Abstract: The paper introduces an important enhancement to the robust decentralized controller design for multi input/multi output uncertain systems within the setting of the Equivalent Subsystems Method (ESM). The class of design problems considered includes guaranteeing robust stability and nominal performance in terms of maximum overshoot of the full system achieved through specified phase margins in equivalent subsystems. The developed design procedure is illustrated by an example.

Keywords: Decentralized controller, Frequency domain control, Nominal performance, Robust stability

1. INTRODUCTION

When designing decentralized control (DC) for performance, performance objectives can be of two basic types, depending on the specific application: a) achieving required performance in the different subsystems (using either independent or dependent design methods); b) achieving a desired performance of the overall system. The Nyquist-based frequency domain decentralized controller design technique for guaranteed performance called “Equivalent Subsystems Method” (ESM) (Kozáková and Veselý, 2009) belongs to the latter group; according to it, the DC design for plants described by continuous-time transfer function matrices is performed using independent design applied to equivalent subsystems that are actually Nyquist plots of decoupled subsystems shaped by a selected characteristic locus of the interactions matrix. It has been proved that local controllers independently tuned for stability and specified feasible performance in terms of degree of stability of equivalent subsystems constitute the decentralized controller that guarantees the specified performance for the full system. In (Kozáková et al., 2010) ESM was used to design digital decentralized PI controller for specified phase margin in equivalent subsystems using their discrete Bode plots, thus guaranteeing maximum overshoot of the full system.

Application of the ESM in the design for robust stability and nominal performance can be found e.g. in (Kozáková and Veselý, 2007; 2009). Similarly as in other references, e.g. (Skogestad and Postlethwaite, 2005) the DC design for robust stability is carried out in a two-stage procedure: first, the DC for performance of the nominal system is designed, and afterwards, fulfillment of the robust stability conditions is examined; if robust stability is not achieved either controller parameters are to be modified, or the redesign is to be carried out with modified performance requirements.

Unlike standard robust approaches, ESM allows to consider full nominal model instead of just a diagonal one, thus reducing conservativeness of resulting robust stability conditions.

In this paper a direct DC design methodology for robust stability is proposed, that directly integrates the robust stability conditions in the ESM thus allowing designing local controllers for robust stability of the full system without trials and errors. The plant-wide performance in terms of maximum peaks of the complementary sensitivity or sensitivity, which corresponds to the maximum overshoot in the full system, is translated into bounds for local designs in terms of phase margins of equivalent subsystems.

The paper is organized as follows: Preliminaries and problem formulation are in Section 2, principles of the Equivalent Subsystems Method (ESM) are revisited in Section 3. Section 4 deals with the main result of the paper – development of the direct robust DC design procedure in which robust stability conditions are directly integrated in the design of local controllers for equivalent subsystems. Theoretical results are demonstrated on an example in Section 5.

2. PRELIMINARIES AND PROBLEM FORMULATION

Consider a MIMO system described by a transfer function matrix $G(s) \in \mathbb{R}^{m \times m}$, and a controller $R(s) \in \mathbb{R}^{m \times m}$ in the standard feedback loop (Fig. 1); $w, u, y, e, d$ are respectively vectors of reference, control, output, control error and disturbance of compatible dimensions. Necessary and sufficient closed-loop stability conditions are given by the Generalized Nyquist Stability Theorem applied to the closed-loop characteristic polynomial (CLCP)

$$\det F(s) = \det(I + Q(s))$$

where $Q(s) = G(s)R(s) \in \mathbb{R}^{m \times m}$. 

