On Computing Quotient Decentralized Fixed Modes

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Abstract: In the decentralized control of linear time-invariant (LTI) systems, the notion of decentralized fixed modes (DFMs) was introduced to capture those modes which are immovable using LTI controllers. However, some DFMs can be moved using more complicated controllers. Recently a so-called “quotient system” associated with the original plant was investigated and it was proven that its DFMs, which we label quotient DFMs (QDFMs), are exactly those DFMs of the original system which cannot be moved using any form of nonlinear time-varying (NLTV) controller. The algorithm provided there to compute the QDFMs requires two steps: first, a partitioning of the sub-systems, using graph theory, and then application of standard tools from decentralized control. However, the first step is fragile, so the goal here is to provide a more robust approach to computing the QDFMs.

Keywords: decentralized control, decentralized fixed modes, quotient decentralized fixed modes, nonlinear control.

1. INTRODUCTION

It is well known that a linear time invariant (LTI) system can be stabilized using decentralized LTI control if and only if the system does not possess any unstable decentralized fixed modes (DFM) (see Davison and Wang (1973)). This paper also considers the case of more general information flow constraints other than decentralized, and in Davison and Chang (1990) the above results are extended to the case of general proper systems.

The use of time-varying controllers in the control of decentralized systems was explored starting in the early 1980’s. It was demonstrated in Anderson and Moore (1981) that some decentralized fixed modes can be moved using time-varying feedback. This lead to other investigations of the use of time-varying control in this context, including that of Wung (1982) and Willems (1989) in continuous-time and Khargonekar and Ozuguler (1994) in discrete-time. It also lead to an attempt to classify DFMs into those which are truly fixed and those which can be moved using a sufficiently sophisticated controller.

In 1985 Ozguner and Davison (1985) considered the class of systems with distinct eigenvalues, none of which are at zero, and introduced the notion of structured DFMs (SDFMs), which are those modes, if any, of a decentralized control system that continue to be DFMs after perturbing the nonzero elements; the DFMs which do not have this property are said to be unstructured DFMs (USDFMs). Indeed, it is shown that for the class of systems under consideration, for almost all sampling periods the associated discretized plant has the same number of structured DFMs as the original continuous-time system, but has no unstructured DFMs. Hence, if there are no structured unstable fixed modes, then a sampled-data controller consisting of a zero-order-hold, a discrete-time LTI compensator, and a standard sampler can be used to stabilize the system. It is interesting to observe that the synthesis approach may not work for systems which lie outside the class under consideration - see Example 2.

In a quest to remove the simplifying assumption of Ozguner and Davison (1985) and to see if nonlinear time-varying controllers offer advantages over linear time-varying controllers, Gong and Aldeen (1997) consider a “quotient system” associated with the nominal plant together with its DFMs, which we label “quotient decentralized fixed modes” (QDFMs). 1 This quotient system was introduced by Corfmat and Morse (1976) and subsequently investigated by Kobayashi and Yoshikawa (1982); to construct it one uses some elementary results from graph theory to partition the system into a set of strongly connected sub-systems. Gong and Aldeen (1997) proves that the QDFMs of the system are exactly those DFMs which are immovable using any form of NLTV control. Last of all, it turns out that, for the class of systems considered in Ozguner and Davison (1985), the notion of QDFM and SDFM are identical.

The main goal topic of this paper is to develop a new test to determine the QDFMs of a system. The proposed test is

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1 The authors of Gong and Aldeen (1997) label them “quotient fixed modes”.

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an algebraic test which has the advantage that it does not require any graphical structure decision making, such as in determining if a system is strongly connected or not, which is always difficult to determine due to the approximate nature of system models and the presence of numerical error. The proposed QDFM test for a given plant is simply a standard DFM determination problem of an associated system, and directly permits one to determine the presence or non-presence of a QDFM in a system. It has the additional advantage that it provides a natural means to explore the “approximate decentralized fixed modes” (ADFMs) as well (see Vaz and Davison (1989)). Last of all, most proofs are omitted for space considerations.

2. MATHEMATICAL PRELIMINARIES OF DECENTRALIZED CONTROL

In this paper we will be dealing with a variety of decentralized control systems, some strictly proper and some not, so we shall present common nomenclature and preliminary results here.

