 \quad \forall v \in \mathcal{H}^+ \forall \tau \in [0, 1]. \quad (13)$$

for any choice of $\theta_k > 0$. Second, by the KYP lemma (12) implies

$$\begin{bmatrix} G(j\omega) \end{bmatrix}^\ast \Pi \begin{bmatrix} G(j\omega) \end{bmatrix} < 0 \quad \forall \omega \in \mathbb{R}.$$

Namely, there exists $\epsilon > 0$ such that

$$\sigma_\Pi(Gw, w) < -\epsilon \|w\|^2 \quad \forall w \in \mathcal{H}^+. \quad (14)$$

Since positively well-posedness is assumed, by Theorem 5, the interconnection of $G$ and $\Delta$ is positively stable. Finally, to see that the use of diagonal $P$ induces no conservatism, notice that (12) is equivalent to

$$\begin{bmatrix} A^TP + PA & PB \\ B^TP & 0 \end{bmatrix} + \begin{bmatrix} C D \\ 0 \end{bmatrix}^T \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} C D \\ 0 \end{bmatrix} < 0 \quad (15)$$

where

$$\begin{bmatrix} A & B \Theta^{-\frac{1}{2}} \\ C & D \Theta^{-\frac{1}{2}} \end{bmatrix} = \hat{G}.$$

Note that $\hat{G}$ is internally positive because $G$ is and the system $\hat{G}$ is positive if $\Theta \geq 0$. By Theorem 5, there exists a diagonal matrix $P > 0$ satisfying (15) if and only if there exists a full symmetric matrix $P > 0$ satisfying (15).

5. ROBUST CONTROL SYNTHESIS

In this section the $H_\infty$ control design using a static state feedback based on Theorem 8 is presented. Instead of repeating the result from Tanaka and Langbort (2010), the result is shown for slightly more general cone systems, rather than the nonnegative systems. Let $M, L, R$ be a nonsingular square matrices. Define cones $\mathcal{M}, \mathcal{L}, \mathcal{R}$ by

$$\mathcal{M} = \{x \in \mathbb{R}^n | Mx \geq 0\},$$

$$\mathcal{L} = \{u \in \mathbb{R}^m | Lu \geq 0\},$$

$$\mathcal{R} = \{y \in \mathbb{R}^p | Ry \geq 0\}.$$

A linear system (5) is called $(\mathcal{M}, \mathcal{L}, \mathcal{R})$-cone-system if $x(t) \in \mathcal{M}, y(t) \in \mathcal{L}$ for all $t \geq 0$ and $u(t) \in \mathcal{R}$. Let $M, L, R$ be a nonsingular square matrices.

Theorem 10. Let an LTI plant $G$ be given by

$$\dot{x} = Ax + B_1w + B_2u,$$

$$y = C_1x + D_{11}w + D_{12}u,$$

and assume that $MB_1L^{-1} \geq 0$, $RD_{11}L^{-1} \geq 0$. Let $\Pi = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$. Then there exists a static state feedback $K$ such that the closed loop system

$$G_K : \dot{x} = (A + B_2K)x + B_1w,$$

$$y = (C_1 + D_{12}K)x + D_{11}w,$$

satisfying

(a) $\begin{bmatrix} G_K(j\omega) \end{bmatrix}^\ast \Pi \begin{bmatrix} G_K(j\omega) \end{bmatrix} < 0 \quad \forall \omega \in \mathbb{R}.$$

(b) $G_K$ is a $(\hat{\mathcal{M}}, \hat{\mathcal{L}}, \hat{\mathcal{R}})$-cone-system if and only if there exists a positive diagonal matrix $Q$ and $\hat{Z} \in \mathbb{R}^{\dim(x) \times \dim(x)}$ such that

- $S = MAM^{-1}Q + MB_2Z$ is Metzler
- $T = RC_1M^{-1}Q + RD_{12}Z \geq 0$, and
- $S + T^T MB_1L^{-1} \geq 0$, $RD_{11}L^{-1} \geq 0$, and
- $B_1^T M^{\frac{1}{2}} - I \quad D_{11}^T R^{-\frac{1}{2}} - I \quad \leq 0$.
Fig. 2. Plant description

