Stochastic Reachability: From Markov Chains to Stochastic Hybrid Systems

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Abstract: Stochastic hybrid systems represent established classes of realistic models of hybrid dynamics subject to random perturbations, autonomous uncontrollable transitions, non-determinism or uncertainty. Stochastic reachability analysis is a key factor in the verification and deployment of stochastic hybrid systems. In this paper, we tackle the stochastic reachability problem from a purely probabilistic perspective. Using the connection between stochastic reachability and optimal stopping, the reach probabilities are estimated using only the probabilistic parameters (transition probabilities, infinitesimal generator) of stochastic hybrid systems. The technique is illustrated first for Markov chains, then for other more complex Markov models.

Keywords: stochastic hybrid systems, reachability, optimal stopping, Markov models.

1. INTRODUCTION

Many practical systems (automobiles, chemical processes, and autonomous vehicles) are very suitable to be described by dynamics that comprise continuous state evolution within a mode of operation and discrete transitions from one mode to another, either controlled or autonomous. Such systems often interact with their environment in the presence of randomness and variability. For these systems, the most used modeling paradigm is provided by stochastic hybrid systems (SHS). SHS can model complex nonlinear dynamics, randomness, multiple modes of operations and support high-level control specifications that are required for design of (semi-)autonomous applications. Intuitively, the SHS dynamics can be described as a sequence of a finite/countable family of diffusion processes controlled by a Markov chain. Modeling and analysis of SHS Bujorianu et al. [2006] have been proved to be a very difficult and challenging task from a mathematical point of view. The stochastic analysis tools employed to study different aspects related to the probabilistic features of SHS are rather complex and sometimes less intuitive. The comprehension of SHS involves understanding how to combine tools available for diffusion processes and Markov chains, in order to obtain valuable characterizations of these systems. But, the fact the switching mechanism between the diffusion paths is also random makes things more complicated and thus the generalization of the well studied diffusion properties is not straightforward. The best characteristic of the SHS dynamics is that it can be described by some relatively ‘good’ Markov processes Bujorianu et al. [2004, 2006]. Moreover, the hybrid nature of these stochastic systems is illustrated by different characterizations of such processes like the expression of their generator, the martingale problem, the differential formula Bujorianu et al. [2004], or the associated partial differential equations like the Kolmogorov backward/forward equations. The influence of the boundaries on the dynamics leads to the fact that these processes cover only a specific subclass of Markov processes, with a very little intersection with other types of processes more popular in the theory of stochastic control. The difficulties arise because the mode boundaries play the role of the guards (if the trajectory reaches the boundary of the existing mode then the system needs to execute a discrete transition in another mode and to start a new continuous evolution). This fact, which is very natural in the evolution of hybrid systems, produces the apparition of the forced (predictable) transitions that destroy the “smoothness” of the transition probabilities. These forced transitions affect mostly the extension/generalization of the approximation methods and of stochastic control techniques from diffusion processes to SHS.

The probability distributions associated to the SHS dynamics make difficult the extension of the verification techniques available for deterministic hybrid systems. The reachability problem concept in the framework of stochastic hybrid systems was set up in Bujorianu et al. [2003]. Since then, different authors have studied this problem for particular classes of SHS Blom et al. [2006], or in the general framework Koutsoukos et al. [2006]. The mathematical foundations of this concept have been addressed in Bujorianu et al. [2003]. Intuitively, the stochastic reachability (SR) analysis aims to assess the probabilities of those system trajectories that visit a target set in finite/infinite horizon time.

In this paper, we study the SR problem in an incremental manner starting with Markov chains and ending with stochastic hybrid processes. The foundation of this study is given by a two-face characterization of the reachability measures. One face is given by the concept of réduite function belonging to the probabilistic potential theory, and the other face is coming from stochastic control and is the well known concept of value function for an opti-
Mal stopping problem (OSP). The connections between Markov process theory, potential theory and optimal control have become more classical in the literature. It was the merit of the famous mathematician J. Doob who identified these connections (using the concept of martingale) and possible the use of the strong analytical tools from potential theory to stochastic analysis.

