Kinematic Control of Robots with Noisy Guidance System

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Abstract: The main component of a kinematic control scheme for robots is the inverse kinematic algorithm. The paper proposes to solve the problem for a general robotic system by approaching it as a state estimation problem for a nonlinear dynamic system. This point of view allows to adopt any of the state estimation algorithms available in the literature. The most attractive reason is that such algorithms can cope with measurement noise and uncertainties, that always affect the guidance system of an advanced robotic system operating in an unstructured environment. The choice made here is to use an Extended Kalman Filter and the convergence analysis of the discrete-time algorithm is carried out in a stochastic framework to explicitly take into account the presence of noise characterised by any type of probability distribution. A simulation case study is presented to show the effectiveness of the algorithm.

Keywords: Mobile robots, Autonomous robots, Estimation and filtering, Bayesian methods

1. INTRODUCTION

The classical kinematic control scheme for controlling advanced robotic systems with application in unstructured environments can be found in many textbooks, e.g. (Siciliano et al., 2008). Basically, the robot is guided by a planner whose input consists of measurements from exteroceptive sensors and whose output is the reference trajectory in the task space. Then the trajectory in the task space is usually converted in references for different variables, that are, usually, directly controlled by a low-level control loop at dynamic level. For example, in a robot manipulator, the reference trajectory for the end effector, guided by an artificial vision system, is converted into a reference trajectory in the joint space by inverting the kinematics. Also in the mobile robotics field, the kinematic control approach has been proposed and successfully adopted for controlling platoon of autonomous vehicles Antonelli and Chiaverini (2006), for control of, e.g., formation, navigation or entrapment/escorting. In any case, the basic component of a kinematic control scheme is the inverse kinematic algorithm. Inverse kinematics of robotic systems has been addressed and solved in a variety of manners both for non-redundant and redundant robots. For a comprehensive review, the reader is referred to (Waldron and Schmiedeler, 2008) and references therein. The alternative approach proposed in this paper can be considered somehow dual with respect to the CLIK (Closed-Loop Inverse Kinematics) algorithm, which poses the inverse kinematic problem as a feedback control problem (Sciavicco and Siciliano, 1986). In fact, it is here posed as a state estimation problem for a nonlinear dynamic system, and the solution is obtained by resorting to the Extended Kalman Filter (EKF). This choice is made because in a real setting, the desired task trajectory to be computed on the basis of information collected from the environment via exteroceptive sensors is corrupted by noise. In such applications of kinematic control, differently from the classical kinematic control of industrial robots, besides the accuracy, another requirement is of fundamental importance, namely the computational burden of the algorithm should allow it to run in real-time. This observation is the main limitation of the approach based on particle filters of Qin and Carreira-Perpiñán (2008). Also, in every sensor-based (visual or force) servoing, when joint space control is used, inverse kinematics has to deal with noisy signals and thus the importance of an inverse kinematic algorithm robust to sensor noise, or whose behaviour is predictable in presence of noise is desirable. The main difficulty in the study of algorithmic solutions to the inverse kinematics problem is related to the discrete-time nature of the dynamic system at hand, combined with its strong nonlinearity, deriving from the kinematics. This paper intends to address the stability of a discrete time inverse kinematics algorithm in a stochastic framework, and solving it by resorting to the EKF. Stochastic stability of the EKF has been originally addressed by Reif et al. (1999). However, these results, even though general, provide only boundedness of the estimation error, while the present paper ensures Lyapunov stability. The proposed inverse kinematic algorithm is shown to be stable with any type of probability distribution of the noise, not necessarily assumed Gaussian. A simulation case study presents an application of mobile robotics, i.e. a platoon of robots escorting a moving target tracked by a vision system.

2. THE INVERSE KINEMATICS PROBLEM

Let $x \in X$ be the vector of task variables of a robotic system with $X$ a domain of $\mathbb{R}^n$ and let $q \in Q$ be the vector of the robotic system configuration with $Q$ a domain of $\mathbb{R}^n$. For example, in a robotic manipulator $x$ is the pose...
of the end effector and \( q \) is the vector of joint positions, whereas in a platoon of mobile robots, \( q \) is the vector of coordinates representing the location of each robot and \( x \) is the vector of suitable task variables depending on the mission. Note that the task space \( X \) and the configuration space \( Q \) are assumed of the same dimension, which generally means that the robot is not redundant. The case of redundant robots can be easily carried to such a case by adopting the well-known extended jacobian method for redundancy resolution (Bailleul, 1985), where an augmenting map is introduced so that the resulting extended direct kinematics equation can be always written in the form

\[
k : Q \subseteq \mathbb{R}^n \to X \subseteq \mathbb{R}^n, \quad x = k(q).
\]

This function will be hereafter called “direct kinematics function” or “task function” without distinction. It is well-known that such a method for redundancy resolution guarantees the so-called repeatability property when inverting the differential kinematics (Shamir and Yomdin, 1988), even though it suffers from the so-called algorithmic singularities. Moreover, it is assumed the absence of both kinematic and algorithmic singularities within the configuration space \( Q \), i.e. the jacobian

\[
J(q) = \frac{\partial k(q)}{\partial q} \in \mathbb{R}^n \times \mathbb{n}
\]

is invertible \( \forall q \in Q \). This matrix will be hereafter called “robot jacobian” or “task jacobian” without distinction. If not differently stated, hereafter the Euclidean norm will be considered as vector norm and the spectral norm (the largest singular value) will be considered as matrix norm. The smallest and largest singular values of a matrix \( A \) will be denoted with \( \sigma(A) \) and \( \bar{\sigma}(A) \), respectively.

