A New Hot-start Interior-point Method for Model Predictive Control

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Abstract: In typical model predictive control applications, a finite-horizon optimal control problem, in the form of a quadratic program (QP), must be solved at each sampling instant with a known initial state. We present a new hot-start strategy to solve such QPs using interior-point methods, where the first interior-point iterate is constructed from a backward time-shifting of the solution to the QP at the previous time-step. There are two difficulties with such a strategy. First, a naive backward shifting of a previous solution can yield an initial iterate on the boundary of the primal-dual feasible region, leading to blocking of the search direction and consequently to very small and inefficient interior-point steps. Second, a backward shifted solution does not provide a set of strictly feasible terminal KKT conditions. In order to address both of these issues, we propose a modification to the basic backward-shifting method which provides simultaneously an initial iterate that satisfies strict feasibility conditions and a strictly feasible set of primal and dual terminal decision variables. Numerical results indicate that the proposed technique yields convergence in fewer iterations than a cold-start interior-point method.

Keywords: Optimization problems, Interior-point methods, Model predictive control.

1. INTRODUCTION

This paper describes a new hot-start strategy in an interior-point method (IPM) when solving a quadratic programming (QP) problem. In predictive control, at each sampling instant with the given state information, a QP problem is solved to obtain a sequence of inputs and only the first input is applied to the plant. This process is repeated at every sample instant with a new initial state. We consider the scenario in which one QP problem has been solved by an IPM at the previous time instant, and we need to solve a slightly perturbed QP from the preceding QP at the current time instant. A strategy in which the information gained in solving the previous problem is used to help select a starting point in an IPM is known as a hot-start strategy.

In contrast with active set methods, interior-point methods have been developed that permit polynomial-time performance guarantees, beginning with the seminal work of Karmarker (1984). However unlike active set methods, IPMs cannot easily exploit opportunities for algorithmic hot-starting given a solution of a previous problem (Wright, 1997; Gondzio and Grothey, 2008). To improve hot-start strategies in IPMs, a lot of work has been done in the last decade (Yildirim and Wright, 2002; John and Yildirim, 2008; Gondzio and Grothey, 2008). In Yildirim and Wright (2002), two hot-start strategies are presented. In each strategy an adjustment in the solution of preceding problem is made to obtain a starting-point based on the perturbed data of the new problem. These adjustments are calculated using least-squares and Newton method. A worst case estimate on the number of IPM iterations required to converge to a solution of a perturbed linear program from the hot-start starting point is also determined. Their estimate mainly depends on the size of perturbation and on the conditioning of the original problem. John and Yildirim (2008) implemented several hot-start strategies in IPMs for LP problems. They concluded that most of the strategies are effective in reducing the computational time for smaller perturbations. Gondzio and Grothey (2008) proposed a new unblocking strategy based on sensitivity analysis of the search direction with respect to the current point. Numerical tests show that, on average, 50–60% of the computations can be saved on a range of LP and QP problems varying from small scale to large scale problems when this unblocking strategy is combined with other hot-start strategies.

To reduce the computational effort for using predictive control in fast processes with short sampling time, many new techniques have emerged (Bemporad et al., 2002; Ferreau et al., 2008; Wang and Boyd, 2010; Shahzad et al., 2010b). The approach presented in Bemporad, Morari, Dua, and Pistikopoulos (2002), which is mainly applicable...
for small-scale problems, solves a large number of QPs offline for all possible initial states of the plant, and then an explicit function is formed using the solutions of the QPs. The hot start strategy proposed by Ferreau, Bock, and Diehl (2008) is based on an active set method. This technique uses the ideas from parametric optimization and exploits the solution of the previous QP under the assumption that the active set does not change much from one QP to the next one. Furthermore, a premature termination is described to meet the requirements of real-time implementations, which in some cases may lead to infeasibility. The hot-starting and early termination of the QP problem also discussed in Wang and Boyd (2010). For hot-starting, the initialization of the QP problem is done using the predictions made in the previous step. The early termination significantly reduces the computations, but on the other hand it may lead to state equation violations.

In each iteration of an IPM, a linear system of equations needs to be solved to find the search direction. This linear system becomes increasingly ill-conditioned as the solution is approached. In Shahzad et al. (2010a), an approximation of this linear system is proposed which results in a well-conditioned and smaller linear system. Based on this well-conditioned IPM, a hot-start strategy is presented in Shahzad et al. (2010b). In this strategy the focus is on reducing the cost of each iteration of IPM rather than reducing the total number of IPM iterations.

