Aeroelastic Modelling of Long-Span Suspension Bridges

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Abstract: The 2D aerodynamic modelling of long-span suspension bridges is considered. We use thin airfoil theory from the aircraft industry and a sectional bridge model with an integrated controllable trailing-edge flap. The relatively less well known aerodynamic properties of leading-edge flap will be studied in detail. The optimal approximation of the classical Theodorsen circulation function will be studied as part of the bridge sectional model building exercise, which can therefore be re-casted in a form suitable for control systems analysis and design. The critical wind speeds for flutter and torsional divergence are predicted precisely. Static winglets are shown to be relatively ineffective to mitigate torsional divergence.

1. INTRODUCTION

The now iconic Tacoma Narrows bridge disaster (1940) was caused by the gradual growth, over a period of approximately 45 minutes, of a torsional flutter oscillation. It was subsequently established that the Tacoma Narrows bridge failure resulted from the use of a structurally and aerodynamically inappropriate squat H-section for the main bridge deck Billah and Scanlan [1991]. Over the intervening period engineers have considered a wide range of possible modifications to the aeroelastic properties of new and existing bridge structures Dowell et al. [2004]. Three different solution paradigms, which may be used in combination, are possible Astiz [1998]: (1) modify the deck’s aerodynamic design by invoking variations in the deck cross-section, (2) modify the design of the suspension system so as to adjust beneficially the bridge’s critical structural mode shapes and modal frequencies, and (3) introduce stationary, or actively controlled aerodynamic flutter suppression surfaces. An early example of aerodynamic and structural design changes, which were used on the Tacoma bridge rebuild (1950), was the introduction of a deep open-truss section that leads to enhanced aerodynamic stability and higher structural torsional stiffness. An alternative approach was proposed for the Severn suspension bridge, where a flat box girder was used. Box girders are lighter and easier to manufacture than stiffening trusses, they enhance structural torsional rigidity, they reduce significantly the wind drag forces and at the same time reduce the weight of the deck; see Wilde and Fujino [1998] and the references therein. Technical interest in these problems might have waned at this point were it not for the unrelenting increase in the length of modern suspended structure bridges. Oft quoted examples include the Akashi-Kaikyo, the Great Belt and the Severn bridges, which have central spans of 1991 m, 1624 m and 988 m respectively. The planned Messina bridge, with a central span of 3300 m, would make it the longest in the world. Although wind loading in a civil engineering context is traditionally thought of as a bluff body phenomenon Scanlan and Tomko [1971], modern suspended span bridges are making use increasingly of streamlined decks and classical flat plate analysis usually gives good estimates of the critical flutter wind speed in these cases. This trend...
is important from an aeroelastic modelling perspective, because it makes traditionally inappropriate thin airfoil theory Bisplinghoff et al. [1955] increasingly relevant in circumstances where supplementary flutter suppression techniques must be considered. Following papers such as Wilde and Fujino [1998], Hansen and Thoft-Christensen [2001], Omenzetter et al. [2000a,b] we make use of thin airfoil theory in our computer models. In the case of the trailing-edge flaps, the requisite force and moment expressions are well known and can be taken from the aircraft industry Theodorsen [1934], Bisplinghoff et al. [1955]. Leading-edge flaps in the aerospace context are largely limited to static deployment and so the aerodynamic properties of these surfaces in unsteady operation are not as widely known. While windward flaps have certainly been used in the civil engineering literature, a detailed thin airfoil model of these surfaces has not so far appeared, and in some cases, we believe, the models being used are incorrect.

Most of the civil engineering literature concentrates on flutter, with other aeroelastic instabilities such as torsional divergence receiving relatively less attention. When designing control systems every potential source of aeroelastic instability must be considered and their conflicting requirements assessed in concert. It has been suggested that structural winglets can be used to mitigate torsional divergence, but the suggested remedies presuppose knowledge of the direction of the wind del Arco and Aparicio [1999]. Furthermore these results are based on the unrealistic assumption that the winglets and main deck are aerodynamically independent. We take account of this aerodynamic interaction and show that static winglets are ineffective in increasing the critical wind speed relating to torsional divergence instability.