Both the direct kinematics function and the robot jacobian are assumed to fulfill the following assumptions:

\begin{enumerate}
  \item[i)] \( \exists \delta > 0 : \|J(q)\| \leq \delta, \forall q \in Q \)
  \item[ii)] \( \exists \delta' > 0 : \|J^{-1}(q)\| \leq \delta', \forall q \in Q \)
  \item[iii)] \( \exists \beta > 0 : \|J(q)J(q)^T\| \leq \beta, \forall q \in Q \)
  \item[iv)] \( \exists \beta' > 0 : \|J(q)J(q)^T\| \geq \beta', \forall q \in Q \)
  \item[v)] the function \( k(q) \) is smooth enough such that \( k(q + \tilde{q}) = k(q) + J(q)\tilde{q} + r_k(q) \),
  where the reminder \( r_k(q) \) is such that \( \exists v_k > 0 : \|r_k(q)\| \leq v_k \|\tilde{q}\|^2, \forall \tilde{q} : q + \tilde{q} \in Q \)
  \item[vi)] for any given matrices \( X \) and \( Y \) with bounded norm, the following matrix function of \( q \)
    \[
    XJ^T(q)\left(J(q)XJ^T(q) + Y\right)^{-1}
    \]
    is smooth enough such that, for \( i = 1, \ldots, n, \exists \rho_i > 0, \nu_i > 0 \) such that
    \[
    c_i(q + \tilde{q}) = c_i(q) + \frac{\partial c_i(q)}{\partial q} \tilde{q} + r_{c_i}(q),
    \]
    with the reminders \( r_{c_i}(q) \) such that \( \|r_{c_i}(q)\| \leq \rho_i \|\tilde{q}\|^2, \forall \tilde{q} : q + \tilde{q} \in Q \)
    and the jacobians such that \( \|\frac{\partial c_i(q)}{\partial q}\| \leq \nu_i, \forall q \in Q \).
\end{enumerate}

Assumptions \( i), vi) \) deserve a more thorough discussion. The degree of smoothness of \( k(q) \) and \( c_i(q) \) needed to satisfy the assumptions can be easily quantified by applying Lemma 2 in the appendix. Basically, the nonlinear functions have to possess second-order derivatives bounded, and in many applications such a requirement is verified. For example, every robot with revolute joints has a direct kinematics function constituted by polynomial combinations of trigonometric functions of the joint variables, therefore their second-order derivatives are bounded.

Let \( x_{dh} \in X \), being \( h \in \mathbb{Z} \) the discrete-time variable, be a desired task space position, the objective of any inverse kinematics algorithm is to find one of the possibly many configurations \( q_{dh} \) such that

\[
x_{dh} = k(q_{dh}),
\]

which is a system of \( n \) nonlinear equations. In order to reformulate the inverse kinematics problem as a state estimation problem, assume that the vector of configuration variables \( q_h \) to be determined from the knowledge of the task variables \( x_h \) is the state of a discrete-time dynamic system whose output is \( x_h \) and with state space representation

\[
q_{h+1} = f(q_h, w_h),
\]

\[
x_h = k(q_h) + v_h,
\]

where the state update function \( f \) is in general nonlinear and \( w_h \) and \( v_h \) are stationary stochastic processes, both assumed uncorrelated with the state variable, with zero mean and covariance matrices \( W > 0 \) and \( V \geq 0 \), respectively. \( V \) is assumed to be positive semi-definite with positive norm. Note that \( V \) is not necessarily positive definite, as this is very common in many applications, when not all the desired task variables come from measurements, but some of them are assigned by the user, and thus not affected by noise. An example will be presented in the case study. A key feature of the proposed algorithm is that the stochastic processes above can have any distribution, while classical EKF assumes them to be Gaussian.

From this point of view, it is straightforward to adopt any of the state estimation algorithms available in the literature. Of course, the main difficulty is the unavailability of the state update function \( f \), but this is the classical problem of every target tracking problem Ristic et al. (2004). In such problems, the usual approach is to assume a simplified model for the state update. Typical choices are the so-called “constant position” and “constant velocity” models. In this paper the former assumption is considered:

\[
q_{h+1} = q_h + w_h
\]

\[
x_h = k(q_h) + v_h.
\]

Future developments foresee the use also of some of the constant velocity model or even more complex models. Considering the desired position in the task space \( x_d \) (assumed constant) as the observed output, this can be corrupted by noise if it is computed by a planner which makes use of exteroceptive sensors. In this paper EKF will be adopted to estimate the state \( q_h \), and the stability of the algorithm will be proved. Moreover, \( x_d \) is assumed belonging to the
reachable task space $X$, in the sense that it exists at least a configuration $q_d$ such that
\[ k(q_d) = x_d, \] (8)
and it is assumed that $q_d$ is an interior point of $Q$.