We start with a general decentralized model given by

\[
\begin{align*}
\dot{x} &= Ax + \sum_{i=1}^{p} B_i u_i \\
y_i &= C_i x + \sum_{j=1}^{p} D_{ij} u_j, \quad i = 1, \ldots, p
\end{align*}
\]

(1)

with \(x(t) \in \mathbb{R}^n\) while \(u_i(t) \in \mathbb{R}^{m_i}\) and \(y_i(t) \in \mathbb{R}^{r_i}\) for \(i = 1, \ldots, p\); we set \(m = \sum_{i=1}^{p} m_i\) and \(r = \sum_{i=1}^{p} r_i\); we represent this model by

\((A; B_1, \ldots, B_p; C_1, \ldots, C_p; D_{11}, \ldots, D_{1p}, \ldots, D_{p1}, \ldots, D_{pp})\).

Associated with this model are

\[
B := [B_1 \cdots B_p], C := \begin{bmatrix} C_1 \\ \vdots \\ C_p \end{bmatrix}, D := \begin{bmatrix} D_{11} & \cdots & D_{1p} \\ \vdots & \ddots & \vdots \\ D_{p1} & \cdots & D_{pp} \end{bmatrix}
\]

In the decentralized context we allow \(u_i\) to depend solely on \(y_i\), whether the controller is linear time-invariant (LTI), linear time-varying (LTV), or nonlinear time-varying (NLTV).

The centralized fixed modes of (1), labelled

\(CFM(A, B, C, D)\),

are those eigenvalues which are immovable using LTI feedback (indeed, they are the modes which are either not controllable or not observable). It is well known that they are immovable by LTI feedback iff they are immovable by static output feedback; indeed, if we set

\[
K_{cen}(A, B, C, D) := \{K \in \mathbb{R}^{mxr} : \det(I - DK) \neq 0\},
\]

or \(K_{cen}\) for short, then it is well known that

\[
CFM(A, B, C, D) = \bigcap_{K \in K_{cen}} \text{sp}((A + BK(1 - DK)^{-1}C)).
\]

The notion of a decentralized fixed mode was introduced in Davison and Wang (1973); the goal is to capture which eigenvalues are immovable using LTI feedback which respects the information flow constraints. To this end, we define the set of feedback gains

\[
K_{dec}(A; B_1, \ldots, B_p; C_1, \ldots, C_p; D_{11}, \ldots, D_{1p}, \ldots, D_{p1}, \ldots, D_{pp}) := \{K \in \mathbb{R}^{m \times r} : K = \text{diag}(K_1, \ldots, K_p)\}
\]

with \(K_i \in \mathbb{R}^{m_i \times r_i}\) satisfying \(\det(I - DK) \neq 0\), or simply \(K_{dec}\) for short.

Definition 1. The decentralized fixed modes of \((A; B_1, \ldots, B_p; C_1, \ldots, C_p; D_{11}, \ldots, D_{1p}, \ldots, D_{p1}, \ldots, D_{pp})\) are given by

\[
\bigcap_{K \in K_{dec}} \text{sp}((A + BK(1 - DK)^{-1}C)) .
\]

So the decentralized fixed modes are those eigenvalues of \(A\) which are immovable using static output feedback which respects the information flow constraints. It is well known that these DFMs are also immovable using LTI dynamic controllers - see Davison and Wang (1973) and Davison and Chang (1990). However, for some systems, some of the DFMs are moveable using more complicated control laws, e.g. see Anderson and Moore (1981), Ozguner and Davison (1985), Willems (1989) and Khargonekar and Ozguler (1994).

The following is a convenient characterization of DFMs.