2. DYNAMIC MODEL

We will now describe the structural and aerodynamic models of the bridge section used in this paper.

2.1 Structural Model

We will make use of a simple sectional model of the bridge deck, since this is all that is required to capture many of the central control-theoretic issues. This model will typically represent the lowest-frequency heave and torsional modes of the bridge structure, which have symmetrical mode shapes Omenzetter et al. [2002]. Referring to the diagram of the system kinematics in Fig. 2, we see that the generalized coordinates are the deck's heave $h$ and pitch angle $\alpha$, and the flap angles $\beta_t$ and $\beta_l$. The flap angles will be treated as inputs, although they will also be system states, since their first and second derivatives will be required in the aerodynamic model. In order to characterize the primary modes, we will assume physical values corresponding to the Akashi Kaikyo bridge Omenzetter et al. [2000b], which is stable for all wind speeds that might be encountered. We assume a deck span of $2b=30\text{ m}$, a mass per unit length of $M_h=33,600\text{ kg/m}$, and a torsional mass moment of inertia per unit length of $J_{\beta_t}=4.97\times 10^6\text{ kg m}^2 \text{ s}^2$. The undamped natural frequencies of the heave and pitch modes are $\omega_h=0.427\text{ rad/s}$ and $\omega_\beta=0.917\text{ rad/s}$ respectively. The structural damping coefficients are assumed to be $\zeta_h=0.0083$ and $\zeta_\beta=0.0072$. An air density of $\rho=1.23\text{ kg/m}^3$ will be assumed.

The heave dynamics is described by:

$$M_h\ddot{h} + 2\omega_h\zeta_h\dot{h} + K_h h = L,$$

in which $L$ is the aerodynamic lift force with $K_h = M_h\omega_h^2$. In the same way, the pitch dynamics is described by:

$$J_\alpha\ddot{\alpha} + 2\omega_\beta\zeta_\beta\dot{\alpha} + K_\beta \alpha = M,$$

in which $M$ is the aerodynamic moment about the deck mid-chord and $K_p = J_\beta\omega_\beta^2$.

![Fig. 2. Kinematic model of the bridge deck. The origin of the inertial axis system is $O$. The wind velocity $U$ is assumed positive to the right, the heave $h$ and lift force $L$ are assumed to be positive downwards, moments $M$ are positive clockwise, as are the pitch and trailing-edge flap angles $\alpha$ and $\beta_t$ respectively. The leading-edge flap angle $\beta_l$ is positive anti-clockwise.](image)

2.2 Aerodynamic Model

The aerodynamic model is based on unsteady thin airfoil theory as was described in the first instance by Theodorsen [1934]. The lift on the system is given by

$$L = \rho b^3\omega^2 \left\{ L_h\frac{h}{b} + L_\alpha\alpha + L_{\beta_t}\beta_t + L_{\beta_l}\beta_l \right\},$$

where $L_h$, $L_\alpha$, $L_{\beta_t}$, and $L_{\beta_l}$ are the aerodynamic derivatives corresponding to the various perturbation variables. From equation (XVIII) in Theodorsen [1934], it follows that

$$L_h = \pi(1 - \frac{2C(k)}{k}),$$

in which $C(k)$ is the Theodorsen function

$$C(k) = \frac{J_1(k) - jY_1(k)}{(J_1(k) + Y_1(k)) - j(J_0(k) - Y_1(k))},$$

where $J_0(k)$, $J_1(k)$, $Y_0(k)$ and $Y_1(k)$ are Bessel functions of the first and second kind respectively, $k = \omega b/U$ the reduced frequency and $j = \sqrt{-1}$. Similarly one may verify

$$L_\alpha = -\pi \left( \frac{2C(k)}{k^2} + \frac{j}{k}(1 + C(k)) \right),$$

$$L_{\beta_t}(c_t) = \frac{jT_4(c_t)}{k} - T_1(c_t) - C(k) \left( \frac{2T_{10}(c_t)}{k^2} + \frac{jT_{11}(c_t)}{k} \right).$$

The functions $T_i(\cdot)$ are defined in Theodorsen [1934]. The leading-edge flap is analysed in the next section.