We start with the techniques available for Markov chains for computing the r´eduite/value function that represents the desired reachability measure. These techniques are using the one-step probabilities (or the stochastic matrix) that define a Markov chain. Then we extend these techniques to continuous time continuous space Markov processes, in particular to SHS. We will not use equidistant time discretisations of the given process and approximations of the stochastic differential equations (SDE) by difference equations since they are difficult to be extended for SHS. In general, it has been proven that the analytical and computational methods available for diffusion processes do not admit easy leveraging to SHS due to the guarded transitions. We will consider Markov chain approximations that can be derived using approximations of the infinitesimal generator of a Markov process. Then convergence results will be employed to prove that these approximations can be used for the computation of the reachability measures. The main contribution of the paper is to derive algorithms for the computation of the reach probabilities, using only the probabilistic apparatus associated to SHS.

2. PROBLEM FORMULATION

This section provides the formal definitions for the SR concept in the framework of SHS. Then this concept is related to other mathematical concepts that can help to obtain different characterizations of this problem and techniques for solving it.

Let us consider a Markov process $M = (x_t, P_x)$ viewed as the realization of an SHS (all the definitions will be provided in the following sections). In practice, if we think this Markov process as the realization of a real system, most of the verification problems can be formulated in terms of stochastic reachability, in the sense that we need to evaluate the probability that the system trajectories avoid an unsafe set or reach a safe one. The trajectories of $M$ can be thought of as elementary events in the underlying probability space $(\Omega, \mathcal{F}, P)$.

Given a measurable set $A$ in the state space (that can be thought an obstacle or an unsafe set) and a time horizon $T \in [0, \infty]$, we define (see Bujorianu et al. [2003]):

$$\text{Reach}_T(A) = \{ \omega \in \Omega \mid \exists t \in [0, T] : x_t(\omega) \in A \}$$  \hspace{1cm} (1)

The SR problem consists of determining the probabilities of such events. The reachability problem is well-defined, i.e. $\text{Reach}_T(A)$, $\text{Reach}_\infty(A)$ are indeed measurable sets, i.e. events belonging to the filtration $\mathcal{F}$. Then the probabilities of reach events are

$$P[\text{Reach}_T(A)] = E[\max_{t \in [0, T]} 1_A(x_t)]$$

$$P[\text{Reach}_\infty(A)] = E[\max_{t \geq 0} 1_A(x_t)]$$  \hspace{1cm} (2)

where $1_A$ is the indicator function of $A$, $P$ is a probability on the measurable space $(\Omega, \mathcal{F})$, and $E$ denotes the expectation w.r.t. $P$. The existence of the maximum in the right hand part of (2) shows that the reach set probability “measures” as functions of the target set are not additive, but sub-additive. Using the first hitting (entrance) time $T_A := \inf\{t > 0 | x_t \in A\}$ of target set $A$: we may express the reach set probabilities as follows $P(T_A < T)$ or $P(T_A < \infty)$. Note that $P$ can be chosen to be $P_x$ if we want to consider the trajectories, which start in $x$.

In this paper, we investigate the properties of the reach set probabilities from two different perspectives: (A) potential theory connected to Markov processes, (B) stochastic control for Markov processes.

The SR problem can be viewed in connection with two concepts: (1) r´eduite of a function (belonging to the potential theory language), (2) optimal stopping problem (for Markov processes) with the reward function given by the indicator function of a subset of the state space. The connection between these two concepts has been studied in the context of the optimal stopping problems for Markov processes El Karoui et al. [1992]. The reach set probability can be characterized either as the r´eduite of a specific function (the indicator function of the target set), or the value function of an OSP. Intuitively, the r´eduite of a function is the smallest excessive function that lies above the given function. For Markov processes, the excessive functions can be thought of as stochastic Lyapunov functions for reasons that will be detailed in the following section. Therefore, in order to obtain feasible computational solutions for the evaluation of the reach set probabilities, we can cross-fertilize the methodologies available to study these concepts in the potential theory framework or in stochastic control.

Analytical solutions can be obtained by deriving the variational inequalities for the value function of the OSP Bujorianu et al. [2007]. From the theoretical viewpoint these are difficult to deal with and, they involve managing a very complex mathematical apparatus with hidden subtleties (at least for SHS). From the practical viewpoint, a huge computational effort might be necessary depending on the system structure and on the characteristics of the system transitions. Approximation methods can be derived using Markov chain approximations of the underlying process. An important cornerstone of these methods is given by the existence of the convergence results for the value function of the OSP, or for the r´eduite function. For SHS, another important aspect is how the transitional structure (discrete/continuous transitions) is reflected by the approximation process. It is well known that the “classical” approximations (Euler-Maruyama) are not very suitable for SHS, since, sometimes, they can not “capture” the discrete transitions given by guards.