Proposition 1. (Davison and Chang (1990)) \(\lambda \in \text{sp}(A)\) is a DFM of \((A; B_1, \ldots, B_p; C_1, \ldots, C_p; D_{11}, \ldots, D_{1p}, \ldots, D_{p1}, \ldots, D_{pp})\) if one of the following conditions hold:

(i) The eigenvalue at \(\lambda\) is not centrally controllable:

\[
\text{rk} \left[ A - MB \right] < n .
\]

(ii) The eigenvalue at \(\lambda\) is not centrally observable:

\[
\text{rk} \left[ A - MC \right] < n .
\]

(iii) There exists a partition of the set of indices \(\{1, \ldots, p\}\) into nonempty sets \(S_1 = \{i_1, \ldots, i_q\}\) and \(S_2 = \{i_{q+1}, \ldots, i_p\}\) satisfying

\[
\begin{bmatrix}
A - \lambda I & B_{i_1} & \cdots & B_{i_q} \\
C_{i_{q+1}} & D_{i_{q+1}i_{q+1}} & \cdots & D_{i_{q+1}i_q} \\
\vdots & \vdots & \ddots & \vdots \\
C_{i_p} & D_{i_pi_{q+1}} & \cdots & D_{i_pi_q}
\end{bmatrix} < n .
\]

Following Corfmat and Morse (1976), it turns out that graph theory can be used to study decentralized systems. One can envision building a directed graph of the general system (1) as follows: there are \(p\) nodes representing the \(p\) control agents and \(p\) sensor agents, with an arc from node \(i\) to node \(j\) if \(D_{ij} + C_j(sI - A)^{-1}B_i \neq 0\). A directed graph is said to be strongly connected if there is a path from every node to every other node along the arcs. Associated with this graph is the topology matrix \(T \in \mathbb{R}^{p \times p}\) defined as follows:

\[
T_{ij} := \begin{cases} 0 & \text{if } C_i(sI - A)^{-1}B_j + D_{ij} = 0 \\ 1 & \text{otherwise.} \end{cases}
\]

It contains sufficient information to construct the graph of the system, and will be very useful later on in the paper.
Proposition 2. (Anderson and Moore (1981)) The directed graph corresponding to 
\((A; B_1, \ldots, B_p; C_1, \ldots, C_p; D_{11}, \ldots, D_{ip}, \ldots, D_{pp})\) is strongly connected \(\iff\) for every partition of the set of indices \(\{1, \ldots, p\}\) into nonempty sets \(S_1 = \{i_1, \ldots, i_q\}\) and \(S_2 = \{i_{q+1}, \ldots, i_p\}\), it satisfies 
\[
\begin{bmatrix}
    D_{i_q+1i} & \cdots & D_{i_q+i_q} \\
    \vdots & \ddots & \vdots \\
    D_{ip_i} & \cdots & D_{ip_{i_q}} \\
\end{bmatrix} + \begin{bmatrix}
    C_{i_q+1} \\
    \vdots \\
    C_{ip_{i_q}} \\
\end{bmatrix} (sI - A)^{-1} \begin{bmatrix}
    B_{i_1} \\
    \vdots \\
    B_{i_q} \\
\end{bmatrix} \neq 0.
\]

Remark 1. If \(p = 2\), then the associated graph is strongly connected \(\iff\) both \(D_{12} + C_1(sI - A)^{-1}B_2\) and \(D_{21} + C_2(sI - A)^{-1}B_1\) are non-zero.

### 3. THE PROBLEM

Here we consider the strictly proper plant 
\[
\dot{x} = Ax + \sum_{i=1}^{p} B_i u_i 
\]
with \((x(t) \in \mathbb{R}^n)\) while \((u_i(t) \in \mathbb{R}^{m_i})\) and \((y_i(t) \in \mathbb{R}^{r_i})\) for \(i = 1, \ldots, p\); we set \(m = \sum_{i=1}^{p} m_i\) and \(r = \sum_{i=1}^{p} r_i\). We label this system by \(S_0 = (A; B_1, \ldots, B_p; C_1, \ldots, C_p; 0; 0, 0)\) and the corresponding topology matrix by \(T_0\). Here we wish to examine which of its DFMs are moveable using NLTV feedback.