The aerodynamic derivatives associated with the moment on the system may be computed using:

$$M = \rho b^3\omega^2 \left\{ M_h\frac{h}{b} + M_\alpha\alpha + M_{\beta_t}\beta_t + M_{\beta_l}\beta_l \right\},$$

in which $M_h$, $M_\alpha$, $M_{\beta_t}$, and $M_{\beta_l}$ are the moment-related aerodynamic derivatives. Direct calculation using equation (XX) in Theodorsen [1934] gives
In the same way the following may be verified

\[ M_\alpha = \frac{\pi}{8} - j\frac{\pi}{2k} + \frac{\pi C(k)}{k^2} \tag{10} \]

and

\[ M_\beta_i(c_i) = -\frac{T_4(c_i)}{k^2} - j\frac{T_1(c_i) - T_6(c_i) - c_i T_4(c_i)}{k} - c_i T_1(c_i) + \left(\frac{C(k) - 1}{k^2}\right) \left(T_{10}(c_i) + \frac{jk T_{11}(c_i)}{2}\right) \tag{11} \]

with the newly introduced \( T_i(\cdot) \) also given in Theodorsen [1934]. Equations (10) and (11) are compared in Fig. 3 with a two-dimensional discrete vortex panel code, which applies the assumptions of thin aerofoil theory. It should be noted that the plots given in Fig. 3 have been multiplied by \( k^2 \) so that the illustrated quantities are properly defined in the steady state \( (k = 0) \).

![Fig. 3. Aerodynamic derivatives for the trailing-edge flap.](image)

The blue dashed curves (red solid) correspond to the real (imaginary) parts of the aerodynamic derivatives computed using thin aerofoil theory. The (red) stars and (blue) hexagons were computed using a discrete vortex panel code.

### 2.3 Leading-Edge Flaps — Pitch and Heave Offsets

The aerodynamic influence of the leading-edge flap may be modelled using superposition and the results in Theodorsen [1934], or what is essentially the same thing, the results in Theodorsen and Garrick [1942] that consider a wing-flap-tab combination. In order to do this, we consider the transformation illustrated in Fig. 4 in combination with equations (22) and (23) in Theodorsen and Garrick [1942] in which \( l, m \) are set to zero. The first step involves the use of a negative value of \( c_i \) to replace \( c_i \), which transforms the wing into the leading-edge flap and the flap into the main deck. This change leaves the main deck at an inclination of \( \alpha + \beta_i \) and with a heave offset of \( c_i b \beta_i \) in the mass centre. In order to correct the heave and pitch offsets, the whole assembly must be rotated through \( -\beta_i \) and elevated through a vertical distance of \( c_i b \beta_i \). The leading-edge flap aerodynamic derivatives are thus given by:

\[ L_{\beta_i}(c_i) = L_{\beta_i}(c_i), \quad L_\alpha = -c_i b L_\alpha, \tag{12} \]

\[ M_{\beta_i}(c_i) = M_{\beta_i}(c_i) - M_\alpha - c_i b M_\alpha. \tag{13} \]

The expressions in (12) and (13) make use of (7) and (11) with \( c_i \) taking the negative value of \( c_i \), \( \beta_i = \beta_i \), with the pitch and heave corrections from (4), (6), (9) and (10).

In order to check these results they are compared with a discrete vortex panel code in Fig. 5. These plots show close agreement between the thin aerofoil results given in (12) and (13) and the vortex code. As before, the results in Fig. 5 have been scaled by \( k^2 \).