Additive uncertainty:
\[
\Pi_{a} := \{ G(s) : G(s) = G_{0}(s) + E_{a}(s), \quad E_{a}(j\omega) \leq \ell_{a}(\omega)\Delta(j\omega) \}
\]
(7)
\[
\ell_{a}(\omega) = \max_{k} \sigma_{\text{max}}[G^{k}(j\omega) - G_{0}(j\omega)], \quad k = 1, 2, \ldots, N
\]
Inverse additive uncertainty
\[
\Pi_{ia} := \{ G(s) : G(s) = G_{0}(s)[I - E_{ia}(s)G_{0}(j\omega)]^{-1}, \quad E_{ia}(j\omega) \leq \ell_{ia}(\omega)\Delta(j\omega) \}
\]
(8)
\[
\ell_{ia}(\omega) = \max_{k} \sigma_{\text{max}}\{G^{k}(j\omega)]^{-1} - [G^{k}(j\omega)]^{-1} \}, \quad k = 1, 2, \ldots, N
\]
Standard feedback configuration with uncertain plant modeled using any unstructured uncertainty form can be recast into the \(M - \Delta\) structure (Fig. 2) where \(\Delta(s) \in \mathbb{R}^{m \times m}\) is the norm-bounded complex perturbation and \(M(s)\) for the uncertainty forms (7),(8) are respectively
\[
M(s) = \ell_{a}(s)R(s)[I + G_{0}(s)R(s)]^{-1}
\]
(9)
\[
M(s) = \ell_{ia}(s)[I + G_{0}(s)R(s)]^{-1}G_{0}(s)
\]
(10)
Conservatism of the robust stability condition (19) can be relaxed by using the additive affine-type uncertainty \(E_{af}(s)\) (Kozáková and Veselý, 2007).
\[
E_{af}(s) = \sum_{i=1}^{p} G_{i}(s)q_{i}
\]
(12)
where \(G_{i}(s) \in \mathbb{R}^{m \times m}, i=0,1, \ldots, p\) are stable matrices, \(p\) is the number of uncertainties defining \(2^{p}\) polytope vertices that correspond to individual perturbed models; \(q_{i}\) are polytope parameters. The related \(\Pi_{af}\) generated by the additive affine-type uncertainty is

\[
\Pi_{af} := \{ G(s) : G(s) = G_{0}(s) + E_{af}(s), \quad E_{af}(j\omega) \leq \ell_{af}(\omega)\Delta(j\omega) \}
\]
\[ \Pi_{af} := \{ G(s) : G(s) = G_0(s) + E_{af} \}, \]
\[ E_{af} = \sum_{i=1}^{p} G_1(s)q_i, \quad (13) \]
\[ q_i \leq q_{i_{\text{min}}} \cdot q_{i_{\text{max}}} \cdot q_{i_{\text{min}}} \cdot q_{i_{\text{max}}} = 0 \]
where \( G_0(s) \) is the „affine“ nominal model. In the matrix form, individual plants from the set \( \Pi_{af} \) can be expressed as follows:
\[ G(s) = G_0(s) + QG_u(s), \quad (14) \]
where \( Q = [I_{q_1} \ldots I_{q_p}]^T \in R^{mx(\times p)}, I_{q_i} = q_iI_{mxm}, \)
\[ G_u(s) = [G_1 \ldots G_p]^T \in R^{mx \times pm}. \]

Similarly to previous uncertainty forms, the feedback loop with uncertain plant modeled using the additive affine type uncertainty in Fig. 3 can be recast into the \( QM_{af} \) structure with
\[ R_{u1} 0 \ldots 0 \\
0 \ldots R_{u2} \ldots 0 \\
\ldots \ldots \ldots \ldots \ldots \ldots \\
0 \ldots 0 \ldots R_{um} \\
G_{11} 0 \ldots 0 \\
0 \ldots G_{22} \ldots 0 \\
\ldots \ldots \ldots \ldots \ldots \ldots \\
0 \ldots 0 \ldots G_{mm} \\
0 \quad G_{12} \ldots G_{1m} \\
G_{21} 0 \ldots 0 \\
\ldots \ldots \ldots \ldots \ldots \ldots \\
G_{m1} G_{m2} \ldots 0 \]
Fig. 3. Standard feedback loop with additive affine-type additive uncertainty.

The stability condition for the \( M_{af} - Q \) system is
\[ \sigma_{\text{max}}(M_{af} Q) < 1 \quad (15) \]
Under the assumption that \( q_0 = q_{i_{\text{min}}} = q_{i_{\text{max}}} \) (14) can further be modified to yield
\[ \sigma_{\text{max}}(M_{af})q_0 \sqrt{p} < 1 \quad (16) \]

2.1 Problem Formulation
Consider an uncertain system with \( m \) subsystems given as a set of \( N \) transfer function matrices obtained in \( N \) working points of plant operation, described by a nominal model \( G_0(s) \) and any unstructured uncertainty form (7),(8) or (12). Let \( G_0(s) \) can be split as follows:
\[ G_0(s) = G_d(s) + G_m(s) \quad (17) \]
where \( \forall s \in D \)
\[ G_d(s) = \text{diag}\{G_1(s)\}_{mxm}, \quad \text{det} G_d(s) \neq 0 \]
\[ G_m(s) = G_0(s) - G_d(s) \]

A decentralized controller (DC)
\[ R(s) = \text{diag}\{R_1(s)\}_{mxm} ; \quad \text{det} R(s) \neq 0 \quad (18) \]
is to be designed to guarantee stability over the whole operating range of the plant specified by either (7), (8) or (13) (robust stability) and a specified performance of the nominal model (nominal performance).