Before proceeding we wish to make precise the notion of a QDFM. Following Gong and Aldeen (1997) we use the approach of Corfmat and Morse (1976), which we summarize here, to construct a quotient system corresponding to (5). It is proven there, using basic results in graph theory, that the graph associated with the decentralized control system can be decomposed uniquely into a number of strongly connected sub-systems which can then be ordered in a very convenient way. The end result is a partition of the input/output channels into \(p(\leq p)\) strongly connected sub-systems, with the \(i^{th}\) sub-system containing \(k_i\) input/output channels. By properly re-labelling the input/output channels and partitioning them to correspond with the strongly connected sub-systems, we end up with a topology matrix in upper block diagonal form:
\[
\begin{bmatrix}
    T_{11} & T_{12} & \cdots & T_{1p} \\
    0 & T_{22} & \cdots & T_{2p} \\
    \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & \cdots & T_{pp} \\
\end{bmatrix}
\]
with \(T_{ii} \in \mathbb{R}^{k_i \times k_i}\) and \(\sum_{i=1}^{p} k_i = p\). Since the DFMs of (5) are independent of the ordering (or labelling) of the index, we assume that this re-ordering was in effect at the outset, so that the above matrix is exactly \(T_0\).

At this point it is convenient to introduce some new notation: set \(k_0 := 0\) and define \(k_i := k_1 + \cdots + k_i, \quad i = 1, \ldots, p\).

We then define
\[
\begin{align*}
B_i & := [B_{k_i+1}, \ldots, B_{k_i}], \\
C_i & := [C_{k_i+1}, \ldots, C_{k_i}], \\
\hat{u}_i & := [u_{k_i+1}, \ldots, u_{k_i}], \\
\hat{y}_i & := [y_{k_i+1}, \ldots, y_{k_i}]
\end{align*}
\]
for \(i = 1, \ldots, p\). Correspondingly to \(T_0\), the so-called quotient system is given by
\[
\dot{x} = Ax + \sum_{i=1}^{p} B_i \hat{u}_i,
\]
\[
\hat{y}_i = \hat{C}_i x, \quad i = 1, \ldots, p.
\]
Here the sub-system associated with the first \(q_1\) inputs/outputs is strongly connected; the sub-system associated with the second \(q_2\) inputs/outputs is strongly connected, and so on, with no path from sub-system \(i\) to sub-systems \(i + 1, \ldots, p\), as captured by the upper block triangular nature of \(T_0\).

### Definition 2. (Gong and Aldeen (1997)) The quotient decentralized fixed modes (QDFMs) of (5) are 
\[
DFM(A; \hat{B}_1, \ldots, \hat{B}_p; \hat{C}_1, \ldots, \hat{C}_p; 0).
\]

So the approach of Gong and Aldeen (1997) to finding the QDFMs is to first use a graphical method to identify the strongly connected sub-systems, yielding the quotient system, followed by a standard algorithm to find the DFMs of this quotient system. The drawback of the approach is that the step of partitioning the system into strongly connected sub-systems is fragile. The goal of this paper is to construct an algorithm to compute the QDFMs in a more direct fashion which avoids the binary decision making process which is part of graph theory; we end up with an algorithm which looks similar to the classical approach to finding DFMs, which has the added advantage that it can be used for analysing ADFMs as well.

Before continuing, we would like to define three decentralized control systems which are associated with the original plant. With
\[
B_i := [B_1, \ldots, B_p; C_1, \ldots, C_p; 0, 0, \ldots, 0],
\]
the first model is obtained by replacing each \(B_i\) by its corresponding controllability matrix, namely
\[
S_{con} = (A; B_1, \ldots, B_p; C_1, \ldots, C_p; 0, 0, \ldots, 0).
\]

With
\[
C_i := \begin{bmatrix}
    C_1 \\
    \vdots \\
    C_{k_i}
\end{bmatrix},
\]
the next model is obtained by replacing each \(C_i\) by its corresponding observability matrix, namely
\[
S_{obs} = (A; B_1, \ldots, B_p; C_1, \ldots, C_p; 0, 0, \ldots, 0).
\]
The last model requires a replacement of \(D_{ij} = 0\) by an associated matrix of Markov parameters, with corresponding padding of the \(B_i\)’s and \(C_i\’s\) with zero: define
\[
\hat{B}_i := [B_1, 0, 0, \ldots, 0], \quad \hat{B} := [B_1, \ldots, B_p], \quad \hat{C}_i := \begin{bmatrix}
    C_1 \\
    \vdots \\
    0
\end{bmatrix}.
\]
\[ \bar{C} := \begin{bmatrix} C_1 \\ \vdots \\ C_p \end{bmatrix}, \quad \bar{D}_{ij} = \begin{bmatrix} 0 \\ C_iB_j \\ \vdots \\ C_iA^{n-1}B_j \end{bmatrix}, \quad \bar{C} := \begin{bmatrix} \bar{C}_1 \\ \vdots \\ \bar{C}_p \end{bmatrix}, \quad \bar{D}_{ij} = \begin{bmatrix} 0 \\ C_iB_j \\ \vdots \\ C_iA^{n-1}B_j \end{bmatrix}, \]

