Modeling and Control of a Nonuniform Vibrating String under Spatiotemporally Varying Tension and Disturbance

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Abstract: In this paper, robust adaptive boundary control is developed for a class of flexible string systems under unknown spatiotemporally varying distributed disturbance and time-varying boundary disturbance. The vibrating string is nonuniform since the spatiotemporally varying tension applied to the system. By using Hamilton’s principle, the nonuniform vibrating string system is represented by a nonlinear nonhomogeneous partial differential equation (PDE) and two ordinary differential equations (ODEs). Robust adaptive boundary control is developed to suppress the vibration of the flexible nonuniform string and compensate the system parameter uncertainties and the disturbance uncertainties. With the proposed control, the state of the string system is proven to converge to a small neighborhood of zero by choosing the design parameters. Simulations are provided to illustrate the effectiveness of the proposed control.

1. INTRODUCTION

In modern mechanical engineering, a large number of flexible systems, How et al. [2009], Ge et al. [1998, 2010], Tee and Ge [2006], such as conveyor belts, cables and chains can be modeled as string-based structures. These flexible systems exhibit vibration in the presence of disturbances. Since the excessive vibration creates unwanted noises, reduces the system quality, leads to limited productivity and results in premature fatigue failure, vibration suppression is well motivated to improve the performance of the system. However, taking into account the unknown spatiotemporally varying disturbance leads to the appearance of oscillations and the spatiotemporally varying tension makes the system nonuniform. Thus, the control problem become more difficult.

From a mathematical point of view, a system with vibration is often considered as a original distribution parameter system (DPS). These systems cannot be modeled by ODEs since the motion of such systems is described by variables depending on both time and space Logan [2006]. The dynamics of DPSs could be described by a hybrid dynamic model involving both PDE and ODE, i.e., PDE representing the dynamics of the string and ODEs representing the dynamics of the lumped tip payload. Such flexible mechanical system is difficult to control due to the infinite dimensionality of the system. A common modeling method for DPSs is based on discretization of the PDE into a finite set of ODEs since there are many control techniques developed for ODE systems. However, the finite dimensional discrete models are approximated by neglecting high order modes, which would result in spillover instability. In order to avoid the spillover phenomenon, boundary control has been developed for various infinite dimensional systems.

Compared with distributed controllers, boundary control is a more practical method to control PDSs. Fewer actuators and sensors located at the boundary of the system are able to satisfy the demands. The boundary control can also be derived from a Lyapunov function which is relevant to the mechanical energy based on the dynamics of the system. Due to the advantages over other control schemes, boundary control has received much attention among the research area. Recent progress in the boundary control is summarized in Belishev [2007]. In Queiroz and Rahn [2002], an overview on the boundary control for DPSs is introduced. Boundary control based on Lyapunov techniques are developed in Queiroz et al. [2000b] to stabilize the vibration. Semi-group theory of the boundary control techniques is investigated in Curtain and Zwart [1995]. Boundary control design for string-based structures has also made a great deal of progress. Authors in Yang et al. [2004] propose a robust adaptive boundary control to reduce the vibration for a moving string with the spatiotemporally varying tension. A PR transfer function in Yang et al. [2005] is used to suppress the vibration of a translating string. In Li et al. [2008], a linear boundary velocity feedback control is designed to ensure exponential stabilization of a nonlinear moving string. An axially moving string with nonlinear behavior resulting from spatially varying tension is investigated in Nguyen and Hong [2010]. In Rahn et al. [1999], a boundary controller for a linear gantry crane model with flexible string-type cable is developed and experimentally implemented.

However, in all these papers, the control schemes have been designed with neglecting the unknown distributed disturbance. After considering the distributed disturbance, the string system is governed by a nonhomogeneous PDE. Additional, a constant axial tension and mass per unite length are assumed in most of papers mentioned above. From a practical point of view, many string systems have a varying tension and a varying mass per unite length which make the string system nonuniform.
2. PROBLEM FORMULATION AND PRELIMINARIES

2.1 Dynamic analysis

Fig. 1 shows a string-based structure extracted from a class of flexible systems. The left boundary of the string is fixed at origin. System parameters and variables are defined as follows:

\[
\begin{align*}
L & \quad \text{Length of the string} \\
M & \quad \text{Mass of the payload} \\
\rho(x) & \quad \text{Mass per unit length of the string at the position } x \\
T(x, t) & \quad \text{Tension of the string at the position } x \text{ for time } t \\
w(x, t) & \quad \text{Displacement of the string at the position } x \text{ for time } t \\
d(x, t) & \quad \text{Unknown time-varying right boundary} \text{ disturbance on the tip payload} \\
f(x, t) & \quad \text{Unknown spatiotemporally varying distributed} \text{ disturbance along the string} \\
u(t) & \quad \text{Boundary control force}
\end{align*}
\]

**Remark 1.** For clarity, notions \((\cdot)' = \frac{\partial (\cdot)}{\partial x}\) and \((\cdot) = \frac{\partial (\cdot)}{\partial t}\) are used in this paper.