We will make use of approximations based on the infinitesimal generator of the underlying stochastic hybrid process. These approximations have a Poisson time stepping such that the influence of the forced transitions is captured. The method does not aim to approximate the underlying hybrid process by discrete processes, but to discretize the generator by some appropriate stochastic kernels that will be later on used in the computation of the reach set probabilities.
3. PRELIMINARIES

In this section we give the necessary background for SHS, the description of their dynamics, and some stochastic analysis that is used for studying SHS. The presentation is given in layered manner. At the bottom, we give the main elements (the hybrid automaton structure) that will be used then to describe the hybrid state space and the hybrid dynamics. The second layer is the description of the state space and the executions of the hybrid automaton. The third layer comes with the description of hybrid executions in the form of a Markov process. The transition probabilities of such a process can be employed to construct different linear operators like those that define the semigroup, resolvent and the infinitesimal generator of the underlying Markov process. These operators represent the main tool to handle the stochastic calculus related to Markov processes.

Stochastic Hybrid Systems

Different modelling paradigms for SHS have been proposed in literature Hu et al. [2000]. Applications of SHS range from air traffic management, biology, to communication networks.

Formally, a stochastic hybrid automaton (SHA) is defined as a tuple $H = (Q, X, F, R, \lambda)$, where

- $Q$ is a finite set of discrete variables;
- $X : Q \to \mathbb{R}^{|X|}$ maps each $q \in Q$ into a mode (an open subset) $X^q$ of $\mathbb{R}^{|X|}$, where $d(q)$ is the Euclidean dimension of the corresponding mode;
- $F : Q \to 2^{\mathbb{R}^{|X|}}$ specifies the continuous evolution of the automaton in terms of stochastic differential equations (SDE) over the continuous state $x^q$ for each mode;
- $R = (R^q)_{q \in Q}$ is a family of stochastic kernels;
- $\lambda : \bigcup_{j \in Q} X^j \to \mathbb{R}^+$ is a transition rate function, which gives the distributions of the jump times.

The executions of an SHA, $H$, can be described as follows: start with an initial point $x_0 \in X^0$, follow a solution of the SDE associated to $X^0$, jump when this trajectory hits the boundary or according with the transition rate $\lambda$ (the jump time is the minimum of the boundary hitting time and the time that is exponentially distributed with the transition rate $\lambda$). Under standard assumptions, for each initial condition $x \in \bigcup_{j \in Q} X^j$, the possible trajectories starting from $x$, form a stochastic process. Moreover, for all initial conditions $x$, the realizations of an SHA make up a Markov process in a general setting.

Let us consider the stochastic process $M = (x_t, P_x)$, which represents the realization (or semantics) of $H$, i.e. all its possible trajectories. We can define in a standard way the probability space $\Omega$ as the set of all trajectories of $M$. As well, for each time $t > 0$, we may define the history of the process $F_t$ in the form of a $\sigma$-algebra. Under mild assumptions on the parameters of $H$, $M$ can be viewed as a family of Markov processes with the state space $(X, B)$, where $X$ is the union of modes and $B$ is its Borel $\sigma$-algebra. Let $B^0(X)$ be the Banach space of bounded positive measurable functions on $X$ with the norm given by supremum. Here, $(P_x)_{x \in X}$ represent the probabilities on the trajectories, i.e. $P_x(x_0 = x) = 1$.

A function $k(x, \Gamma)$ defined on $X \times B(X)$ is a transition function (or stochastic kernel) if: (i) $k(x, \cdot)$ is a probability on $X$ for each $x \in X$, and (ii) $k(\cdot, \Gamma)$ is measurable for each $\Gamma \in B(X)$. The family of stochastic kernel in the definition of the hybrid automaton is defined as follows: $R^t : \mathbb{R}^n \times \bigcup_{j \in Q, q \in Q} X^j \to [0, 1]$; where $B(X^j)$ is the Borel $\sigma$-algebra of $X^j$, $(R^t)_{t \in \mathbb{Q}}$ play the role of the reset map from the case of deterministic hybrid systems.