The main contribution of this paper is to present a new hot-start strategy in solving an MPC problem which reduces the IPM iterations substantially. It is well-known that an initial guess of unknown variables which satisfies the feasibility conditions and is also close to the central path where the product of Lagrange and slack variables of the given problem are equal, converges quickly (Wright, 1997, Chapter 11). In order to get such a starting point, the idea of reducing constraint limits along the horizon known as constraint tightening is used in this paper. The constraint tightening approach has been successfully used in robust model predictive control to avoid infeasibility and instability (Gossner et al., 1997; Chisci et al., 2001; Richards and How, 2006).

This paper is organized as follows. We start in Section 2 with an outline of the receding horizon regulator problem with quadratic objective and linear constraints using a discrete-time state space process. The first-order optimality conditions of the receding horizon regulator problem known as Karush–Kuhn–Tucker (KKT) conditions are also described. In Section 3, we describe a new hot-start strategy based on an IPM. In Section 4, numerical results are presented to show the effectiveness of our proposed algorithm. Finally, some conclusions are drawn.

2. PROBLEM DESCRIPTION

Consider the finite-horizon optimal control problem

\[ \min_{(x,u)} \frac{1}{2} \left( x_N^TPx_N + \sum_{i=0}^{N-1} (x_i^TQx_i + u_i^TRu_i) \right) \]

subject to:

\[ x_0 = x \]

\[ -x_{i+1} + Ax_i + Bu_i = 0, \quad \forall i \in \{0, \ldots, N-1\} \]

\[ Cx_i + Du_i \leq d_i, \quad \forall i \in \{0, \ldots, N-1\} \]

\[ Yx_N \leq f, \quad \text{(MPC)} \]

where \( x_i \in \mathbb{R}^n \) is the state vector and \( u_i \in \mathbb{R}^m \) is the input vector at the \( i^{th} \) time instant, \( C \in \mathbb{R}^{l \times n}, D \in \mathbb{R}^{l \times m} \) and \( Y \in \mathbb{R}^{p \times n}. \)

The vectors \( d_i \) are defined such that

\[ d := d_0 > d_1 > d_2 \cdots > d_{N-1} > 0, \]

with their successive differences \( \delta_i \) defined as

\[ \delta_i := d_i - d_{i-1} > 0 \quad \forall i \in \{1, \ldots, N-1\}. \]  \hspace{1cm} (1)

The objective is to find, over a finite horizon of length \( N \) and for a given initial state \( x \), a sequence of optimal control inputs

\[ u^*(x) := (u_0, u_1, \ldots, u_{N-1})^T(x) \]  \hspace{1cm} (2)

and the corresponding state trajectory

\[ x^*(x) := (x_0, x_1, \ldots, x_N)^T(x). \]  \hspace{1cm} (3)

These optimal solutions can be applied to the system in the usual receding-horizon manner.

2.1 Supporting Assumptions

We make the following standard assumptions about the cost function and constraints in problem (MPC):

**Assumption 1.**

a. \( P > 0, R > 0 \) and \( Q \succeq 0. \)

b. The set \( \mathcal{Z} := \{(x,u) \mid Cx + Du \leq d\} \) is compact and contains the origin in its interior.

c. The matrices \( (P, Q, R, K) \) satisfy the following Lyapunov condition:

\[ (A + BK)^\top P(A + BK) - P + Q + K^\top RK \preceq 0. \]  \hspace{1cm} (4)

d. \( \mathcal{X}_f := \{x \mid Yx \leq f\} \) is compact, contains the origin in its interior, and is a constraint admissible positively invariant set under the control law \( u = Kx. \)