To solve the above formulated problem, a frequency domain robust decentralized controller design technique has been developed (Kozáková and Veselý, 2009); the core of it is the Equivalent Subsystems Method (ESM).

3. EQUIVALENT SUBSYSTEMS METHOD

The Equivalent Subsystems Method (ESM) is an original Nyquist-based DC design method for stability and guaranteed performance of the full system. According to it, local controller designs are performed independently for so-called equivalent subsystems that are actually Nyquist plots of decoupled subsystems, shaped by one selected characteristic locus of the interactions matrix. Local controllers of equivalent subsystems independently tuned for stability and specified feasible performance constitute the decentralized controller guaranteeing specified performance of the full system. Unlike standard robust approaches, the proposed technique considers full mean parameter value nominal model, thus reducing conservatism of resulting robust stability conditions. In the context of robust DC design (Kozáková and Veselý, 2009) the Equivalent Subsystems Method is applied to design DC for the nominal model (Fig. 4).

![Fig. 4. Standard feedback loop under decentralized controller](image-url)
\begin{equation}
I + R(s)[G_d(s) - P(s)] = I + R(s)G^{eq}(s) = 0
\end{equation}

where

\begin{equation}
G^{eq}(s) = \text{diag}\{G_i^{eq}(s)\}_{m \times m}
\end{equation}

is a diagonal matrix of equivalent subsystems

\begin{equation}
G_i^{eq}(s) = G_i(s) - p_i(s), \quad i = 1, 2, ..., m;
\end{equation}

As all matrices are diagonal, on subsystems level (22) breaks down into \( m \) equivalent characteristic polynomials

\begin{equation}
\text{CLCP}_i^{eq}(s) = I + R_i(s)G_i^{eq}(s) \quad i = 1, 2, ..., m
\end{equation}

The point at issue is how to select \( P(s) \)? According to (21a), the closed-loop system is at the limit of instability if

\begin{equation}
det[ P(s) + G_m(s)] = det[ p_i(I + G_m)] = 0, \quad i = 1, 2, ..., m
\end{equation}

From stability viewpoint, \( P(s) \) in (26) ”counterbalances” interactions \( G_m(s) \) if \( p_i \) is equal to any of the characteristic functions. Stability conditions for the closed-loop in Fig.4 are summarized in the following theorem.

**Theorem 2.** The closed-loop in Fig. 4 comprising the system (17) and the decentralized controller (18) is stable if and only if there exists a diagonal matrix \( P(s) = \text{diag}\{p_i(s)\}_{m \times m} \) such that

1. \( \det[ p_i(I + G_m)] = 0, \quad i = 1, 2, ..., m; \)
2. all equivalent characteristic polynomials (25) have roots with \( \text{Re}\{s\} < 0; \)
3. \( N[0, \text{det } F(s)] = n_q \)

where \( \text{det } F(s) = I + G(s)R(s) \) and \( n_q \) is the number of open loop poles with \( \text{Re}\{s\} > 0. \)

Comparing (2) with (26), the system (17) under the DC (18) will be at the limit of instability if the diagonal matrix \( P(s) \) is chosen to have identical entries equal to any selected characteristic function \( g_i(s) \) of \([G_m(s)]\) (Kozáková et al. 2009a,b), i.e. if

\begin{equation}
P(s) = -g_k(s)I, \quad k \in \{1, ..., m\} \text{ is fixed.}
\end{equation}

In such a case \( \forall s \in D \)

\begin{equation}
\det[ P(s) + G_m(s)] = \prod_{i=1}^{m}[-g_k(s) + g_i(s)] = 0
\end{equation}

From stability viewpoint, the matrix \( G_m(s) \) in (19) and (21a) can be substituted by \([P(s)]\). Hence, for \( P(s) \), that satisfies (27) and (28), the following CLCP’s are equivalent with respect to stability and performance:

\begin{equation}
det[ I + G(s)R(s)] \Leftrightarrow \det[ R^{-1}(s) + G_k(s) - P(s)]]
\end{equation}

The related equivalent subsystems generated according to (24) are

\begin{equation}
G_{ik}^{eq}(s) = G_i(s) + g_k(s) \quad i = 1, 2, ..., m
\end{equation}

The design technique resulting from **Corollary 1** enables to design local controllers of equivalent subsystems using any SISO frequency-domain design method, e.g. the Neymark D-partition method (Kozáková and Veselý, 2009), standard Bode diagram design etc.