Hybrid Processes

Stochastic Analysis Elements: For the analysis of SHS, we need to make use of the different characterizations of Markov processes. In the following, the most important linear operators associated to a Markov process are briefly presented. They have been proved to be very useful in the context of continuous time, continuous space stochastic processes. Their presence in this paper is justified by the fact that these operators are not standard in the theory of discrete processes (like Markov chains), and the reader familiar only with discrete processes might have difficulties in understanding the stochastic analysis characterizations for SHS developed in the continuous context.

Let us consider the Markov process $M = (x_t, P_x)$. The following mathematical objects can be defined:

- Transition probability function: $p_t(x, A)$, $A \in B$, $t > 0$.
- Operator semigroup: $P = (P_t)_{t \geq 0}$ defined by $P_t f(x) = \int f(y) p_t(x, dy)$, $f \in L^1$, $\forall x \in X$; where $E_x$ is the expectation w.r.t. $P_x$.
- Operator resolvent: The operator resolvent $(V_\alpha)_{\alpha \geq 0}$ is the Laplace transform of the semigroup $P$, i.e. $V_\alpha f(x) = \int_0^\infty e^{-\alpha t} P_t f(x) dt, x \in X$. (3)

Let denote by $V$ the initial operator $V_0$ of $V$, which is known as the kernel operator of the Markov process $M$.

Infinitesimal generator: $L$ is the derivative of $P_t$ at $t = 0$. Let $D(L) \subset B_0(X)$ be the set of functions $f$ for which the limit $\lim_{t \to 0} \frac{1}{t}(P_t f - f)$ exists (denoted by $L f$). $D(L)$ is known as the domain of the generator. Traditionally, Markov processes have been described by their generators and the corresponding evolutions by the operator semigroups/resolvents.

Realization of an SHS: Suppose now $M$ represents the realization of an SHS $H$. We have proved that under standard assumption $M$ is a Borel right process Bujorianu et al. [2006]. The most important property is that $M$ is a strong Markov process. This property is equivalent with the following one: If $f$ is an $\alpha$-excessive function for the semigroup $P$, then the sample path $t \to f(x_t(\omega))$ is a.s. (almost sure) right continuous. Moreover, the sample paths of $M$ are right continuous with left limit (RCLL), i.e. are cadlag (the French abbreviation for RCLL).
Recall that a nonnegative function \( f \in B^b(X) \) is called \( \alpha \)-excessive (\( \alpha \geq 0 \)) if \( e^{-\alpha t} P_t f \leq f \) for all \( t \geq 0 \) and \( e^{-\alpha t} P_t f \not\rightarrow f \) as \( t \searrow 0 \). If \( \alpha = 0 \), a 0-excessive function is simply called excessive function. The excessive functions can be characterized also using the operator resolvent (3) as follows: \( f \in B^b(X) \) is an excessive (or a \( \mathcal{V} \)-excessive) function if \( \alpha V f \leq f \) for all \( \alpha > 0 \), and \( \sup \{ \alpha V f : \alpha > 0 \} = f \). In the theory of Markov processes, the \( \alpha \)-excessive functions play the role of the superharmonic functions from the theory of partial differential equations (a function \( f \geq 0 \) is superharmonic w.r.t. the Laplace operator if \( \Delta f \leq 0 \)). Moreover, sometimes the excessive functions are called stochastic Lyapunov functions since they have the supermartingale property on the paths, i.e. if \( f \) is an excessive function then \( f(x_t) \) is a supermartingale.

Let us denote the cone of excessive functions by \( \mathcal{E}_M \). We assume also that \( M \) is transient, i.e. there exists a strictly positive Borel function \( q \) such that \( V q \) is bounded. The transience of \( M \) means that any process trajectory which will visit a Borel set of the state space it will leave it after a finite time. The transience hypothesis guarantees that the cone \( \mathcal{E}_M \) is rich enough to be used.