2.2 KKT Conditions for Problem (MPC)

There are two standard approaches to solving the finite horizon optimal control problem (MPC). In the first approach, the states are eliminated from the objective function and constraints using the state equations, resulting in a small, but dense Hessian (Maciejowski, 2002). The computational complexity of this approach in each iteration of an IPM is \( \mathcal{O}(l + m)mn^2N^2. \) In the second approach, the state equations are treated as equality constraints and the states are considered as optimization variables, which results in a large, but sparse Hessian. In this formulation, it was shown by Rao et al. (1998) that the computational complexity in each iteration of an IPM can be reduced.
to $O((n + m)^3 + l(m + n)^2)N)$, using a Riccati recursion scheme to solve the resulting linear system. In this paper we follow the second approach, so state variables are treated as decision variables, and the KKT conditions for the associated QP of (MPC) problem are

$$Qx_i - y_i + A^T y_{i+1} + C^T z_i = 0 \quad \forall i \in \{0, \ldots, N - 1\}$$
$$Rx_i + B^T y_{i+1} + D^T z_i = 0 \quad \forall i \in \{0, \ldots, N - 1\}$$
$$Px_N - y_N + Y^T \zeta_N = 0$$

$$x_0 = x$$
$$Ax_i + Bu_i - x_{i+1} = 0 \quad \forall i \in \{0, \ldots, N - 1\}$$
$$Cx_i + Du_i + s_i = d_i \quad \forall i \in \{0, \ldots, N - 1\}$$
$$Yx_N + \zeta_N = f$$

$$(s_i)D(\zeta_i) = 0, \quad (s_i, \zeta_i) \geq 0 \quad \forall i \in \{0, \ldots, N - 1\}$$
$$D(\zeta_i)D(\zeta_N) = 0, \quad (\zeta_N, \zeta_N) \geq 0.$$ 

Remark 5. Note that the dimensions of the path parameters $\hat{\sigma}$ and $\tilde{\sigma}_i$ are not the same. The terms $\tilde{\sigma}_i$ are strictly

3. HOT-START STRATEGY

3.1 Hot-start conditions at the next time step

Given a solution to the KKT conditions for problem (MPC), our goal is to find a strictly interior point for the same problem, but using the new initial condition

$$\tilde{x} = Ax_0 + Bu_0. \quad (5)$$

This point can then be used as a hot-start initial condition for an interior point algorithm for use with an MPC control law. A strictly interior point for this problem will satisfy the following conditions:

$$Q\tilde{x}_i - \tilde{y}_i + A^T \tilde{y}_{i+1} + C^T \tilde{z}_i = 0 \quad \forall i \in \{0, \ldots, N - 1\}$$
$$R\tilde{x}_i + B^T \tilde{y}_{i+1} + D^T \tilde{z}_i = 0 \quad \forall i \in \{0, \ldots, N - 1\}$$
$$P\tilde{x}_N - \tilde{y}_N + Y^T \tilde{\zeta}_N = 0$$

$$\bar{x}_0 = \tilde{x}$$
$$Ax_i + Bu_i - \bar{x}_{i+1} = 0 \quad \forall i \in \{0, \ldots, N - 1\}$$
$$C\bar{x}_i + Du_i + \bar{s}_i = d_i \quad \forall i \in \{0, \ldots, N - 1\}$$
$$Y\bar{x}_N + \bar{\zeta}_N = f$$

$$(\bar{s}_i)D(\bar{\zeta}_i) = \bar{\sigma}_i, \quad (\bar{s}_i, \bar{\zeta}_i) > 0 \quad \forall i \in \{0, \ldots, N - 1\}$$
$$D(\bar{\zeta}_i)D(\bar{\zeta}_N) = \bar{\sigma}_N, \quad (\bar{\zeta}_N, \bar{\zeta}_N) > 0. \quad (6)$$

In the above, the terms $\bar{\sigma}_i$ are positive vectors whose values correspond to the component-wise products of the slack variables and Lagrange multiplier terms. If every element of every one of these vectors were equal, then the above would correspond to a point on the central path.

Remark 3. Throughout, vectors written as $x_i$ denote components of an optimal solution to problem (MPC) with initial condition $x$. Components of a feasible and strictly interior point for (6) are denoted $\bar{x}_i$. In the following section we define a set of ancillary variables to aid the construction of a solution to (6); these variables will be denoted $\tilde{x}_i$.