In the originally developed ESM version (Kozáková et al. 2009a; 2009b) based on (26) – (30) it was proved that local controllers independently tuned for stability and a specified feasible degree of stability of equivalent subsystems constitute the decentralized controller guaranteeing the same degree of stability for the full system. In (Kozáková et al. 2010) the performance specification applied in ESM was based on the relationship between phase margins of equivalent subsystems and maximum overshoot of the full system. This performance specification is used in the next development

#### 4. ROBUST DECENTRALIZED CONTROLLER DESIGN

This section deals with implementation of the ESM in the decentralized controller design for robust stability and nominal performance applicable for uncertain systems described as a set of transfer function matrices. The nominal model can be calculated either as the mean value parameter model (Skogestad and Postlethwaite, 2005), or the “affine” model, obtained within the procedure for calculating the affine-type additive uncertainty (Kozáková and Veselý, 2007; 2008). Unlike the standard robust approach to DC design which considers diagonal model as the nominal one (interactions are included in the uncertainty), the ESM method applied in the design for nominal performance allows to consider the full nominal model. Model uncertainty is described by any unstructured uncertainty form (7), (8) or (13).

In (Kozáková and Veselý, 2009) a two-stage robust DC design methodology was proposed based on ESM and fulfillment of the M-A structure stability conditions. The direct DC design for robust stability and nominal performance is the main result of this paper.

**4.1 Direct decentralized controller design for robust stability and nominal performance**

If the robust stability conditions (11) or (16) are directly integrated in the ESM, local controllers of equivalent subsystems are designed with regard to robust stability of the full system. For this purpose, a suitable performance specification for the full system is the maximum peak of the complementary sensitivity \( M_T \) corresponding to maximum overshoot in the full system as well as in individual equivalent subsystems where it can be translated into lower bounds for their phase margins according to (31) (Skogestad and Postlethwaite, 2005)

\begin{equation}
PM \geq 2 \arcsin \left( \frac{1}{2M_T} \right) \geq \frac{1}{M_T} \text{[rad]}
\end{equation}

where \( PM \) is the phase margin, and \( M_T \) is the maximum peak of the complementary sensitivity \( T(s) \)
\[ T(s) = G(s)R(s)[I + G(s)R(s)]^{-1} \]  

(32)

Because for MIMO systems

\[ M_T = \sigma_{\text{max}}(T), \]  

(33)

the upper bound for the nominal complementary sensitivity

\[ T_0(s) = G_0(s)R(s)[I + G_0(s)R(s)]^{-1} \]  

can be derived by substituting into (1) the uncertain system model (additive uncertainty is considered in the following development) where \( G_0(s) \) is the nominal model:

\[ \det[I + (G_0 + \ell_a A)R] = \det[I + \ell_a AR(I + G_0 R)^{-1}] \]

(34)

The first term on the r.h.s. of (34) is the CLCP of the nominal system that corresponds to the CLCP\(^{\text{eq}}\) according to (29); condition for stability of the second term can be determined using the small gain theorem: hence, it is stable if and only if the nominal closed loop is stable and

\[ \|\ell_a AR(I + G_0 R)^{-1}\| < 1 \]  

(35)

Considering the spectral norm and the singular value properties, (35) can be readily manipulated to yield the final condition (38). Bounds for other uncertainty forms can be derived by analogy.

In case of inverse uncertainty forms, robustness bounds are obtained in terms of the maximum peak of the sensitivity

\[ M_S = \sigma_{\text{max}}(S) \]  

where

\[ S(s) = [I + G(s)R(s)]^{-1} \]

(36)

using the lower bounds for PM in the form (Skogestad and Postlethwaite, 2005)

\[ PM \geq 2 \arcsin \left( \frac{1}{2M_S} \right) \geq \frac{1}{M_S} \{\text{rad}\} \]  

(37)

Upper bounds for \( \sigma_{\text{max}}[T_0(\omega)] \) or \( \sigma_{\text{max}}[S_0(\omega)] \) for additive-type uncertainties are summarized below.

Additive uncertainty:

\[ \sigma_{\text{max}}[T_0(\omega)] < \frac{\sigma_{\text{min}}[G_0(\omega)]}{\ell_a(\omega)} = L_A(\omega) \quad \forall \omega \]

(38)

Additive affine-type uncertainty:

\[ \sigma_{\text{max}}[T_0(\omega)] < \frac{1}{q_0 \sqrt{p}} \frac{\sigma_{\text{min}}[G_0(\omega)]}{\sigma_{\text{max}}[G_0(\omega)]} = L_{AF}(\omega) \quad \forall \omega \]

(39)