and define \( S_{mar} \) by
\[ (A; B_1, \ldots, B_p; C_1, \ldots, C_p; \bar{D}_{11}, \ldots, \bar{D}_{1p}, \ldots, \bar{D}_{p1}, \ldots, \bar{D}_{pp}). \]

Associated with these systems are their related topology matrices (which lie in \( \mathbb{R}^{p \times p} \)). We adopt the natural notation of \( T_{con}, T_{obs} \) and \( T_{mar} \).

**Proposition 3.** \( T_0 = T_{con} = T_{obs} = T_{mar} \).

4. THE STRONGLY CONNECTED CASE

We start with the simplest case, namely when the graph associated with the plant is strongly connected.

**Theorem 1.** (Gong and Aldeen (1997)) If \( S_0 \) is strongly connected then the QDFMs of \( S_0 \) equals \( CFM(A, B, C, 0) \).

At this point we can present a result linking the QDFMs of \( S_0 \) with the DFMs of \( S_{con}, S_{obs} \) and \( S_{mar} \). While this has little direct utility, its real use is in proving a comparable result in the non-strongly connected case.

**Theorem 2.** Suppose that \( S_0 \) is strongly connected.
(i) The QDFMs of \( S_0 \) are the DFMs of \( S_{mar} \).
(ii) If every eigenvalue of \( A \) has only one Jordan block, then the QDFMs of \( S_0 \) are the DFMs of \( S_{con} \).
(iii) If every eigenvalue of \( A \) has only one Jordan block, then the QDFMs of \( S_0 \) are the DFMs of \( S_{obs} \).

To prove the above result we need the following:

**Proposition 4.** If \( S_0 \) is strongly connected, then so are \( S_{con}, S_{obs} \) and \( S_{mar} \).

**Proof of Theorem 2:**

(\( \triangleright \)) Let \( \lambda \) be a QDFM of (5); by Theorem 1, this means that \( \lambda \in CFM(A, B, C, 0) \). Then either (2) or (3) holds.

If (2) holds, then it follows immediately from Proposition 1 and the definition of \( S_{obs} \) that \( A \) is a DFM of \( S_{obs} \). Since it is clear that
\[ \text{rk} \left[ A - \lambda I \quad B \right] = \text{rk} \left[ A - \lambda I \quad B \right] < n, \]

it follows immediately from Proposition 1 and the definition of \( S_{mar} \) that \( \lambda \) is a DFM of \( S_{mar} \). Last of all, with a little work it follows that
\[ \text{rk} \left[ A - \lambda I \quad B \right] = \text{rk} \left[ A - \lambda I \quad B \right] < n, \]

so it follows immediately from Proposition 1 and the definition of \( S_{con} \) that \( \lambda \) is a DFM of \( S_{con} \).

Now suppose that (3) holds. Using analogous arguments to the above, we can conclude that \( \lambda \) is a DFM of \( S_{con}, S_{obs} \) and \( S_{mar} \).

(\( \triangleright \)) Now we prove the reverse inclusion.

(ii): Suppose that \( A \) has one Jordan block per eigenvalue and that \( \lambda \) is a DFM of \( S_{con} \). By Proposition 1 it follows that one of three things must be true:

(a) \( \text{rk} \left[ A - \lambda I \quad B_1 \cdots B_p \right] < n \).
(b) \( \lambda \) is not centrally observable; (3) holds.
(c) There exists a partition of the set of indices \( \{1, \ldots, p\} \) into non-empty sets \( S_1 = \{i_1, \ldots, i_q\} \) and \( S_2 = \{i_{q+1}, \ldots, i_p\} \) satisfying
\[ \text{rk} \left[ \begin{array}{cccc} A - \lambda I & B_1 & \cdots & B_q \\ C_{i_1 + 1} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ C_{i_p} & 0 & \cdots & 0 \end{array} \right] < n. \] (7)

If (a) holds, then it follows from the fact that the columns of \( B_i \) are contained in the columns of \( B_i \) that (2) holds, so \( \lambda \) is a QDFM of \( S_0 \) by Theorem 1.