Fig. 3. Sector bound condition

Proof. Define an LTI system

\[
\tilde{G}_K = \left[ \begin{array}{c} M(A + B_2K)M^{-1} \\ R(C_1 + D_{12}K)M^{-1} \\ MB_1L^{-1} \\ RD_{11}L^{-1} \end{array} \right]
\]

and a Hermitian matrix

\[
\tilde{\Pi} = \left[ \begin{array}{cc} R^{-1} & 0 \\ 0 & L^{-1} \end{array} \right] \Pi \left[ \begin{array}{cc} R^{-1} & 0 \\ 0 & L^{-1} \end{array} \right].
\]

Then (a) and (b) are equivalent to

\[
(a') \left[ \begin{array}{c} \tilde{G}_K(j\omega) \\ I \end{array} \right] = 0 \forall \omega \in \mathbb{R}
\]

\[
(a') \tilde{G}_K \text{ is internally positive.}
\]

An application of Theorem 8 and a change of variables \( Q = P^{-1}, Z = KM^{-1}P^{-1} \) yields the desired LMI condition.

The form of the LMI problem in Theorem 10 suggests that the structured control synthesis (a nonconvex problem in general systems) is tractable when \( M = I \). More precisely, one can seek a controller gain \( K \) with an arbitrary sparse structure by imposing the same structure on the LMI variable \( Z \). The desired controller is then reconstructed by \( K = ZQ^{-1} \). Such a controller exists if and only if corresponding LMI problem is feasible.

6. EXAMPLE

In this section a robust control design method using tools developed so far is presented. Consider a system on Fig. 2 and let \( \Delta_1, \Delta_2 \) be scalar uncertainties satisfying the sector bound condition of Fig. 3. Notice that each uncertainty block satisfy a standard IQC

\[
\alpha^2 ||v||^2 - ||\Delta(v)||^2 \geq 0 \forall v \in \mathcal{R}
\]

and a nonnegative IQC

\[
\beta^2 ||v||^2 - ||\Delta(v)||^2 \geq 0 \forall v \geq 0.
\]

For technical reasons, we always assume the well-posedness, and further the positively well-posedness when \( G_{cl} \) is a nonnegative operator. Our objective is to design a static state feedback controller \( K \) which achieves robust stability. Since the uncertainty block in Fig. 2 has a structure, we apply the \( \mu \)-synthesis, which can be formulated as an LMI problem in the case of static state feedback design. In the following, two approaches are compared.

Approach 1: (Standard IQC)

The IQC (16) is used. The controller synthesis exists if there exists a symmetric matrix \( P > 0 \), a real matrix \( K \) of compatible dimension, \( \Theta = diag(\theta_1, \theta_2) > 0 \) such that

\[
\begin{bmatrix}
A_{cl}^TP + PA_{cl}PB_1 \\
B_{cl}^TP & 0
\end{bmatrix} + \begin{bmatrix}
C_{cl}D_1 & 0 \\
0 & 1
\end{bmatrix}^T \Theta \begin{bmatrix}
0 & 0 \\
0 & -(1/\alpha^2)\Theta
\end{bmatrix} \begin{bmatrix}
C_{cl}D_1 \\
0 & 1
\end{bmatrix} < 0
\]

where \( A_{cl} = A + B_2K, \ C_{cl} = C + D_2K \). By standard manipulations and replacements \( P^{-1} = Q, KP^{-1} = Z, \alpha\Theta^{-1} = \Lambda \), the above condition becomes

\[
\begin{bmatrix}
(AQ + B_2Z)^T & -(1/\alpha)\Lambda \\
\Lambda D_{11}^T & \Lambda D_{11}^T
\end{bmatrix} < 0
\]

Hence a robust controller exists if (18) admits a solution \( Q > 0 \), a real matrix \( Z \) of a proper dimension, and \( \Lambda = diag(\lambda_1, \lambda_2) > 0 \) satisfying

- LMI (18) with \( \alpha \) being replaced by \( \beta \)
- \( AQ + B_2Z \) is Metzer
- \( CQ + D_2Z \geq 0 \).