3.2 Ancillary Control Problem

In order to assist in constructing a hot-start solution that is a strictly interior point, we define ancillary decision variables $\{\tilde{x}_i, \hat{u}_i, \hat{s}_i, \hat{z}_i, \hat{\xi}_N, \hat{\zeta}_N\}$ such that

$$Q\tilde{x}_i - \tilde{y}_i + A^T \tilde{y}_{i+1} + C^T \tilde{z}_i = 0 \quad \forall i \in \{1, \ldots, N - 1\}$$
$$R\tilde{x}_i + B^T \tilde{y}_{i+1} + D^T \tilde{z}_i = 0 \quad \forall i \in \{1, \ldots, N - 1\}$$
$$P\tilde{x}_N - \tilde{y}_N + \tilde{\zeta}_N = 0$$

$$\hat{x}_1 = 0$$
$$A\hat{x}_i + B\hat{u}_i - \hat{x}_{i+1} = 0 \quad \forall i \in \{1, \ldots, N - 1\}$$
$$\hat{x}_N = 0$$
$$C\hat{x}_i + D\hat{u}_i + \hat{s}_i = d_i \quad \forall i \in \{1, \ldots, N - 1\}$$
$$D(\hat{s}_i)D(\hat{\zeta}_i) = 1\hat{\sigma}_i, \quad (\hat{s}_i, \hat{z}_i) > 0 \quad \forall i \in \{1, \ldots, N - 1\} \quad (7)$$

Remark 4. The ancillary decision variables $\{\tilde{x}_i, \hat{u}_i, \hat{s}_i, \hat{z}_i\}$ define a strictly interior point, at a point along the central path parameterized by the scalar value $\hat{\sigma}$, for the ancillary control problem:

$$\min_{(x, u)} \frac{1}{2} \left( \tilde{x}_N^T P\tilde{x}_N + \sum_{i=1}^{N}(\tilde{x}_i^T Q\tilde{x}_i + \hat{u}_i^T R\hat{u}_i) \right)$$

subject to:
$$\tilde{x}_i = 0$$
$$-\tilde{x}_{i+1} + A\tilde{x}_i + B\hat{u}_i = 0, \quad \forall i \in \{1, \ldots, N - 1\}$$
$$C\tilde{x}_i + D\hat{u}_i \leq \hat{d}_i, \quad \forall i \in \{1, \ldots, N - 1\} \quad (Anc)$$

where

$$\hat{u} := (\hat{u}_1, \hat{u}_2, \ldots, \hat{u}_{N-1}),$$
$$\tilde{x} := (x_1, x_2, \ldots, x_N).$$

Note that unlike the original MPC problem (MPC), the ancillary problem (Anc) is defined over the interval $\{1, \ldots, N\}$. This is because the ancillary decision variables will be used to construct a strictly feasible interior point for an IP algorithm based on a backward shifting of the previous solution. The variables corresponding to time 0 are not required.

Unlike the nominal problem (MPC), the ancillary problem defined above has a terminal equality constraint $\tilde{x}_N = 0$, and a corresponding equality multiplier $\tilde{\psi}_N$. The reason for this is so that the backward shifting procedure described in the next section will work correctly when one defines

$$\tilde{x}_N \leftarrow (A + BK)(x_N + \tilde{x}_N)$$

and tries to ensure that $\tilde{x}_N \in \mathcal{X}_f$.

Remark 5. Note that the dimensions of the path parameters $\hat{\sigma}$ and $\hat{\sigma}_i$ are not the same. The terms $\hat{\sigma}_i$ are strictly
positive vectors, where each element of each vector is (potentially) different. In the ancillary problem (Anc) the corresponding values are equal to the scalar \( \hat{\sigma} \) throughout, i.e. a solution to the ancillary control problem’s conditions will represent a point on the central path.

### 3.3 Supporting Results

**Stiemke’s Theorem and Compact Sets** The following results will prove useful in specifying hot-start conditions with constraint-admissible terminal conditions:

**Theorem 6.** (Stiemke’s Theorem). For any \( A \in \mathbb{R}^{m \times n} \), exactly one of the following conditions holds:

A) \( \exists z > 0 \) such that \( A^T z = 0 \) or

B) \( \exists x \in \mathbb{R}^n \) such that \( Ax \leq 0 \) and \( Ax \neq 0 \).

**Remark 7.** The above theorem comes from Stiemke (1915), though a more accessible modern reference is Broyden (2001). The wording above is identical to that in Broyden (2001) with the (trivial) exception that the inequality in (B) is reversed.