If (b) holds, then it follows immediately that \( \lambda \in DFM(A, B, C, 0) \). By Theorem 1, \( \lambda \) is a QDFM of \( S_0 \).

Now suppose that (c) holds, and define
\[ \bar{C} := \begin{bmatrix} C_{i_1 + 1} \\ \vdots \\ C_{i_p} \end{bmatrix}, \quad \bar{B} := [B_{i_1} \cdots B_{i_p}]. \]

Now choose \( T_1 \) full column rank so that \( Im(T_1) = Im(\bar{B}) \), and then choose \( T_2 \) of full column rank so that \( T := [T_1 \ T_2] \in \mathbb{R}^{n \times n} \) is invertible. If we now carry out a similarity transformation on the system we end up with
\[ T^{-1}AT = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \quad \bar{T}^{-1}\bar{B} = \begin{bmatrix} \bar{B}_{11} \\ 0 \end{bmatrix}, \]

\[ \bar{C}T = [\bar{C}_{11} \ \bar{C}_{12}] \]

with \( A_{11}, \bar{B}_{11} \) being controllable, and, more importantly, with \( \bar{B}_{11} \) having full row rank. It follows that

\[ \text{rk} \left[ A - \lambda I \quad \bar{B} \right] = \text{rk} \left[ \begin{array}{ccc} A_{11} - \lambda I & A_{12} & \bar{B}_{11} \\ 0 & A_{22} - \lambda I & 0 \\ C_{11} & C_{12} & 0 \end{array} \right] \]

\[ = \text{rk} \left[ \begin{array}{ccc} 0 & 0 & \bar{B}_{11} \\ 0 & A_{22} - \lambda I & 0 \\ \bar{C}_{11} & \bar{C}_{12} & 0 \end{array} \right] \]

\[ \geq \text{rk}[\bar{C}_{11}] + \text{rk}[A_{22} - \lambda I] + \text{rk}[\bar{B}_{11}] \]

\[ = n - 1 + \text{rk}[\bar{C}_{11}] \]

Now \( \bar{C}(sI - \bar{A})^{-1}\bar{B} \) is non-zero by Proposition 2 and the fact that \( S_0 \) is strongly connected; since
\[ \bar{C}(sI - \bar{A})^{-1}\bar{B} = \bar{C}_{11}(sI - A_{11})^{-1}\bar{B}_{11} \]
as well, we conclude that \( \bar{C}_{11} \neq 0 \). Hence, \( \text{rk}[\bar{C}_{11}] \geq 1 \), so we conclude that (7) never holds, so this case is impossible.

(iii) This case is similar to part (ii).

(i) This proof requires a detailed examination of \( \bar{D}_{ij} \), but it must be omitted due to space constraints. □
5. THE NON-STRONGLY CONNECTED CASE

In order to handle the non-strongly connected case, we first find a special state-space representation of the quotient system shown to exist by Corfmat and Morse (1976). Adopting the geometric notation used there, let \( X_1 \) denote the controllable sub-space associated with the first input of the quotient system:

\[
X_1 := \langle A|B_1 \rangle.
\]

Next, choose \( X_2 \) to be independent from \( X_1 \) so that \( X_1 + X_2 \) denotes the controllable sub-space associated with the first two inputs of the quotient system:

\[
X_1 + X_2 := \langle A|B_1 \rangle + \langle A|B_2 \rangle.
\]

More generally, given \( i \in \{2, ..., \tilde{p} - 2\} \) and \( X_1, ..., X_i \) which are independent and span \( \langle A|B_1 \rangle + \cdots + \langle A|B_i \rangle \), we choose \( X_{i+1} \) so that it is independent of \( X_1, ..., X_i \) and satisfy

\[
X_1 + \cdots + X_{i+1} := \langle A|B_1 \rangle + \cdots + \langle A|B_{i+1} \rangle.
\]

Last of all, we choose \( X_{\tilde{p}} \) to be independent of \( X_1 + \cdots + X_{\tilde{p}-1} \) and sum to \( \mathbb{R}^n \). Of course, some of the \( X_i \)'s could be zero (and therefore of dimension zero).