Taking into account that the state update equation (6) is already linear with the identity matrix as system matrix and that the jacobian of the nonlinear observation function in (7) is exactly the robot jacobian in (2), the EKF equations are set in the classical three-steps form as

- **prediction step**
  \[ q_{h+1|h} = q_h, \quad q_{00} = q_i \]
  (9)

- **Kalman gain computation step**
  \[ K(q_{h+1|h}) = \]
  \[ P_{h+1|h}J^T(q_{h+1|h})(J(q_{h+1|h})P_{h+1|h}J^T(q_{h+1|h}) + V)^{-1} \]
  (11)

- **update step**
  \[ q_{h+1} = q_{h+1|h} + K(q_{h+1|h}) (x_d - k(q_{h+1|h}) + v_h) \]
  (12)

\[ P_{h+1|h} = (I - K(q_{h+1|h})J(q_{h+1|h})) P_{h+1|h} \]
(13)

where $q_i$ is the initial robot configuration, $I$ is the $n \times n$ identity matrix, $q_{h+1|h}$ and $P_{h+1|h}$ denote the expected value and covariance matrix estimates, respectively, at time step $h + 1$ given the observations up to time step $h$, and $q_{h+1|h+1}$ and $P_{h+1|h+1}$ denote the expected value and covariance matrix estimates, respectively, at time step $h + 1$ given the observations up to time step $h + 1$.

In the next section, the convergence of the EKF above will be proven by showing that the expected value estimate $\bar{q}_{h+1|h}$ asymptotically converges to one of the possibly many solutions of (3) with $x_{d,0} = x_d = \text{const.}$ except for a bias term, and, in turn, the expected value of the error in the task space
\[ \bar{e}_h = E[x_d - k(q_{h+1|h})] = x_d - E[k(q_{h+1|h})] \]
(14)
asymptotically converges to a constant, being $E[\cdot]$ the expectation operator. It is important to underline that the estimator will be shown to be biased when only noisy measurements are available. This limitation of the EKF is well-known in the literature (Julier et al., 2000) and it is caused by the classical approximation
\[ E(k(q)) = k(q) \]
and the assumption that the estimated state variable is still Gaussian. The same limiting assumption is also made in classical EKF convergence results, e.g. (Krener, 2002). Many ways to overcome the problem have been proposed, e.g. the Unscented Kalman Filter (UKF) and the Particle Filters (PF). For a thorough review of these estimation techniques the reader is referred to e.g. Ristic et al. (2004). One of the novel contributions of this paper is that it will not make use of the above assumptions, by completely avoiding to approximate the estimated state with a Gaussian random process, although it uses an EKF. Moreover, it will provide an upper bound estimate of the error bias related to the noise power and to the filter design parameters, thus very helpful in the design process.

The presentation of a biased estimator is more a value of the present paper rather than a limitation, since it means that by resorting to more sophisticated filtering techniques the inverse kinematics problem with noisy measurements can be easily tackled, once it has been formulated as a state estimation problem. The purpose of this paper is to clearly and rigourously show how the most classical state estimation algorithm for nonlinear systems, the EKF, can be effectively used, with all its limitations, to solve the inverse kinematics problem of a generic robotic system in a stochastic framework. This will pave the way to the use of more recent and powerful state estimation techniques to solve the same problem.

3. CONVERGENCE ANALYSIS

To make the convergence analysis more easily readable and to optimize the computations of the inverse kinematics algorithm, the following change of notation is introduced in the filter equations (9)–(13)

\[ q_{h|h} \rightarrow q_h, \quad P_{h|h} \rightarrow P_h \]
\[ J(q_{h+1|h}) = J(q_{h+1|h}) \rightarrow J_h, \quad K(q_{h+1|h}) = K(q_h) \rightarrow K_h. \]

With these new symbols, it is easy to see that (9)–(13) are equivalent to the discrete-time nonlinear dynamic system with state space equations

\[ P_{h+1} = (I - K_h J_h)(P_h + W) \]
(15)
\[ q_{h+1} = q_h + K_h (x_d - k(q_h) + v_h), \]
(16)
being
\[ K_h = (P_h + W J_h^T J_h K_h + V)^{-1}, \]
(17)
and output equation
\[ e_h = x_d - k(q_h), \]
(18)
with initial conditions
\[ q_0 = q_i \]
(19)
\[ P_0 = O. \]
(20)

The expected value of the error in (18) will be shown to converge to a steady-state value different from zero, but whose norm decreases for a decreasing norm of the noise covariance matrix. Furthermore, the convergence of the algorithm will be proved by proving that the expected value of the state variable reaches an asymptotically stable equilibrium. Before stating the theorem ensuring this result and whose proof will exploit some useful lemmas in the Appendix, it is necessary to make some preliminary considerations. First of all, to take into account the nonlinear dependence of the Kalman gain on the state variable $q_h$ (see (17)), let $g$ denote the nonlinear part of (16), thus defined as follows
\[ g(q_h, v_h) = K_h (x_d - k(q_h) + v_h), \]
(21)
which is assumed smooth enough to apply Lemma 5. In virtue of Lemma 5, the expected value of $g(q_h, v_h)$ can be computed

1 Hereafter the symbol $\hat{x}$ will denote the expected value of the random variable $x$. 
\[
E[g(q_h, v_h)] = g(\bar{q}_h, \bar{v}_h) + E[r_g(q_h, v_h)] = K(\bar{q}_h)(x_d - k(\bar{q}_h)) + \bar{r}_g,
\] (22)
where \( \bar{r}_g \) is the expected value of the reminder \( r_g \), and the symbol \( K(\bar{q}_h) \) indicates the Kalman gain computed using the exact value of the state variable, i.e.
\[
K(q_h) = (P_h + W)J^T(q_h) \left\{ J(q_h)P_h + W \right\}J(q_h) + V \right\}^{-1}.
\] (23)

Still in view of Lemma 5, and due to the assumption of measurement noise \( v_h \) uncorrelated with the state variable \( q_h \), \( r_g \) in (22) is bounded as follows
\[
\|\bar{r}_g\| \leq 2n_v\|\text{cov}[(q_h^T, v_h^T)^T]\| \leq 2n_v\|\text{blockdiag}(P_h, V)\| \leq 2n_v\|P_h\|^2 + \|V\|^2)^{1/2},
\] (24)
where the symbol \( \text{cov}[x] \) denotes the covariance matrix of the multivariate random variable \( x \), and a standard property of the spectral norm of block diagonal matrices has been used (see Fact 9.9.49 in Bernstein (2005)). As it will be shown in the next theorem, the norm of this reminder tends to zeros as \( \|V\| \) tends to zero.