**Lemma 8.** Suppose that \( A \in \mathbb{R}^{m \times n} \) and the set

\[
S := \{ x \mid Ax \leq b \}
\]

is compact with \( 0 \in \text{int} S \). Then for every \( b \in \mathbb{R}^n \) there exists \( z \in \mathbb{R}^m \) such that

\[
A^T z = b, \quad z > 0.
\]

**Proof.** First consider whether the linear system \( A^T z = b \) is solvable:

Since \( S \) is compact, it follows that there does not exist any \( x \neq 0 \) such that \( Ax = 0 \), otherwise \( S \) would contain the line \( \{ \lambda x \mid \lambda \in \mathbb{R} \} \). Therefore \( \mathcal{N}(A) = \{ 0 \} \) and consequently \( \mathcal{R}(A^T) = \mathbb{R}^n \). This proves that there exists some \( z \in \mathbb{R}^m \) such that \( A^T z = b \).

Next consider whether there exists some \( \pi > 0 \) such that \( A^T \pi = 0 \):

Following a similar line of reasoning as above the compactness of \( S \), there does not exist any \( x \neq 0 \) such that \( Ax \leq 0 \). Therefore Stiemke’s theorem guarantees the existence of \( \pi > 0 \) such that \( A^T \pi = 0 \).

The proof is completed by choosing \( z = \hat{z} + \alpha \pi \) with \( \alpha > 0 \) large enough so that

\[
A^T (\hat{z} + \alpha \pi) = b, \quad (\hat{z} + \alpha \pi) > 0.
\]

### 3.4 Constructing a new strictly interior point

**Shifted Variables** In order to construct a new feasible and strictly interior point for (6), we first make the following backward-shifting substitutions based on the previous optimal solution and the solution to the ancillary problem (Anc).

\[
\tilde{u}_0, \ldots, \tilde{u}_{N-2} \leftarrow \{ u_1, \ldots, u_{N-1} \} + \{ \tilde{u}_1, \ldots, \tilde{u}_{N-1} \}
\]
\[
\tilde{x}_0, \ldots, \tilde{x}_{N-1} \leftarrow \{ x_1, \ldots, x_N \} + \{ \tilde{x}_1, \ldots, \tilde{x}_N \}
\]
\[
\tilde{y}_0, \ldots, \tilde{y}_{N-1} \leftarrow \{ y_1, \ldots, y_N \} + \{ \tilde{y}_1, \ldots, \tilde{y}_N \} \quad (8)
\]
\[
\tilde{z}_0, \ldots, \tilde{z}_{N-2} \leftarrow \{ z_1, \ldots, z_{N-1} \} + \{ \tilde{z}_1, \ldots, \tilde{z}_{N-1} \}
\]
\[
\tilde{s}_0, \ldots, \tilde{s}_{N-2} \leftarrow \{ s_1, \ldots, s_{N-1} \} + \{ \tilde{s}_1, \ldots, \tilde{s}_{N-1} \}
\]

This leaves the following variables still unassigned:

\[
(\tilde{u}_{N-1}, \tilde{x}_N) : \text{ new control move and terminal state}
\]
\[
(\tilde{y}_N) : \text{ new terminal state equality multiplier}
\]
\[
(\tilde{z}_{N-1}, \tilde{s}_{N-1}) : \text{ cannot be determined by shift due to dimension mismatch}
\]
\[
(\tilde{z}_N, \tilde{s}_N) : \text{ new terminal inequality multipliers.}
\]

Our goal is to assign these variables such that we get an IP-solver initializer that is a strictly interior point.

An obvious choice for the pair \((\tilde{u}_{N-1}, \tilde{x}_N)\) is to use the control/state that would result from application of the terminal control law \( u = Kx \), i.e.

\[
(\tilde{u}_{N-1}, \tilde{x}_N) \leftarrow \left( K\tilde{x}_{N-1}, (A + BK)\tilde{x}_{N-1} \right)
\]

\[
\equiv \left( (Kx_N + \hat{x}_N)(A + BK)(x_N + \hat{x}_N) \right).
\]

Note that if a terminal equality constraint is imposed, then the term \( \hat{x}_N \) is not required since it is zero anyway. However if a terminal inequality constraint is used instead then it is required; cf. the comments towards the end of Remark 4.