Now we would like to form a state space representation consistent with the above partitioning of the state space. Using standard facts about controllability, it is easy to see that the state-space representation is of the form:

\[
\begin{pmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\vdots \\
\dot{x}_{\tilde{p}}
\end{pmatrix} =
\begin{pmatrix}
A_{11} & A_{12} & \cdots & A_{1p} \\
0 & A_{22} & \cdots & A_{2p} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & A_{pp}
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
\vdots \\
x_p
\end{pmatrix} +
\begin{pmatrix}
B_{11} & B_{12} & \cdots & B_{1p} \\
0 & B_{22} & \cdots & B_{2p} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & B_{pp}
\end{pmatrix}
\begin{pmatrix}
u_1 \\
u_2 \\
\vdots \\
u_p
\end{pmatrix},
\tag{8}
\end{align}
\]

with \((A_{11}, B_{11}), (A_{22}, B_{22}), ..., (A_{pp}, B_{pp})\), etc., all controllable, which can be combined with the structure of the topology matrix to show that \( C \) is also upper block triangular:

\[
\begin{pmatrix}
y_1 \\
y_2 \\
\vdots \\
y_p
\end{pmatrix} =
\begin{pmatrix}
C_{11} & C_{12} & \cdots & C_{1p} \\
0 & C_{22} & \cdots & C_{2p} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & C_{pp}
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
\vdots \\
x_p
\end{pmatrix}.
\tag{9}
\]

Since the QDFMs and DFMs of (5) are independent of the choice of state-space representation (with the dimension fixed), **without loss of generality** we assume at the outset that (5) has the form (8)-(9).

Gong and Aldeen (1997) present their result for the general case in terms of the plant (5) with the input/output indexing and the state-space representation chosen so that it is as in (8)-(9). In this representation the information flow is clear - it flows upwards from one strongly connected component to the next, but it does not flow back down. Hence, when it comes to stability all we need to do is consider \( p \) separate sub-systems. This leads to

**Theorem 3.** (Gong and Aldeen, 1997) The QDFMs of \( S_0 \) equals \( \bigcup_{i=1}^{\tilde{p}} CFM(A_{ii}, B_{si}, C_{ii}, 0) \).

It turns out that our earlier result (Theorem 2) holds with the strongly connected assumption removed.

**Theorem 4.** In all cases:

(i) The QDFMs of \( S_0 \) are the DFMs of \( S_{\text{mar}} \).

(ii) If every eigenvalue of \( A \) has only one Jordan block, then the QDFMs of \( S_0 \) equals the DFMs of \( S_{\text{con}} \).

(iii) If every eigenvalue of \( A \) has only one Jordan block, then the QDFMs of \( S_0 \) equals the DFMs of \( S_{\text{obs}} \).

6. EXAMPLES

In this section we provide several illustrative examples. Before doing so, we observe that there are numerous algorithms to compute the DFMs of an LTI system. Hence, we adopt the algorithm of Davison and Chang (1990), since that algorithm provides a “continuous” measure of the size of an ADFM, unlike algorithms which are “binary”, such as algorithms which require a rank condition to be satisfied. In this algorithm one only needs to apply a finite number of specified output feedback gain of the form

\[
u = K(i)y, \quad i = 1, ..., q,
\]

to the system, and then check \( \bigcap_{i=1,...,q} \text{sp}[A + BK(i)C]\). (This differs from the classical algorithm of Davison and Wang (1973) in which “random gains” are applied.) For example, in the case of \( p = 2 \) with scalar input and outputs \((m_1 = m_2 = r_1 = r_2 = 1)\), we have

\[
K(1) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad K(2) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad K(3) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.
\]

**6.1 Example 1**

The first example is chosen to illustrate the proposed algorithms. Consider the system

\[
\dot{x} = \begin{bmatrix} -1 & 0 & -1 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u_2
\]

\[
y_1 = [0 1 0] x, \quad y_2 = [1 0 0] x.
\]

In this case when we apply the DFM algorithm mentioned at the beginning of the section, we obtain:

\[
\text{sp}[A + BK(1)C] = \{-1, -2, -3\},
\]

\[
\text{sp}[A + BK(2)C] = \{-1, -2, -3\},
\]

\[
\text{sp}[A + BK(3)C] = \{-102, 98.005, -2\},
\]

which means that \(-2\) is a DFM.