Hereafter, the covariance matrix \( W \) of the process noise will be assumed scalar, namely \( W = wI, w > 0 \), being \( I \) the \( n \times n \) identity matrix. Considering the point of view under which the state update model in (4) has been introduced, such assumption is not restrictive, since this matrix is more a design parameter rather than an actual model parameter. Moreover, for notation economy, the norm of the covariance matrix \( V \) will be hereafter indicated with the scalar \( v \). Nevertheless, the matrix can be full to take into account that the used sensors can have different noise characteristics and can also be correlated to each other. With this notation, before stating the theorem, some useful functions of \( v \) and \( w \), under the assumption that \( w > v\delta^2 \), are defined as follows:
\[
s(v, w) \triangleq \frac{w\delta^2}{w - v\delta^2},
\] (25)
\[
m(v, w) \triangleq \frac{2n_v\sqrt{1 + \delta^2(v, w)}}{w - v\delta^2},
\] (26)
\[
t(v, w) \triangleq \frac{\sqrt{1 + \delta^2(v, w)}}{w - v\delta^2},
\] (27)
which have the following asymptotic properties
\[
s(v, w) \to \delta^2, \text{ as } w \to \infty \text{ or as } v \to 0 \quad \text{or as } v \to 0 \quad \text{or as } w \to \infty \text{ or as } v \to 0 \quad \text{or as } v \to 0.
\] (28)(29)(30)
The theorem below will ensure that the expected value \( q_h \) of the state variable reaches an asymptotically stable equilibrium. The dynamic equation of the expected value \( \bar{q}_h \) of the state variable \( q_h \) is
\[
\bar{q}_h = \bar{q}_h + K(\bar{q}_h)(x_d - k(\bar{q}_h)) + \bar{r}_g.
\] (31)
This nonlinear system admits an equilibrium point if the nonlinear equation
\[
K(\bar{q}_h)(x_d - k(\bar{q}_h)) + \bar{r}_g = 0
\] (32)
admits solutions. Under this assumption, call \( \bar{q}_d \) one of such solutions, then it is
\[
x_d - k(\bar{q}_d) = -K^{-1}(\bar{q}_d)\bar{r}_g.
\] (33)
where the inverse of the Kalman gain computed as in (17) always exists since \( W \) is assumed to be positive definite and \( P_h \) is positive semi-definite by construction. Interestingly enough, the norm of \( x_d - k(\bar{q}_d) \) tends to zero as \( v \) tends to zero since it does \( \bar{r}_g \), as already anticipated and as it will be shown in the theorem. In this limit case (\( v = 0 \) and thus \( \bar{r}_g = 0 \)), (32) certainly admits solutions since, by assumption, \( x_d \) belongs to the reachable task space of (1), i.e. (8) holds. Therefore, in view of the smoothness properties of the left-hand side of (32) and the assumption that \( q_d \) is an interior point of \( Q \), by resorting to the degree theory (Berger and Berger, 1968), it can be shown that, if \( v \) is small enough, it admits solutions even when \( v \neq 0 \) and thus \( \bar{r}_g \neq 0 \). With reference to the nonlinear dynamic system (15)–(18), the next theorem ensures that the expected value \( \bar{q}_h \) of the state variable converges, in the sense that \( q_d \) is a locally asymptotically stable equilibrium point of the nonlinear discrete-time system (31). Also, the theorem ensures that the estimated covariance matrix \( \tilde{P}_h \) is always bounded and the bound tends to zero as \( v \to 0 \).

Theorem 1. Under the assumptions \( ij, iij, vj, vij \), if the measurement noise covariance matrix has norm \( v \) small enough and selecting the process noise covariance matrix \( W \) in (15) such that
\[
W = wI, \quad \text{with } w > v\delta^2,
\] (34)
then, the nonlinear dynamic system (31) has a locally asymptotically stable equilibrium in \( \bar{q}_d \), solution of (32), and the covariance matrix estimate in (15) is always bounded, i.e.
\[
\|\tilde{P}_h\| \leq \psi, \quad \forall h \geq 0, \quad \text{with } \psi = \psi(v, w).
\] (35)