![Fig. 4. Transformation of the Theodorsen-Garrick wing-flap-tab configuration.](image)

(A): The wing pitch angle is \( \alpha \), the aileron angle is \( \beta_i \) and the tab angle is \( \beta_i \). The wind speed is \( U \) (from the left), the heave \( h \) and lift \( L \) are positive downwards, while moments and angles are positive clockwise. The wing chord is \( 2b \), the width of the tab and aileron are described in terms of \( c_i \) and \( c_i \) respectively. (B): The wing-ailer-on-tab configuration is transformed into the controlled bridge deck by making \( c_i \) negative, thereby forcing the flap hinge to the left of the origin. In this new configuration the aileron becomes the bridge deck, the wing the leading-edge flap and the tab the trailing-edge flap. In order to re-level the bridge, and return its mass centre to the correct position, pitch and heave corrections must be applied.

![Fig. 5. Aerodynamic derivatives for the leading-edge flap.](image)

The blue dashed curves (red solid) correspond to the real (imaginary) parts of the aerodynamic derivatives computed using Theodorsen-Garrick potential theory. The (red) stars and (blue) hexagons were computed using a linear discrete vortex code.

### 2.4 Rational Approximations

The traditional (and somewhat arcane) approach to aeroelastic stability analysis is based on finding iteratively wind speed(s) for which sinusoidal solutions to equations (1) and (2) exist. Standard algorithms, including the \( k \)-method and the \((p - k)\)-method Hodges and Pierce [2002], require the repeated evaluation of the Theodorsen function, which
is a combination of Bessel functions, Theodorsen [1934], Bisplinghoff et al. [1955]. This approach has several disadvantages: Firstly, one must seek out the resonant frequencies one at a time (one of which might be zero). Second, one must distinguish between single- and coupled-mode cases — this becomes particularly inconvenient when high-order modal combinations are possible. Thirdly, during a design exercise, one cannot form a clear picture as to how design changes are influencing the aeroelastic stability of the structure as a whole, which may possibly include multiple flutter modes as well as multiple divergence modes. Motivated by a desire to study transient (non-steady) phenomena, Jones and Sears introduced operational methods (and the Laplace transform) into aeroelastic theory Jones [1938], Sears [1940]. To that end Jones introduced the following rational approximation to the (steady-state) Theodorsen function

\[
J(\hat{s}) = \frac{1}{2\pi} \left\{ 2\pi - \frac{0.330\pi \hat{s}}{0.0455 + \hat{s}} - \frac{0.670\pi \hat{s}}{0.3 + \hat{s}} \right\},
\]

in which \(\hat{s} = \frac{s}{\omega_n}\) is the reduced Laplace transform variable (\(s\) is the Laplace transform variable).

By invoking linear least squares approximation methods, the authors found an accurate, stable and minimum phase quartic rational approximation to \(C(k)\) whose numerator and denominator coefficients are given in Table 1. The Nyquist diagrams of the Theodorsen, Jones and quartic approximation are shown in the left-hand part of Fig. 6. It is self evident that the quartic approximation is very accurate with the lower order Jones function somewhat less so. If one considers the quartic approximation to be a system, then its step response, shown on the right-hand of Fig. 6, is the well known Wagner step response curve.

![Fig. 6. The Theodorsen function and its rational approximations. In the left-hand diagram the Theodorsen function, Theodorsen [1934], is the (blue) dot-dash curve, the quartic approximation the (red) dashed curve, and the Jones function, Jones [1938], the (black) dotted curve. The right-hand diagram is the step response of the quartic approximation - this is the Wagner step response curve Bisplinghoff et al. [1955].](image)

![Fig. 7. Root-loci of the bridge section. The wind speed is swept from 30 m/s to 80 m/s, with the low-speed end of the root loci marked with (blue) diamonds and the high-speed ends marked with (red) hexagons. The pitch mode goes unstable at approximately 52 m/s, while the torsional divergence mode goes unstable at approximately 70 m/s.](image)