We have proved that for an appropriate domain \( D_v(\mathcal{L}) \), the extended generator of an SHS has the following integro-differential form \( \mathcal{L} f(x) = \mathcal{L}_{conf} f(x) + \mathcal{L}_r f(x) \) where \( \mathcal{L}_{conf} f(x) \) has the standard form of the diffusion infinitesimal operator. What makes this generator “special” is its domain that contains at least the set of second order differentiable functions that satisfy the following boundary condition:

\[
\begin{align*}
  f(x) &= \int f(y) R(x, dy), \quad x \in \partial X. 
\end{align*}
\]

4. CHARACTERIZATIONS OF SR

Suppose that the target set \( A \) is a Borel measurable set. In this case, we aim to characterize the reach set probabilities, in connection with the well studied concepts available for studying Markov processes, aiming to derive computational solutions. This section provides some characterizations of the SR problem, based on potential theory, on one hand, and on stochastic control, on the other hand.

Rédoute and optimal stopping problem

Suppose that the notations from the previous section are in force. Now we introduce the potential theory concept called reduced function or réductive. For any \( f : X \rightarrow \mathbb{R}_+ \), we denote by \( R f \) the function

\[
R f := \inf \{ u \in \mathcal{E}_M | u \geq f \}
\]

(5)

Usually, this function is called the réductive of \( f \) w.r.t. the resolvent \( \mathcal{V} \). The réductive of \( f \) differs from \( f \) only on a negligible set. For any subset \( A \) of \( X \) and \( v \in \mathcal{E}_M \), the function \( R_A v = R(1_A v) \) is called the réductive of \( v \) on \( A \), where the operator \( R \) is defined by (5). The convention in use is \( 0 \cdot (+\infty) = (+\infty) \cdot 0 = 0 \).

In potential theory, in the proof of the existence of the réductive for a function \( f \), it is usually assumed that \( f \) is the difference of two excessive functions. The réductive is an useful concept that defines for an arbitrary function the smallest excessive function that lies above. In Markov process theory, the existence of the réductive of a bounded measurable function \( g : X \rightarrow \mathbb{R} \) (w.r.t. the resolvent \( \mathcal{V} \)) is proved using different ideas from the ones used in potential theory. In fact, this existence is based on the following equality El Karoui et al. [1992]:

\[
R g(x) = \sup\{ E_x [ g(\gamma_S) 1_{\{S<\infty\}} ]; \; S \text{ stopping time} \}.
\]

(6)

It is easy to see that right hand side of the equality (6) is related with the optimal stopping problem associated with a Markov process.

The OSP definition for a strong Markov process \( M \) taking values in a Lusin space is briefly recalled in the following. Let \( \Sigma \) denote the set of stopping times (finite or not) w.r.t. the filtration \( \{ F_t \} \) (i.e. \( \tau \in \Sigma \Leftrightarrow \forall t, \{ \tau \leq t \} \in F_t \)). Consider \( g \in B^b(X) \) called the reward function (the interpretation being that if we stop the process at a point \( x \in X \) we obtain a reward \( g(x) \)). Obviously, the definition of OSP requires some integrability conditions over the paths of \( M \) (see, for example El Karoui et al. [1992], for more details). Let \( (\gamma_{\tau})_{\tau \geq 0} \) be the reward process defined by \( \gamma_{\tau} := g(x_{\tau}), \tau \geq 0 \). The value function (in the terminology of Davis [1993]) is

\[
v(x) := \sup \{ E_{x} g(\tau) | \tau \in \Sigma \}.
\]

(7)

The value function has been characterized in terms of the minimal excessive function lying above the reward function El Karoui et al. [1992], Shiryaev [1976]. In the light of the definitions presented in the previous subsection, this means that the réductive of \( g \) coincides with the value function (7). Moreover, the value function can be characterized by means of the smallest supermartingale (called Snell’s envelope) dominating the reward process Mertens [1972].

Connection with the SR

In this subsection, we characterize the reach set probabilities in terms of the réductive, or equivalently, in terms of the value function for an OSP.

Now, let us consider a target set \( A \in B(X) \) in the context of the reachability problem. It is easy to observe that if we set the reward function \( g \) to be equal with the indicator function of \( A \), i.e. \( g := 1_A \) and the reward process is \( g := 1_A(x_{\tau}), \tau \geq 0 \). We make use now of (2) to express the reach set probabilities. To relate these with the concepts discussed above, we need few additional results related to the concept of essential supremum of an arbitrary family of random variables.