Proof. For brevity, only a sketch of the proof will be given. First, the covariance matrix estimate \( \tilde{P}_h \) is shown to be always bounded starting from the following chain of inequalities
\[
\|I - K_hJ_h\| \leq \left\| J_h^{-1}VJ_h^{-T}\right\| \left\| \left( J_h^{-1}VJ_h^{-T} + P_h + W \right)^{-1}\right\| \leq \frac{v\|J_h^{-1}\|}{\|J_h^{-1}VJ_h^{-T} + P_h + W\|},
\] (36)
where, the first and second inequalities follow from standard properties of matrix norm. Then, applying the properties of the smallest singular value and in view of assumptions \( ii) \) and (34), (36) becomes
\[
\|I - K_hJ_h\| \leq \frac{v\delta^2}{w} \leq \frac{\varepsilon}{1 - \varepsilon}, \quad \forall h \geq 0.
\] (37)
Next, considering the update equation (15) of the covariance matrix \( P_h \) and the inequality (37) shown before, it results
\[
\|P_{h+1}\| \leq \|I - K_hJ_h\|\|P_h + W\| \leq \varepsilon\|P_h\| + \varepsilon w,
\] (38)
where, again standard matrix norm properties have been applied. In virtue of (37), it is \( \varepsilon \in (0, 1) \) and, by applying Lemma 6, it results
\[
\|P_h\| \leq \psi = \frac{\varepsilon w}{1 - \varepsilon}, \quad \forall h \geq 0,
\] (39)
where the bound can be expressed as a function of the design parameter \( w \) and the norm \( v \) of the noise covariance matrix
\[
\psi = \frac{wv^2}{w-v^2} = v(w, w).
\]
(40)
The boundedness of the covariance matrix estimate implies that also the reminder \( \bar{r}_y \) in (24) is bounded, i.e.
\[
\| \bar{r}_y \| \leq 2nw_\phi(v^2 + v^2)^{1/2} = wn(v, w).
\]
(41)
In a similar way, an upper bound of the Kalman gain is obtained as
\[
\| K_{\bar{y}} \| \leq \left( \frac{n}{\sigma_\phi^2} + \frac{1}{\varepsilon} \right) \| q \|, \quad \forall h \geq 0,
\]
where, using the definition of \( \varepsilon \) in (37), the bound can be expressed as a function of \( v \) and \( w \) as
\[
\mu = \frac{wv^2}{w-v^2} = \delta_s(v, w),
\]
(43)
which tends to \( \delta_s^2 \) as \( v \to 0 \). Also, the inverse of the Kalman gain is shown bounded, i.e.
\[
\| K_{\bar{y}}^{-1} \| = \left\| \left( J_h(P_h + W)J_h^T + V \right) J_h^{-1}(P_h + W)^{-1} \right\|
\leq \left( \| J_h \| \| P_h + W \| \| J_h^T \| + v \right) \frac{\delta_s^2}{2(P_h + W)} \leq \frac{(\psi + w)^2}{w} \delta_s'(w, v, w, \varepsilon), \quad \forall h \geq 0.
\]
(44)
This implies also that the difference \( x_d - k(\bar{q}_h) \) in (33) is bounded, in fact
\[
\| x_d - k(\bar{q}_h) \| \leq \| \bar{q}_h \| \| \bar{r}_y \| \leq \nu l(v, w)m(v, w),
\]
(45)
where the bounds in (41) and (44) have been exploited.

As usual when invoking a Lyapunov argument, instead of studying the stability of the equilibrium point \( \bar{q}_h \) of the system (31), it is more convenient to study the stability of the origin, which is equilibrium point of the system obtained with the change of variable
\[
\bar{q}_h = q_h - \bar{q}_d \iff \bar{q}_h = \bar{q}_d + \bar{q}_h.
\]
(46)
Before computing the dynamics in the new state variable, it is convenient to write the function \( k(\bar{q}_h) \) as
\[
k(\bar{q}_h) = k(\bar{q}_d) + J(\bar{q}_d)\bar{r}_k + r_k,
\]
(47)
with \( \| r_k \| \leq \nu_k \| \bar{q}_h \| ^2 \) and where assumption \( v \) has been considered. Also the Kalman gain \( K(\bar{q}_h) \) in (23) can be rewritten as
\[
K(\bar{q}_h) = K(\bar{q}_d) + \bar{q}_h = (p_1(\bar{q}_d + \bar{q}_h) \cdots p_n(\bar{q}_d + \bar{q}_h)),
\]
(48)
being
\[
p_j(\bar{q}_d + \bar{q}_h) = p_j(\bar{q}_d) + \frac{\partial p_j}{\partial q} \bar{q}_d + r_{pj},
\]
(49)
with
\[
\| r_{pj} \| \leq \nu_j \| \bar{q}_h \| ^2, \quad \| \frac{\partial p_j}{\partial q} \| \bar{q}_d \| \leq \nu_j, \quad j = 1, \ldots, n
\]
and where assumption \( v \) has been considered. Therefore, the Kalman gain can be rewritten as
\[
K(\bar{q}_h) = K(\bar{q}_d) + M(\bar{q}_h) + R(\bar{q}_h),
\]
(50)
where
\[
R(\bar{q}_h) = (r_{p_1} \cdots r_{p_n}),
\]
(51)
\[
M(\bar{q}_h) = \left( \frac{\partial p_1}{\partial q} \bar{q}_d \cdots \frac{\partial p_n}{\partial q} \bar{q}_d \right) \bar{q}_h
\]
(52)
which both have bounded norm in virtue of Lemma 3 and Lemma 4, respectively, i.e.
\[
\| M(\bar{q}_h) \| \leq \nu \| \bar{q}_h \|, \quad \nu = n \sum_{j=1}^{n} \nu_j
\]
(53)
\[
\| R(\bar{q}_h) \| \leq \rho \| \bar{q}_h \| ^2, \quad \rho = \sqrt{n} \sum_{j=1}^{n} \rho_j.
\]
(54)
It can be then shown that the dynamics in the new state variable can be written as
\[
\bar{q}_h+1 = \bar{A}\bar{q}_h + \bar{n}(\bar{q}_h),
\]
(55)
where
\[
\bar{A} = I - K(q_d)J(q_d)
\]
(56)
\[
\bar{n}(\bar{q}_h) = (M(\bar{q}_h) + R(\bar{q}_h))(x_d - k(\bar{q}_d) - J(\bar{q}_d)\bar{q}_d - r_k) - K(q_d)r_k.
\]
(57)
In virtue of (37), the norm of the matrix \( \bar{A} \) is less than one, therefore all its eigenvalues have magnitude lower than one as ensured by Fact. 5.10.14 in Bernstein (2005). Furthermore, the nonlinear part of the dynamics can be shown bounded by fixing a ball of radius \( \phi \) around the origin, i.e. \( \| \bar{q}_h \| \leq \phi \), as
\[
\| \bar{n}(\bar{q}_h) \| \leq \bar{\nu} \| \bar{q}_h \|,
\]
(58)
being
\[
\bar{\nu} = \delta_{\nu_k} s(v, w) + \nu v t(v, w) m(v, w) + \nu \delta_v + \nu \nu_\phi \phi^2 + \nu v \delta_v + \nu v \nu_\phi \phi^2 + \nu v \nu_\phi \phi^3.
\]
(59)
It is easy to see that \( \bar{\nu} \) can be arbitrary small by selecting \( \phi \) and \( v \) small enough, as it is assumed in the theorem hypothesis. The asymptotic stability of the origin can be proved by taking the Lyapunov function candidate
\[
V_h = \bar{q}_h^T \bar{q}_h
\]
and recalling that \( \| \bar{A} \| \leq \varepsilon < 1 \), hence \( \bar{A} \) has all its eigenvalues with magnitude less than one. □