To determine the QDFMs we apply Theorem 4 and find the DFMs of \( S_{\text{mar}} \); for space constraints we cannot provide the details, except to say that each \( D_{ij} \in \mathbb{R}^{6 \times 3} \). When we apply the above algorithm, we end up determining that \( S_{\text{mar}} \) has no DFMs. We conclude that the original system has no QDFMs. This agrees with Gong and Aldeen (1997), i.e. since the system is clearly strongly connected, and centrally controllable and observable, by Theorem 1.
there are no QDFMs. The benefit of our algorithm is that we avoid checking whether or not the system is strongly connected.

6.2 Repeated Eigenvalue Example

Consider the system

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix}
+ 
\begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix}
+ 
\begin{bmatrix}
1 \\
0 \\
0 \\
0
\end{bmatrix}
+ 
\begin{bmatrix}
u_1 \\
u_2
\end{bmatrix},
\]

This system has two DFMs at 1.

Using Theorem 4, the QDFMs are simply the DFMs of \( S_{mar} \); if we form it and compute these modes, it is easy to check that the set is empty, which means there are no QDFMs.

Since the \( A \) matrix has repeated eigenvalues, the hypothesis of Ozguner and Davison (1985) is violated, so the controller design methodology proposed there is not guaranteed to work. If we proceed anyway and form the discretized plant (using a zero-order-hold), we see that it has a decentralized fixed mode (at \( e^{h} \), with \( h \) denoting the sampling period), which demonstrates that the design methodology will not work. Although some systems of the above sort can be handled by first applying some regularizing output feedback, the above is not one of them.

6.3 Non-Strongly-Connected Case

Consider the decentralized control system

\[
\begin{bmatrix}
-1 & -1 & 7 & -10 \\
0 & -2 & 0 & 0 \\
0 & -3 & 0 & 0 \\
0 & 7 & -10
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix}
+ 
\begin{bmatrix}
-2 \\
0 \\
-1 \\
-1
\end{bmatrix}
+ 
\begin{bmatrix}
u_1 \\
u_2
\end{bmatrix}
+ 
\begin{bmatrix}
1 \\
0 \\
0 \\
0
\end{bmatrix}^T u_3,
\]

\[
y_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} x
\]

\[
y_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} x
\]

which has a DFM of -2. If we form \( S_{mar} \), we find that it has no DFMs, so by Theorem 4 the original system has no QDFMs. It can be confirmed that this system is not strongly connected, and we clearly have computed the QDFMs without a graphical first step.

6.4 Another Non-Strongly-Connected Case

Now consider another decentralized control system

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & 3
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}
+ 
\begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix}
+ 
\begin{bmatrix}
u_1 \\
u_2
\end{bmatrix}
+ 
\begin{bmatrix}
0 \\
1 \\
1
\end{bmatrix}

\]

\[
y_1 = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} x
\]

\[
y_2 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} x
\]

7. SUMMARY AND CONCLUSIONS

Here we consider the calculation of the QDFMs; in the case of distinct non-zero eigenvalues, these are exactly the SDFMs introduced in Ozguner and Davison (1985). The QDFMs are those modes which are immovable by any form of nonlinear time-varying feedback.

In Gong and Aldeen (1997), where the notion of QDFMs are introduced, an algorithm to compute them is provided. It requires two steps: first, a partitioning of the sub-systems, using graph theory, and then application of standard tools from decentralized control. We provide three new tests for computing these quantities: the first two are restricted to systems which have one Jordan Block per eigenvalue, while the last one has no such restriction. In all three cases, the test is simply that of finding the DFMs of a related system. Since our approach avoids the fragile first step of partitioning, we end up with a more robust approach to computing the QDFMs.

REFERENCES