**Substitution of Variables** Substitution of terms from (8) into (6) gives the following set of conditions for all \( i \in \{ 0, \ldots, N-2 \} \):

\[
Q(x_{i+1} + \tilde{x}_{i+1}) - (y_{i+1} + \tilde{y}_{i+1}) + A^T (y_{i+2} + \tilde{y}_{i+2}) + C^T (z_{i+1} + \tilde{z}_{i+1}) = 0
\]
\[
R(u_{i+1} + \tilde{u}_{i+1}) + B^T (y_{i+2} + \tilde{y}_{i+2}) + D^T (x_{i+1} + \tilde{x}_{i+1}) = 0
\]
\[
Q(x_N + \hat{x}_N) - (y_N + \hat{y}_N) + A^T \hat{y}_N + C^T \hat{z}_N = 0
\]
\[
RK_N x_N + B^T \hat{y}_N + D^T \hat{z}_N = 0
\]
\[
P(A + BK)x_N - \hat{y}_N + Y^T \tilde{z}_N = 0
\]
\[
A(x_{i+1} + \tilde{x}_{i+1}) + B(u_{i+1} + \tilde{u}_{i+1}) - (s_{i+2} + \tilde{s}_{i+2}) = 0
\]
\[
Ax_N + BK_N x_N - (A + BK_N)x_N = 0
\]
\[
C(x_{i+1} + \tilde{x}_{i+1}) + D(u_{i+1} + \tilde{u}_{i+1}) + (s_{i+1} + \tilde{s}_{i+1}) \leq (d_{i+1} + \tilde{d}_{i+1})
\]
\[
C\tilde{s}_{N-1} + D\tilde{u}_{N-1} + \tilde{s}_{N-1} = d_{N-1}
\]
\[
Y\tilde{x}_N + \tilde{z}_N = f
\]

**Strictly interior end conditions** It should be obvious that, after simple regrouping of terms, all of the equations above that do not contain unassigned shifted terms are satisfied.

This leaves:

\[
Q\tilde{x}_{N-1} - \tilde{y}_{N-1} + A^T \tilde{y}_N + C^T \tilde{z}_{N-1} = 0 \quad (10a)
\]
\[
R\tilde{u}_{N-1} + B^T \tilde{y}_N + D^T \tilde{z}_{N-1} = 0 \quad (10b)
\]
\[
P\tilde{x}_N - \tilde{y}_N + Y^T \tilde{z}_N = 0 \quad (10c)
\]
\[
C\tilde{x}_{N-1} + D\tilde{u}_{N-1} + \tilde{s}_{N-1} = d_{N-1} \quad (10d)
\]
\[
Y\tilde{x}_N + \tilde{z}_N = f \quad (10e)
\]

where values for \((\tilde{x}_N, \tilde{y}_N)\) are known, and the remaining values \((\tilde{z}_{N-1}, \tilde{z}_N, \tilde{y}_N, \tilde{s}_{N-1}, \tilde{z}_N)\) are yet to be assigned.

In order to ensure the preceding conditions can be satisfied at a strictly interior point, we require the following further assumptions about the terminal conditions and constraints in (MPC):
Assumption 9. (Contractiveness). The constraints and terminal conditions of problem (MPC) satisfy:

- There exists $\alpha < 1$ such that $(A + BK)x_f \subset \alpha x_f$.
- $(Cx, Kx) \in \text{interior}(Z)$ for all $x \in X_f$.

By virtue of Assumption 9, it follows that (10d) and (10e) are satisfied with $\bar{s}_{N-1} > 0$ and $\bar{\xi}_N > 0$. We are also free to select $\bar{y}_N = 0$ (actually any choice of $\bar{y}_N$ will work). This leaves three equations to be satisfied via appropriate choice of $\bar{(z)_N, \bar{z}_{N-1}}$, with strict positivity constraints $\bar{(z)_N, \bar{z}_{N-1}} > 0$:

\[
\begin{bmatrix} C^T \\ D \end{bmatrix} \bar{z}_{N-1} = \begin{bmatrix} \bar{y}_{N-1} - Q\bar{x}_{N-1} \\ -B^T\bar{y}_N - R\bar{u}_{N-1} \end{bmatrix} \quad (11a) \\
Y^T\bar{z}_N = -P\bar{x}_N \quad (11b)
\]

The unknowns $\bar{z}_{N-1}$ and $\bar{z}_N$ can be obtained by solving a linear program (LP) of the form

\[
\min_{\bar{z}_{N-1}, \bar{z}_N} \bar{z}_{N-1}^T1_z + \bar{z}_N^T1_\zeta \\
\text{subject to:} \ (\bar{z}_{N-1}, \bar{z}_N) > 0 \text{ and } (11).
\]

where $1_z$ and $1_\zeta$ are vectors of ones with appropriate dimensions. In the above formulation, we have treated $\bar{y}_N$ as fixed; alternatively, we can solve a slightly different LP in which $\bar{y}_N$ is an unconstrained decision variable.