Theorem 1. Let \( (Y_i)_{i \in I} \) be a family of real valued random variables (with a possibly uncountable index set \( I \)). There exists a random variable \( \overline{Y} \) with values in \( \mathbb{R} \), which is unique up to null events, such that: (i) For all \( i \in I \), \( Y_i \leq \overline{Y} \) a.s.; (ii) If \( Y \) is a random variable with values in \( \mathbb{R} \) satisfying \( Y_i \leq Y \) a.s., for all \( i \in I \), then \( \overline{Y} \leq Y \) a.s. Moreover, there is a countable subset \( J \subset I \) such that \( \overline{Y} = \sup_{i \in J} Y_i \).

The random variable \( \overline{Y} \) is called the essential upper bound (or essential supremum) of the family \( (Y_i)_{i \in I} \) and is denoted by \( \text{ess} \sup_{i \in J} Y_i \).
Proposition 2. Let \((Y_i)_{i \in I}\) be a family of nonnegative valued random variables with the lattice property, i.e. for all indices \(i, j \in I\) there exists an index \(k \in I\) such that \(Y_k \geq Y_i \vee Y_j\) a.s. The following assertions hold: (i) There exists a sequence \((\varepsilon_n)_{n \in \mathbb{N}}\) of indices such that \((Y_n)_{n \in \mathbb{N}}\) is nondecreasing (up to null events), and 
\[
esup_{i \in I} Y_i = \sup_{n \in \mathbb{N}} Y_n = \lim_{n \to \infty} Y_n, \text{ a.s.}
\] (ii) We have:
\[
E(\esssup_{i \in I} Y_i) = \esssup_{i \in I} E Y_i.
\]

Now we return in the framework of SR. Let us make the following notations: \(Y_T := \max_{x \in [0,T]} I(A_k(x)), Y := \max_{x \geq 0} I(A_k(x))\). Due to the measurability of the reach events \(Y_T\) and \(Y\) are indeed random variables. It is clear that they satisfy the conditions of the Th.1 for the families of random variables \((y_k)_{k \in [0,T]}\) and \((y_k)_{k \geq 0}\) where \(y_k\) is the reward process. Therefore, they are nothing else, but the essential supremum of these families, i.e. \(Y_T = \esssup_{k \in [0,T]} I(A_k(x)), Y = \esssup_{k \geq 0} I(A_k(x))\). We can apply the Prop.2, and obtain the following result of major importance for SR.

Theorem 3. Let \(A \in \mathcal{B}(X)\) and \(x \in X\) an initial state. If the families of random variables \((y(x))_{x \in [0,T]}, (y(x))_{x \geq 0}\) satisfy the lattice property, then the reach set probabilities can be expressed as follows:
\[
\begin{align*}
P_x[\text{Reach}_T(A)] &= \sup_{t \in [0,T]} E_x[I(A(x_t))], \\
P_x[\text{Reach}_\infty(A)] &= \sup_{t \geq 0} E_x[I(A(x_t))].
\end{align*}
\]

The lattice property for these families is not trivially satisfied due to the fact that the trajectories of the systems may have jumps, and the set \(A\) might have “holes” (i.e. \(A\) is not convex). One can look for the right hypotheses that will ensure the lattice property of these families, but it is not our goal here. However, using the argument of Mertens [1972], we can “randomize” (8), in order to prove that
\[
\begin{align*}
P_x[\text{Reach}_T(A)] &= \sup_{\tau \in \mathcal{F}_x} E_x[I(A(x_\tau))], \\
P_x[\text{Reach}_\infty(A)] &= \sup_{\tau \geq 0} E_x[I(A(x_\tau))].
\end{align*}
\]

These relations show that the reach set probabilities can be thought as value functions for some specific optimal stopping problems (with finite/infinite horizon time), when the reward is the indicator function of the target set.

5. COMPUTATION OF THE REDUITE

In this section, we present some algorithms used in probabilistic potential theory for the computation of the réduite. We start with the simple case of Markov chain, then the technique is extended to more complex Markov processes.