Inverse additive uncertainty:

\[ \sigma_{\text{max}}[S_0(\omega)] < \frac{1}{\phi_{\omega}(\omega)\sigma_{\text{max}}[G_0(\omega)]} = L_{I\text{AF}}(\omega), \quad \forall \omega \]

(40)

The derived upper bounds (38), (39) or (40) for the nominal can be directly integrated in the ESM due to the fact that performance achieved in equivalent subsystems is simultaneously guaranteed for the full system. The main steps of the resulting design procedure include: 1. specification of maximum overshoot that guarantees robust stability of the full system in terms of \( M_T = \sigma_{\text{max}}(T_0) \) or \( M_S = \sigma_{\text{max}}(S_0) \); 2. translating \( M_T \) (or \( M_S \)) into minimum phase margins required in equivalent subsystems using (31) or (37), and 3. designing local controllers independently for individual single input – single output equivalent subsystems using any frequency domain approach for SISO systems described e.g. in (Tantaris et al., 2002; Emami and Watkins, 2009). A design procedure based on SISO Bode diagram is illustrated in the next section.

5. EXAMPLE

Consider a laboratory plant consisting of two DC motors interconnected so that each armature voltage \( (\omega_1, \omega_2) \) affects rotor speeds of both motors \( (\omega_1, \omega_2) \). The plant was identified in three operating points, and is given as a set \( \Omega = \{G_1(s), G_2(s), G_3(s)\} \) where

\[
G_1(s) = \begin{bmatrix} -0.402s + 2.690 & 0.006s - 1.680 \\ s^2 + 2.870s + 1.840 & s^2 + 11.570s + 3.780 \\ 0.003s - 0.720 & -0.170s + 1.630 \\ s^2 + 9.850s + 1.764 & s^2 + 1.545s + 0.985 \end{bmatrix}
\]

\[
G_2(s) = \begin{bmatrix} -0.342s + 2.290 & 0.005s - 1.510 \\ s^2 + 2.070s + 1.840 & s^2 + 10.570s + 3.780 \\ 0.003s - 0.580 & -0.160s + 1.530 \\ s^2 + 8.850s + 1.764 & s^2 + 1.045s + 0.985 \end{bmatrix}
\]

\[
G_3(s) = \begin{bmatrix} -0.423s + 2.830 & 0.006s - 1.930 \\ s^2 + 4.870s + 1.840 & s^2 + 12.570s + 3.780 \\ 0.004s - 0.790 & -0.200s + 1.950 \\ s^2 + 10.850s + 1.764 & s^2 + 1.945s + 0.985 \end{bmatrix}
\]

In calculating the affine nominal model \( G_0(s) \), all possible allocations of \( G_1(s), G_2(s), G_3(s) \) into the \( 2^3 = 4 \) polytope vertices were examined (24 combinations) yielding 24 affine nominal model candidates and related transfer functions matrices \( G_i(s) \) needed to complete the description of the uncertainty region. The selected affine nominal model \( G_0(s) \) is the one guaranteeing the smallest additive uncertainty calculated according to (7):

\[
G_0(s) = \begin{bmatrix} -0.413s + 2.759 & 0.004s - 1.807 \\ s^2 + 3.870s + 1.840 & s^2 + 12.570s + 3.780 \\ 0.004s - 0.757 & -0.187s + 1.791 \\ s^2 + 10.350s + 1.764 & s^2 + 1.745s + 0.985 \end{bmatrix}
\]

The upper bound for \( T_0(s) \) calculated according to (37) is plotted in Fig. 5. Its worst (minimum) value \( M_T = \min_{\omega} L_{AF}(\omega) = 1.556 \) corresponds to \( PM \geq 37.48^\circ \) according to (31).
The Bode diagram design of local controllers for guaranteed PM was carried out for equivalent subsystems $G_{ij}^{eq}(s), G_{2i}^{eq}(s)$ generated according to (30) using characteristic locus $g_i(s)$ of the matrix of interactions $G_m(s)$, i.e. $G_{ij}^{eq}(s) = G_i(s) + g_2(s)$ $i,j = 1,2$. Bode diagrams of equivalent subsystems are in Fig. 6. Applying the PI controller design from Bode diagram for the required phase margin $PM = 39^\circ$ has yielded the following local controllers:

$$R_1(s) = \frac{3.367 s + 1.27}{s} \quad R_2(s) = \frac{1.803 s + 0.491}{s}$$

Bode diagrams of compensated equivalent subsystems in Fig. 7 prove the achieved phase margin. Robust stability was verified using the original $M_{eq}$ condition (16) with $p=2$ and $q_0=1$; as depicted in Fig. 8, the closed loop under the designed controller is robustly stable.

**REFERENCES**