4. CASE STUDY

The case study presented in this section is close to an actual application involving sensor-guided robots, and thus the desired position in the task space is actually affected by noise. Consider a platoon of three mobile robots moving on a plane, whose positions with respect to a fixed reference frame can be represented by the vector \( q \) obtained by stacking the Cartesian coordinates \((q_{2i-1}, q_{2i}), i = 1, 2, 3\) of each robot, namely \( q = (q_1, q_2, q_3, q_4, q_5, q_6)^T \).
The objective of the mission is the entrapment/escorting of a moving target whose position \((x_{d1}, x_{d2})\) is measured by a vision (or GPS) system, thus the centroid of the platoon has to follow the target, while the robots have to maintain a fixed distance among each other. Following the approach in Antonelli and Chiaverini (2006), the platoon of robots can be controlled by defining the vector of task variables \(x = (x_1, x_2, x_3, x_4, x_5, x_6)^T\), whose first two entries are the coordinates of the platoon centroid, the next three entries are proportional to the squared distances from one robot to another, and the last entry is aimed at fixing the orientation of the robots formation. Therefore, the task function \(x = k(q)\) is defined as follows

\[
\begin{pmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5 \\
x_6
\end{pmatrix} = \begin{pmatrix}
\frac{1}{3}(q_1 + q_3 + q_5) \\
\frac{1}{3}(q_2 + q_4 + q_6) \\
\frac{1}{2}((q_1 - q_3)^2 + (q_2 - q_4)^2) \\
\frac{1}{2}((q_1 - q_5)^2 + (q_2 - q_6)^2) \\
\frac{1}{2}((q_1 - q_5)^2 + (q_4 - q_6)^2) \\
q_1 - q_3
\end{pmatrix},
\tag{60}
\]

whose Jacobian, not reported for brevity, has rank 6 as soon as the robots are at a positive distance and \(q_2 \neq q_4\) and its norm is bounded as long as the robots are constrained to move in a finite region of the plane. Moreover, the Hessian matrices of the task function and of each Jacobian column are constant, hence assumptions \(i, ii, v, vi\) are verified.

The mission can be fulfilled by assigning a desired value to each task variable, i.e.

\[
x_d = \begin{pmatrix} x_{d1} \\ x_{d2} \\ 1/2L^2 \end{pmatrix} = \begin{pmatrix}
1/2L^2 \\
1/2L^2
\end{pmatrix}^T,
\tag{61}
\]

where \(L\) is the desired distance among the robots. Obviously, only the first two task variables are affected by measurement noise, which is assumed to have a uniform distribution with symmetric bounds equal to \(\pm \sqrt{3}\) times a 3 cm standard deviation in both directions, thus the noise covariance matrix is

\[
V = \text{diag}(9 \cdot 10^{-4}, 9 \cdot 10^{-4}, 0, 0, 0, 0).
\tag{62}
\]

The vision system provides the target position (with a frequency of 30 Hz), which is assumed to follow a path described by the following functions of the time variable

\[
x_{d1h} = 1 + 2T/t_f h
\tag{63}
\]
\[
x_{d2h} = 1 + 0.2 \sin(2T\pi/t_f h) + 1.8T/t_f h,
\tag{64}
\]

being \(T = 1/30\) s the sampling time and \(t_f = 100\) s the total duration of the target motion. The platoon configuration at the initial time instant is far more than 50 cm from the desired platoon configuration, and the desired distance among the robots has been fixed at \(L = 40\) cm. The configuration \(q\) of the three robots needed to accomplish the entrapment/escorting mission is computed according to the algorithm (15)–(18), using the parameter value \(w = 8.2 \cdot 10^{-6}\). The results are reported in Fig. 1, where the paths of the robots are clearly much less noisy than the desired path and the expected value of the task space error is very small.

**REFERENCES**


![Fig. 1. Mobile robots and target paths at \(h = 1000\), \(h = 2000\), \(h = 3000\), with \(w = 8.2 \cdot 10^{-6}\).](image)


Appendix A. SOME USEFUL LEMMNAS

The first lemma establishes the degree of smoothness required to some of the nonlinear functions involved in the paper, in particular those defined in (1), (21) and assumption vi).