The Case of Markov Chains

Let \(\{x_n\}_{n = 0, 1, 2, ...}\) be a discrete-parameter Markov chain defined on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) with values in \(X\), and let its natural filtration be \(\mathcal{F}_n = \sigma(x_k | k \leq n)\). A stochastic kernel \(k(x, \Gamma)\) is a transition function for a time-homogeneous Markov chain \((x_n)\), if \(\mathbb{P}(x_{n+1} \in \Gamma | \mathcal{F}_n) = k(x_n, \Gamma)\). The potential kernel associated to \(k\) (or to the Markov chain) is defined by \(G := \sum_{n=0}^{\infty} k^n (k^0 = I)\). For Markov chains, the definition of excessive function is simplified. A measurable positive function \(f\) is called excessive w.r.t. \(k\) if \(f \geq k f\); where \(k f(x) := \int f(y) k(x, dy)\). The reduced function or réduite of a bounded measurable function \(f\) is defined by (5).

Proposition 4. (Réduite Algorithm). Bouleau et al [1994]

Let \(g\) be a measurable and bounded real valued function on \(X\). Let \(\{g_n\}\) be the sequence defined recurrently as follows: \(g_0 = g^*; g_{n+1} = g_n \wedge k u_n\). Then the sequence \(\{g_n\}\) is increasing and converges to \(R g, \text{ i.e. } g_n \nearrow R g\).

It can be proved that \(R g\) is measurable and \(R g = g \wedge R g\). If \(x \in [g < R g] \cup [g \leq 0]\), then \(Rg(x) = k(Rg)(x)\). Moreover, the réduite operator \(R\) is nonlinear, increasing (\(g \leq v \Rightarrow Rg \leq Rv\)), subadditive (\(R(g+v) \leq Rg + Rv\)), positively homogeneous (\(R(tu) = t(Rg)\), \(t \geq 0\)), and continuous on ascending sequences (\(g_n \nearrow g \Rightarrow Rg_n \nearrow Rg\)).

The Case of a Markov Process

Let us consider a Markov process \(M = (x_t, P_x)\) with the operator semigroup \(\mathcal{P} = (P_t)_{t \geq 0}\) and with excessive function cone \(\mathcal{E}_M\). For \(t > 0\) and \(g\) a positive measurable bounded function, we write \(R(t)g\) for the réduite of \(g\) w.r.t. the kernel \(P_t\).

Using the extended generator \(L\) of the semigroup \(\mathcal{P}\), we may approximate the réduite of a function \(g\) that belongs to the domain of the extended generator \(D(L)\). The definition of domain of the extended generator can be found in the literature in different sources [see, e.g. Davis [1993]]. The most known is the one based on martingale property: \(g \in D(L)\) if there exists a Borel measurable function \(h\) such that \(V_n[h] < \infty\) and \(g = V_n(\alpha g - h)\), \(\alpha > 0\). The function \(h\) is unique up to changes on a set of potential zero (\(A \in \mathcal{B}\) is of potential zero if \(V_n 1_A = 0, \alpha > 0\)). Explicitly, we have: \(h = \lim_{n \to \infty} n(nV_n g - g)\).

Proposition 5. Bouleau et al [1994]

Let \(g \in D(L)\) with \(L g\) bounded and \(t \geq 0\). Then: (i) \(Rg = \lim_{n \to \infty} \uparrow R(t/2^n)g\); (ii) \(Rg - R(t)g \leq |t| \|Lg\|\).

The last result can be very useful when the expression of the operator resolvent of SR is known, like in the case of diffusion processes.

6. COMPUTATION OF SR PROBABILITIES

Now we return to the framework of SHS. The executions of such systems are complex Markov models that allow predictable jumps. The existence of these predictable jumps involves a peculiarity of the extended generator. This is reflected in (4) that should be satisfied by all functions belonging to the domain of this generator. It is clear that the indicator functions of some measurable sets do not belong to this domain. Therefore, it is clear that for the approximation of the ëduite corresponding to such indicator functions can not follow the scheme proposed in the Prop.5. Then, the natural idea is to approximate the whole process by Markov chains, and to use convergence results available for the réduite. For stochastic hybrid processes, we will use approximations with Poisson time stepping rather than approximations with fixed time stepping (Euler-Maruyama). The reason is that the discrete jumps can be captured in a better way when the sampling time is given in a Poisson fashion.
Let $M = (x_t, P_x)$ the Markov hybrid process for an SHS, $H$. The construction of an approximation sequence of jump processes for $M$ needs two ingredients: (1) A sequence of homogeneous Markov chains $(\alpha^n)$; (2) A sequence of Poisson processes $(\theta^n)$. Each $\alpha^n = (\alpha^n_k)_{k=0,1,2,\ldots}$ is a Markov chain with some initial distribution $\nu$ and the transition probability function, $K_n$ defined by $K_n(x,dy) := nV_n(x,dy)$, where $V_n$ is computed from formula (3). Each $\theta^n = (\theta^n_k)_{k\geq 0}$ is a Poisson process with the parameter $n$, independent of $\alpha^n$.