Since the sets $X_f$ and $Z$ are both assumed compact, it follows from Lemma 8 that both of these conditions can be satisfied with $\bar{(z)_N, \bar{z}_{N-1}} > 0$.

Remark 10. In constructing the ancillary variables $(\bar{x}_i, \bar{u}_i)$, it was assumed that a point exactly on the central path was required. This reduces flexibility in the choice of multipliers $\{z_i\}$ and $\zeta_N$, and thereby can lead to non-zero values for $(x_i, u_i, y_i)$.

If a central path solution is not required, then we are free to choose $(x_i, u_i, y_i) = 0$, and then choose strictly positive values $(\{z_i\}, \zeta_N)$; this is again made possible due to Lemma 8.

4. NUMERICAL RESULTS

Consider the DT LTI system

\[
x^+ = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 1/2 \\ 1 \end{bmatrix},
\]

with state and input constraints

\[
(x, u) \in Z = \{(x, u) \mid -8 \leq x \leq 2, |u| \leq 1\}.
\]

In order to define an MPC controller for this system, choose a horizon length $N = 15$ and select $Q = I$, $R = 1$ and

\[
P = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}.
\]

A terminal controller $K = (-0.4 -1)$ is selected such that Lyapunov condition (4) is satisfied.

In order to implement our hot-starting procedure, we impose in our MPC problem a time-varying tightening of the state/input constraint $(x, u) \in Z_i := Z \cdot (0.98)^i$ with an associated terminal constraint

\[
x_N \in X_f = \begin{bmatrix} 1 & 0 \\ -0.3714 & -0.9285 \\ 0.3714 & 0.9285 \\ -0.9635 & -0.2676 \end{bmatrix} x \leq \begin{bmatrix} 1.3573 \\ 0.5048 \\ 1.5489 \\ 1.9514 \end{bmatrix}
\]

that is positively invariant with respect to the constraint set $Z_{N-1}$ under the controller $u = K x$. It also satisfies the condition $(A + BK)x_f \subset \alpha x_f$ where $\alpha = 0.9$.

We assume that the state of the system is initialized at $x = [1 - 4]^T$, and implement our control law both with and without hot-starting. Since the initial state estimate $\hat{x}$ could be different from the predicted one given by (5), therefore we have considered an uncertainty $\Delta \hat{x}$ in $\hat{x}$. It is assumed that $||\Delta \hat{x}||_\infty \leq 0.5$. To evaluate the performance of the proposed hot-start strategy simulations are carried out with and without this uncertainty, which is generated randomly with uniform distribution. The state and input trajectories are shown in Fig. 1 and Fig. 2, respectively.

The state and input trajectories are identical for cold start and hot-start without uncertainty, but different with uncertainty, because at each time instant a randomly generated uncertainty is added in the initial state estimate $\hat{x}$. Fig. 3 shows that the number of interior point iterations required in the hot-start strategy is reduced substantially.

5. CONCLUSIONS

We have presented a new hot-start strategy based on an interior-point method to solve a sequence of QPs arising in model predictive control. It is shown that the proposed scheme reduces the number of interior-point iterations considerably.
Fig. 2. The input trajectories of cold-start and hot-start IPMs.

Fig. 3. The number of IPM iterations with hot-start are significantly reduced when compared to the cold-start IPM iterations.

There are a number of parameters that affect the convergence of the interior point algorithm under hot-starting. Further investigations can be made to look at:

i) The central path parameter \( \hat{\sigma} \) used to define the ancillary solution.

ii) The minimum value allowed for the shifted variables \( \tilde{z}_{\mathbf{N-1}} \) and \( \zeta_{\mathbf{N}} \).

Furthermore, to improve the performance of the proposed strategy when the initial state estimate is different from the predicted one, adjustments in the starting-point can be made according to the disturbance in initial state. Such corrections can be computed by a Newton method as mentioned by Yildirim and Wright (2002). The computational cost of computing these corrections is roughly equal to the one IPM iteration.

REFERENCES