Of course, a nonnegative initial state \( x_0 \geq 0 \) is assumed for the stability in this case.

Consider a second order system with

\[
A = \begin{bmatrix}
-1 & -2 \\
-1 & 1
\end{bmatrix}, B_1 = I, B_2 = \begin{bmatrix}
1 & 3 \\
2 & 5
\end{bmatrix}, C_1 = I, D_1 = \begin{bmatrix}
1 & -1 \\
1 & -2
\end{bmatrix}, D_2 = \begin{bmatrix}
0 & 0 \\
3 & 10
\end{bmatrix}
\]

Notice that the open loop system itself is not a nonnegative system. Suppose that the sector bound condition of the uncertainty is given by \( \alpha = 0.4, \beta = 0.2 \). A numerical analysis shows that Approach 1 is infeasible. This means that an IQC obtained by the usual sector bounding does not admit a robust stabilizing controller. On the other hand, Approach 2 is found to be feasible. In other words, by restricting the search to the class of controllers which attain internal positivity and taking the fact that a smaller gain bound of uncertainty is available in the positive region, a stabilizing controller can be found. Of course Approach 2 is not always applicable because first of all there has to be a class of controllers that achieve internal positivity of the closed loop system. This is related to the nonnegative orthonormal feedback holdability discussed by Haddad et al. (2010). Also, even if there exists such a class, the search space of controllers in Approach 2 is smaller than in Approach 1. Depending on this restriction and the description of uncertainty such as \( (\alpha, \beta) \), Approach 2 may or may not provide a better result than Approach 1. Thus Approach 2 can be an alternative when Approach 1 is not applicable. For example, notice that Approach 2 is applicable even if there is no description about the uncertainty block is available on the region \( v < 0 \).

The controller obtained by Approach 2 achieves stability on the nonnegative orthonormal but not necessarily global stability. A possible way to overcome this restriction is to use other control laws first to drive the state to the nonnegative orthonormal until the controller is switched to the one obtained by Approach 2.

Another important advantage of Approach 2 is that a diagonal structure can be imposed on the LMI variable \( Q \) without introducing conservatism. Hence one can readily
synthesize a structured controller design regime. In above numerical example, one can search for a controller gain of the form $K = \begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix}$ that achieves positive stability by imposing the same structure on the LMI variable $Z$ in Approach 2. By numerical analysis it can be found feasible, yielding a controller gain $K = \begin{bmatrix} 0.500 & 0 \\ 0 & 0.675 \end{bmatrix}$.

7. CONCLUSION AND FUTURE WORKS

The refined IQC framework for the nonnegative systems is presented. The robust control design using the nonnegative IQC framework is proposed. An efficient numerical test for the quadratic separation is available when the quadratic form is a small gain type, i.e., $\Pi = \begin{bmatrix} I & 0 \\ 0 & -\gamma^2I \end{bmatrix}$. Examining the possibility of extension to more general $\Pi$ is the subject of future work.

Appendix A. PROOF OF LEMMA 7

Define two metric spaces of linear operators $X = \{Q : \mathcal{H}^+ \to \mathcal{H}^+\}$, $Y = \{Q : \mathcal{H} \to \mathcal{H}\}$ with metrics

$$
\|Q_1 - Q_2\|_X = \sup_{u \in \mathcal{H}^+, \|u\| = 1} \|(Q_1 - Q_2)u\|_\mathcal{H} \\
\|Q_1 - Q_2\|_Y = \sup_{u \in \mathcal{H}, \|u\| = 1} \|(Q_1 - Q_2)u\|_\mathcal{H}.
$$

Also define an extension $\tilde{X}$ of $X$ by

$$
\tilde{X} = \left\{ Q : \mathcal{H} \to \mathcal{H}^+ \mid \tilde{Q}u = \begin{cases} Qu & \text{if } u \in \mathcal{H}^+ \\
0 & \text{otherwise} \end{cases}, Q \in X \right\}.
$$