Using these ingredients, we then define, for each $n \geq 1$, a continuous-time Markov jump process $\rho^n_t := \alpha^n_{\theta^n_t}$, $t \geq 0$.

This means that the jump times of the process $(\rho^n_t)$ are given by the arrival times of the Poisson process $(\theta^n_t)$ and its values between jumps are provided by the Markov chain $(\alpha^n_k)$. The infinitesimal generator of $(\rho^n_t)$ can be expressed as follows: $L^n u(x) = n \int_{K_n} [u(y) - u(x)] nV_n(x,dy)$, where $X_n = X \cup \{\Delta\}$ (where $\Delta$ is thought of as a deadlock point for $M$). It can be shown that $(\rho^n_t)$ converges to $(x_t)$ w.r.t. the Skorokhod topology.

The following convergence result for the reach set probabilities does not depend on the characterization of these measures as appropriate réduit/value functions.

**Theorem 6.** Bujorianu [2009] If the sequence $(\rho^n_t)_{n \geq 1}$ of strong Markov processes converges weakly to $(x_t)$ (under $P_x$) as $n \to \infty$, then $P_x[\text{Reach}_{n}(A)] \to P_x[\text{Reach}_{\infty}(A)]$ for any $A \in B(X)$.

Note that in most of the references regarding the OSP, many times such convergence results require additional hypotheses that might not be satisfied by stochastic hybrid processes. To circumvent this difficulty we use the previous result whose proof has not been based on the characterization of SR as an OSP (for more details, see Bujorianu [2009]). The main result of this section is the following:

**Theorem 7.** For $A \in B(X)$, the reach set probabilities can be approximated as follows: $P_x[\text{Reach}_{\infty}(A)] = \lim_{n \to \infty} P_x[\text{Reach}_{n}(A)]$; where $R_{(x_t)}$ is the réduit operator corresponding to the jump kernel $K_n$.

Using the Reduite Algorithm, the réduit $R_{(x_t)}$ can be computed as follows: $R_{(x_t)} = \lim_{n \to \infty} I_{\{g_n\}}$, where $g_0 := I_A$; $g_{m+1} := g_m \land K_n g_m$. Remark that, in using this algorithm, it is essential to know the expression of the stochastic kernel $K_n$. It can be proved that $K_n := nV_n = n(nI - L)^{-1}$, $n \geq 1$, where $I$ is the identity operator Davis [1993]. Moreover, $V_n$ is the potential kernel of the process $M$ killed with the exponential rate $n$. Therefore, further approximations of the generator $L$ can be employed to obtain evaluations of the reach set probabilities with different degrees of accuracy.

7. CONCLUSIONS

In this paper, we develop new computational methods for SR for Markov models using the characterization of this problem as an OSP. In particular, for SHS, because of the fact that their realizations cover only a particular subset of the class of (right) Markov processes, finding computational methods for reach probabilities has proved to be a difficult and challenging task. In the SHS framework, analytical methods to compute reach set probabilities viewed as the value functions for some appropriate optimal stopping problems are not easy to handle with due to the fact that such processes have discontinuities governed by some guards. Our approach is purely probabilistic. We use that characterization of the value function of an OSP as the réduit of the reward function. The réduit operator belongs to the probabilistic potential theory arsenal associated to Markov processes. For Markov chains, there exist nice approximations algorithms for the réduit operator. Extensions of these algorithms to more complex Markov models are also available.

The behaviour of a complex SHS can be approximated by simpler Markov processes, using the infinitesimal generator of the underlying stochastic hybrid process. For these simpler Markov processes, the réduit approximations go quite smoothly. At the end, the computation of the reach set probabilities for SHS depends very much on our ability to discretize the generator associated to an SHS.

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