Lemma 2. Given a vector function defined in a domain $D \subseteq \mathbb{R}^n$, $f : x \in D \rightarrow f(x) \in \mathbb{R}^m$ with the Hessian matrices $H(f_i(x))$ of all its components $f_i(x)$, $i = 1, \ldots, m$ norm-bounded uniformly in $D$, i.e.

$$\exists \nu_i > 0 : \|H(f_i(x))\| \leq \nu_i, \forall x \in D, \quad i = 1, \ldots, m.$$ (A.1)

then, $\forall \bar{x} \in D : x + \bar{x} \in D$ and the whole line from $x$ to $\bar{x}$ belongs to $D$, it is

$$f(x + \bar{x}) = f(x) + \frac{\partial f(x)}{\partial x} \bar{x} + r_f(x)$$ (A.2)

and the reminder $r_f(x)$ is such that

$$\exists \nu_f > 0 : \|r_f(x)\| \leq \nu_f \|\bar{x}\|^2.$$ (A.3)

Proof. The proof is a direct consequence of Taylor’s theorem for functions of several variables Lang (2006) written with a second order remainder. This theorem ensures that it always exists a point $\xi$ on the line connecting $x$ and $\bar{x}$, if this entirely belongs to the domain $D$, such that

$$f(x + \bar{x}) = f(x) + \frac{\partial f(x)}{\partial x} \bar{x} + \frac{1}{2} \left( \begin{array}{c} x^T H(f_i(\xi)) \bar{x} \\ \bar{x}^T H(f_2(\xi)) \bar{x} \\ \vdots \\ \bar{x}^T H(f_m(\xi)) \bar{x} \end{array} \right).$$

Denoted the last term of the right-hand side with the symbol $r_f$, this can be easily upper bounded as

$$\|r_f(x)\| \leq \sqrt{m} \|r_f(x)\|_\infty = \sqrt{m/2} \max_i \|\bar{x}^T H(f_i(\xi)) \bar{x}\| \leq \sqrt{m/2} \max_i \|H(f_i(\xi))\| \|\bar{x}\|^2 \leq \sqrt{m/2} \max_i \nu_i \|\bar{x}\|^2.$$}

In the following two lemmas, the symbol $e_i$ will denote the unit vector with all zero entries but the $i$-th entry equal to one.

Lemma 3. Given $n$ vector functions $r_j : q \in Q \subseteq \mathbb{R}^n \rightarrow r_j(q) \in \mathbb{R}^m$, $j = 1, \ldots, n$ smooth enough to satisfy hypothesis of Lemma 2 such that

$$\exists \rho_j > 0 : \|r_j(q)\| \leq \rho_j \|q\|^2, \forall q \in Q, j = 1, \ldots, n,$$ (A.4)

then, the $m \times n$ matrix $R(q) = (r_1(q) \cdots r_n(q))$ is such that

$$\|R(q)\| \leq \rho \|q\|^2, \forall q \in Q$$ with $\rho = \sqrt{m} \sum_{j=1}^n \rho_j$. (A.5)

Proof. By definition of infinity norm and Euclidean norm, each column $r_j(q)$ of $R(q)$, in view of (A.4), is such that

$$\|r_j(q)\|_\infty = \max_i e_i^T r_j(q) \leq \|r_j(q)\| \leq \rho_j \|q\|^2, \forall q \in Q.$$ (A.6)

and for all $j = 1, \ldots, n$. Again by definition and in view of (A.6), the infinity norm of the matrix $R(q)$ is such that

$$\|R(q)\|_\infty = \max_{j=1}^n \sum_{i=1}^m \|e_i^T r_j(q)\| \leq \max_{j=1}^n \sum_{i=1}^m \|r_j(q)\|_\infty \leq \sum_{j=1}^n \rho_j \|q\|^2.$$ (A.7)

The claim immediately follows from the well-known property of matrix norms

$$\|R(q)\| \leq \sqrt{m} \|R(q)\|_\infty.$$ (A.8)

Lemma 4. Given $n$ real matrices of dimensions $m \times n$, $A_j = (A_{1j} \cdots A_{nj})^T$, such that

$$\exists \nu_j > 0 : \|A_j\| \leq \nu_j, \quad j = 1, \ldots, n$$ (A.9)

and a real vector $x = (x_1 \cdots x_n)^T$, then

$$\|(A_1 x \cdots A_n x)\| \leq \nu \|x\|,$$ with $\nu = \sqrt{m} \sum_{j=1}^n \nu_j$. (A.10)

Proof. Recalling the definition of the infinity norm of a matrix, it is

$$\|A_j\|_\infty = \max_{k=1}^n \sum_{i=1}^m |e_i^T A_j e_k|,$$ (A.11)

hence, the infinity norm of the matrix $(A_1 x \cdots A_n x)$ is such that

$$\|(A_1 x \cdots A_n x)\|_\infty = \max_{i=1}^n \sum_{k=1}^n |e_i^T A_j e_k| x_k = \max_{i=1}^n \sum_{k=1}^n \sum_{j=1}^n |e_i^T A_j e_k| x_k \leq \max_{i=1}^n \sum_{j=1}^n |e_i^T A_j e_k| \|x\|_\infty \leq \max_{j=1}^n \sum_{k=1}^n |A_j| \|x\|_\infty \leq \sum_{j=1}^n |A_j| \|x\|_\infty \leq \sum_{j=1}^n \rho_j \|q\|^2.$$ (A.11)
\[
\leq \sqrt{n} \sum_{j=1}^{n} \|A_j\|\|x\|_{\infty} \leq \sqrt{n} \sum_{j=1}^{n} \nu_j \|x\|_{\infty}, \quad (A.11)
\]

where assumption (A.8) has been exploited. The claim immediately follows from the well-known properties of matrix and vector norms

\[
\| (A_1 x \ldots A_n x) \| \leq \sqrt{n} \| (A_1 x \ldots A_n x) \|_{\infty}
\]

\[
|\|A_j\|\|x\|_{\infty} \leq \sqrt{n} \sum_{j=1}^{n} \nu_j \|x\|_{\infty}, \quad (A.11)
\]

where assumption (A.8) has been exploited. The following lemma is the basis for proving the convergence of the proposed inverse kinematics algorithm.