Notice that $\tilde{X}$ is a subset of $Y$. If $\tilde{Q}_1, \tilde{Q}_2 \in \tilde{X}$ are associated with $Q_1, Q_2 \in X$ by

$$
\tilde{Q}_i = \begin{cases} Q_iu & \text{if } u \in \mathcal{H}^+ \\
0 & \text{otherwise} \end{cases}
$$

then

$$
\|\tilde{Q}_1 - \tilde{Q}_2\|_Y = \sup_{u \in \mathcal{H}, \|u\| = 1} \|(\tilde{Q}_1 - \tilde{Q}_2)u\|_\mathcal{H} = \|(Q_1 - Q_2)u\|_\mathcal{H} = \|Q_1 - Q_2\|_X.
$$

Step 1: $X$ is complete if $\tilde{X}$ is complete. Suppose that $\tilde{X}$ is complete. Let $\{Q_k\}$ be a Cauchy sequence in $X$ and $\{\tilde{Q}_k\}$ be an associated sequence in $\tilde{X}$ by (A.1). From (A.2), $\{\tilde{Q}_k\}$ is a Cauchy sequence in $\tilde{X}$. By assumption $\{Q_k\}$ has its limit point $Q_\infty$ in $\tilde{X}$ which has a representation

$$
\tilde{Q}_\infty = \begin{cases} Q'u & \text{if } u \in \mathcal{H}^+ \\
0 & \text{otherwise} \end{cases}
$$

for some $Q' \in X$. By taking $Q_\infty = Q'$, we have $Q_\infty \in X$ and $Q_k \to Q_\infty$ as $k \to \infty$. Thus $X$ is complete.

Step 2: $\tilde{X}$ is closed in $Y$. We will show that $Y - \tilde{X}$ is open in $Y$. Suppose $Q_0 \in Y - \tilde{X}$. Then there exists $u_0 \in \mathcal{H}$, $\|u_0\| = 1$ such that $v_0 = Q_0u_0 \in \mathcal{H} - \mathcal{H}^+$. This implies that there exists a Lebesgue measurable support $I_0 \in [0, \infty)$ such that $v_0(t) < 0 \forall t \in I_0$ and $\int_{I_0} |v_0|^2dt = \epsilon > 0$.

Suppose $Q \in \tilde{X}$. Then we have $v = Qu_0 \in \mathcal{H}^+$ and

$$
\|Q - Q_0\|_Y = \sup_{u \in \mathcal{H}, \|u\| = 1} \|(Q - Q_0)u\|_\mathcal{H} = \sup_{u \in \mathcal{H}, \|u\| = 1} \int_{[0, \infty)} \|(Q - Q_0)u\|^2dt \geq \int_{I_0} \|v - v_0\|^2dt \geq \int_{I_0} \|v_0\|^2dt = \epsilon.
$$

The last inequality holds since $v(t) \geq 0$, $v_0(t) < 0 \forall t \in I_0$. Therefore, any $Q \in Y$ with $\|Q - Q_0\|_Y < \epsilon$ belongs to $Y - \tilde{X}$.

From Step 2 and the fact that $Y$ is complete, $\tilde{X}$ is complete since a closed subset of a complete metric space is complete. By Step 1 this further implies that $X$ is complete. In Step 3, we complete the proof by constructing $(I - Q)^{-1}$.

Step 3: Define a Cauchy sequence $T_k \in X$ by $T_k = \sum_{i=0}^{k-1} Q^i$. Then by the submultiplicative inequality

$$
\|T_k(I - Q) - I\|_X = \|(I - Q)T_k - I\|_X = \|Q^{k+1}\|_X = \|Q\|^{k+1} \to 0 \text{ as } k \to \infty.
$$

Since $X$ complete $T_\infty := \lim_{k \to \infty} T_k$ exists in $X$ and $T_\infty(I - Q) = (I - Q)T_\infty = I$. Thus $(I - Q)^{-1} = T_\infty \in X$ exists.

REFERENCES