### 2.5 Open-Loop Response

The uncontrolled system can be represented in terms of a generalized state-space model of the form \(E\dot{x} = Ax\) which is an assembly of equations (1), (2), (3), (8) and a rational approximation to (5). A root-locus diagram for the open-loop system is shown in Fig. 7 in which the wind speed \(U\) is the varied parameter. As one would expect, there are two oscillatory structural (flutter) modes due to the heaving and pitching of the bridge deck. As the wind speed increases, the heave mode damping increases and the corresponding complex pair of eigenvalues moves further into the left-half plane. Although the pitch mode is stabilized by the moderate winds in the positive x-direction, it becomes unstable when the wind speed increases above 52 m/s. There is also a real structural mode that represents the pitching of the bridge deck due to the steady-state aerodynamic moment given in (10) — this mode is referred to as the torsional divergence mode. When the wind speed exceeds 70 m/s, the bridge deck will pitch up and lift into the wind like the unconstrained motion of a piece of paper in a draft.

![Fig. 7. Root-loci of the bridge section. The wind speed is swept from 30 m/s to 80 m/s, with the low-speed end of the root loci marked with (blue) diamonds and the high-speed ends marked with (red) hexagons. The pitch mode goes unstable at approximately 52 m/s, while the torsional divergence mode goes unstable at approximately 70 m/s.](image)
and (10) that the net torsional stiffness is zero when the wind speed reaches a value of
\[ U_{tc} = \sqrt{\frac{K_p}{\rho \pi b^2}}. \] (15)

Direct substitution of the bridge deck data into (15) yields \( U_{tc} = 69.33 \text{ m/s} \), which agrees with both Figures 7 and 8. Although this is in excess of most 100 year return extreme winds, the control challenge is the retention of a positive value of the torsional stiffness at all relevant wind speeds, while simultaneously suppressing flutter instabilities.

Fig. 8. Root-loci of the bridge section with only the pitch dynamics present. The wind speed is swept from 30 m/s to 80 m/s, with the low-speed end of the root loci marked with (blue) diamonds and the high-speed ends marked with (red) hexagons. The torsional divergence mode goes unstable at approximately 70 m/s as is predicted by equation (15).

It has been suggested that the negative aerodynamic moment that leads to an unstable torsional divergence mode might be counteracted by a trailing-edge winglet del Arco and Aparicio [1999]. In the present context winglets are small aerofoil surfaces set at zero (or small) fixed pitch angles relative to the main deck to which they are rigidly attached and are usually located in the vicinity of its trailing (or leading) edge. While winglets can certainly generate stabilizing moments, several issues have to be kept in mind. Firstly, the moment generated by the winglet alone on itself is small and destabilizing, and a net stabilizing moment on the whole system can only be achieved by exploiting the winglet’s lift in combination with a deliberately introduced moment arm. Secondly, a trailing-edge winglet becomes destabilizing if the wind direction reverses — this is somewhat akin to putting the flight feathers on the wrong end of an arrow. Thirdly, the aerodynamic efficacy of the winglet is degraded by the aerodynamic interference generated by the main deck. The effect of the winglet will be reduced as the distance between the winglet and the bridge deck is reduced. Fourthly, the deployment of symmetrically located winglets (at the leading and trailing edges), to provide an effect independent of the wind direction, destabilises the torsional divergence mode.

Suppose as shown in the sketch in Fig. 9 one treats the winglet as an aerodynamically independent thin aerofoil, as in del Arco and Aparicio [1999]. As before, the critical wind speed for torsional divergence is determined as the speed at which the overall torsional stiffness vanishes. That is when
\[ K_p \alpha = M + M_w + rbL_w, \]
in which \( M \) is the quasi-steady-state aerodynamic moment on the main deck, and \( L_w \) and \( M_w \) are the quasi-steady-state aerodynamic lift and moment, respectively, generated by the trailing-edge winglet. Substituting the appropriate steady-state aerodynamic moment and lift quantities gives
\[ U_{tc} = \sqrt{\frac{K_p}{\rho \pi b^2 \left[ 1 + \left( \frac{rb}{\pi} \right)^2 - 2r \left( \frac{rb}{\pi} \right) \right]}}, \] (16)
in which \( b_w \) is the winglet chord. In the case that \( b_w = 2 \text{ m} \) and \( r = 0.8 \) the critical wind speed is given as 77.3 m/s by (16); this figure turns out to be unrealistically high.