**Lemma 5.** Given a vector function defined in a domain \( D \subseteq \mathbb{R}^n \), \( g : z \in D \rightarrow g(z) \in \mathbb{R}^n \) smooth enough to satisfy hypothesis of Lemma 2, thus

\[
g(z + \tilde{z}) = g(z) + \frac{\partial g(z)}{\partial z} \tilde{z} + r_g(z),
\]

where \( r_g \) is such that

\[
\exists \nu_g \geq 0 : \|r_g(z)\| \leq \nu_g \|z\|^2, \quad \forall z \in D
\]

and given a zero mean vector \( x \) of \( n \) random variables, characterised by any type of joint probability density function \( f_X(x) \), with a positive semi-definite covariance matrix \( X \), then the expected value of the random variable \( y = g(x_0 + x) \), being \( x_0 \) any deterministic constant \(^2\), is

\[
y = E[g(x_0 + x)] = g(x_0) + E[r_g(x)]
\]

with \( E[r_g(x)] \| \leq \nu r_g \| X \| \).

**Proof.** By definition, the covariance matrix of the multivariate random variable \( x \) is

\[
X = \int \int \cdots \int (\alpha \alpha^T f_X(\alpha)) d\alpha_1 \ldots d\alpha_n \quad (A.12)
\]

and, in view of its symmetry and positive semi-definiteness, its singular value decomposition can be written in the form

\[
X = U^T \Sigma U,
\]

with \( U \) orthonormal matrix and \( \Sigma = \text{diag}(\sigma_1, \ldots, \sigma_n) = UXU^T \), therefore from (A.12) it follows that

\[
\Sigma = U \int \int \cdots \int (\alpha \alpha^T f_X(\alpha)) d\alpha_1 \ldots d\alpha_n U^T =
\]

\[
= \int \int \cdots \int U (\alpha \alpha^T U^T f_X(\alpha)) U d\alpha_1 \ldots d\alpha_n. \quad (A.14)
\]

By applying the change of variable

\[
U \alpha = \beta \Leftrightarrow \alpha = U^T \beta,
\]

whose jacobian \( U \) has determinant equal to one, the diagonal matrix above can be written as

\[
\Sigma = \int \int \cdots \int (\beta \beta^T f_X(U^T \beta)) d\beta_1 \ldots d\beta_n,
\]

and thus each entry on its diagonal (singular values of the original covariance matrix) is

\[
\sigma_i = \int \int \cdots \int (\beta_i \beta_i^T f_X(U^T \beta)) d\beta_1 \ldots d\beta_n, \quad i = 1, \ldots, n. \quad (A.17)
\]

Now, the claim of the lemma can be easily proved. In fact, the expected value of \( y \) is

\[
y = E[g(x_0 + x)] = E \left[ g(x_0) + \frac{\partial g}{\partial x} \bigg| x_0 \right] (x + r_g(x)) =
\]

\[
= g(x_0) + \frac{\partial g}{\partial x} \bigg|_{x_0} + E[r_g(x)] =
\]

\[
= g(x_0) + E[r_g(x)] \quad (A.18)
\]

and by definition of expected value it is

\[
\|E[r_g(x)]\| \leq E\left( \|g(x_0 + x)\| \right) \leq \nu r_g \| X \|
\]

where the property of the expression of \( g \) recalled in the statement of the Lemma has been exploited along with a standard property of integrals. Now, by applying the same change of variables as in (A.15) from which immediately follows that \( \|\alpha\|^2 = \|\beta\|^2 \), in view of (A.17), the inequality in (A.19) is

\[
E[r_g(x)] \leq \nu r_g \left( \int \cdots \int \|\beta\|^2 f_X(U^T \beta) d\beta_1 \ldots d\beta_n \right) =
\]

\[
= \nu r_g \left( \sum_{i=1}^{n} \int \cdots \int \beta_i^2 f_X(U^T \beta) d\beta_1 \ldots d\beta_n \right) =
\]

\[
\nu r_g \left( \sum_{i=1}^{n} \sigma_i \right) \leq \nu r_g \|\alpha\|^2 \leq \nu r_g \|X\|, \quad (A.20)
\]

having chosen as matrix norm the largest singular value.

The Lemma below, whose proof is straightforward, is a special case of classical results on recurrence inequalities that can be found in (Lakshmikantham and Trigiante, 2002).

**Lemma 6.** Let \( b_k \) be a non-negative sequence satisfying

\[
b_{k+1} \leq \alpha b_k + c, \quad (A.21)
\]

where \( \alpha \) and \( c \) are non-negative real numbers. If \( b_0 \leq a_0 \), being \( a_0 \) the initial condition of the dynamic system

\[
a_{h+1} = \alpha a_h + c, \quad (A.22)
\]

then

\[
b_h \leq a_h, \quad \forall h \geq 0. \quad (A.23)
\]