We will now show that the efficacy of a trailing-edge winglet as a means of increasing the torsional divergence speed is degraded when aerodynamic interactions between the winglet and the main deck are taken into account. Indeed, a winglet used in this way is relatively ineffective and one cannot assume aerodynamically independence from the main deck in any practical configuration. This fallacious supposition (aerodynamic independence) is used in del Arco and Aparicio [1999], where one finds the over optimistic claim: “… thus a bridge could actually be designed to have a divergence speed as high as desired. This is the basis of the argument that torsional divergence is no longer a critical criterion if using aerodynamic appendages …. As is now demonstrated by vortex panel code calculation, the winglet’s lift potential is significantly compromised by the downwash from the main deck. Moreover, this interference extends surprisingly far out and will be key consideration at any practical separation distance. An initial calculation shows that the winglet’s pitch aerodynamic derivative \( CL_{\alpha} \) reduces from its theoretical value of \( 2\pi \), achieved when separated from the main deck by approximately one hundred deck chords, to roughly half that value when the winglet-deck separation is of the order of a deck semi-chord. The computation shown in Fig. 10 assumes that a fixed winglet of chord \( b/10 \) is placed parallel to the main deck with its centre at a variable distance vertically below the trailing edge (of the main deck); the coordinates of the centre of the winglet are thus given by \((b \cdot z_{gap})\), in which \( z_{gap} \) is variable.

Another aspect of the effectiveness of the winglet, as dictated by its interactions with the main deck, is illustrated in Fig. 11 in terms of a ‘winglet effectiveness indicator’ function, which is the ratio of actual moment generated by the deck and winglet to the moment generated by the deck alone. For the winglet to be useful this indicator should be significantly smaller than unity indicating a reduction in the adverse moment; great than unity indicates destabilization. The first region where this moment effectiveness factor is equal to unity (indicating an ineffective winglet
Fig. 10. Lift-pitch aerodynamic derivatives for the winglet and winglet-deck assembly. The blue pentagrams show the steady-state lift-pitch derivative for the whole deck-winglet assembly as computed using a vortex code. The red hexagrams show the steady-state lift-pitch derivative for the winglet alone (× 10); all coefficients non-dimensionalised by the bridge-deck chord. The separation between the winglet and the main deck is given in terms of main deck chords.

Fig. 11. Winglet effectiveness as a moment generator. ‘Moment effectiveness’ is the ratio of actual moment generated by the deck and winglet to the moment generated by the deck alone.

contribution) is the area close to the bridge deck and in the vicinity of the mid point. This is expected, since the moment arm associated with the winglet’s lift force is small. The contours also show that the moment effectiveness factor remains near unity above and below the trailing edge of the main deck, because the winglet’s moment generating potential remains small unless the associated moment arm is large. A large moment effect is achieved when the winglet is located in the region beyond the decks leading edge, but this is both impractical and destabilizing.

4. CONCLUSIONS

This paper addresses the problem of modelling and validation of long-span suspension bridges with controllable flaps at both ends. Since leading-edge flaps appear not to have been treated in the literature, formulae describing their characteristics are not available ‘off the shelf’ and so these results are derived from first principles and are given in Section 2.3. The rational approximation of the Theodorsen function is not a new topic, Jones [1938], but the required level of fidelity varies from application to application. Approximation results suitable for our purposes are given in Section 2.4, where a stable and minimum phase fourth-order approximation is provided. We also have shown that static winglets are ineffective in increasing the critical wind speed relating to torsional divergence instability.

REFERENCES