When \(\rho(x)\) and \(T(x, t)\) are not constants, dynamic equations of the nonuniform string system in Fig.1 can be derived by using Hamilton’s principle Goldstein [1951]. Hamilton’s principle is represented by

\[
\int_{t_1}^{t_2} [\delta E_k(t) - \delta E_p(t) + \delta W(t)] dt = 0, \tag{1}
\]

where \(\delta\) denotes the variational operator, \(t_1\) and \(t_2\) are two time instants and \(t_1 < t < t_2\) is the operating interval.

The kinetic energy \(E_k(t)\) can be represented as

\[
E_k(t) = \frac{1}{2} M \left[ \dot{w}(L, t) \right]^2 + \frac{1}{2} \int_0^L \rho(x) \left[ \dot{w}(x, t) \right]^2 dx, \tag{2}
\]

where \(x\) and \(t\) represent the independent spatial and time variables respectively.

The potential energy \(E_p(t)\) due to a spatiotemporally varying tension \(T(x, t)\) can be obtained from

\[
E_p(t) = \frac{1}{2} \int_0^L T(x, t) \left[ w'(x, t) \right]^2 dx, \tag{3}
\]

where the tension \(T(x, t)\) of the string can be expressed as

\[
T(x, t) = T_0(x) + \lambda(x) [w'(x, t)]^2, \tag{4}
\]

where \(T_0(x) > 0\) is the initial tension, and \(\lambda(x) \geq 0\) is the nonlinear elastic modulus Qu [2001].

The virtual work done by the external force including the spatiotemporally varying distributed disturbance \(f(x, t)\) on the string, time-varying boundary disturbance \(d(t)\) and boundary control force \(u(t)\) on the tip payload is given by

\[
\delta W(t) = \int_0^L f(x, t) \delta w(x, t) dx + [d(t) + u(t)] \delta w(L, t). \tag{5}
\]

Applying Hamilton’s principle \((1)\), we obtain the governing equation of the nonuniform string system as

\[
\rho(x) \ddot{w}(x, t) - \left\{ T(x, t) + 3\lambda(x) [w'(x, t)]^2 \right\} w''(x, t) = f(x, t), \quad \forall (x, t) \in (0, L) \times [0, \infty), \tag{6}
\]

with \(\lambda(x) \geq 0\).

**Remark 2.** With consideration of the unknown spatiotemporally varying distributed disturbance \(f(x, t)\) and tension \(T(x, t)\), the governing equation of the nonuniform string system Eq. \((6)\) is described as a nonlinear nonhomogeneous PDE.

**Assumption 1.** For the unknown disturbances \(f(x, t)\) and \(d(t)\), we assume that there exists constants \(f, d \in R^+\), such that \(|f(x, t)| \leq f, \forall (x, t) \in (0, L) \times [0, \infty)\) and \(|d(t)| \leq d, \forall t \in [0, \infty)\).

**Remark 3.** This is a reasonable assumption as the disturbances \(f(x, t)\) and \(d(t)\) have finite energy and hence are bounded, i.e., \(f(x, t) \in L_2((0, L)]\) and \(d(t) \in L_2.\) For the robust adaptive control, the knowledge of the exact values for \(f(x, t), d(t), f\) and \(d\) is not required.

**Assumption 2.** We assume that \(\rho(x), T_0(x)\) and \(\lambda(x)\) are bounded by known, constant lower and upper bounds as follows:

\[
\underline{\rho} \leq \rho(x) \leq \bar{\rho}, \quad \underline{T_0} \leq T_0(x) \leq \bar{T_0}, \quad \underline{\lambda} \leq \lambda(x) \leq \bar{\lambda}. \tag{9,10,11}
\]

2.2 Preliminaries

For the convenience of stability analysis, we present the following lemmas and properties for the subsequent development.

**Lemma 1.** Rahn [2001] Let \(\phi_1(x, t), \phi_2(x, t) \in R\) with \(x \in [0, L]\) and \(t \in [0, \infty)\), the following inequalities hold:

\[
|\phi_1 \phi_2| = \left| \int_0^1 \phi_1 \phi_2 \right| \leq \frac{1}{\delta} \phi_1^2 + \phi_2^2, \tag{12}
\]

\[
\forall \phi_1, \phi_2 \in R \quad \text{and} \quad \delta > 0.
\]
Lemma 2. Hardy et al. [1959] Let \( \phi(x, t) \in \mathcal{R} \) be a function defined on \( x \in [0, L] \) and \( t \in [0, \infty) \) that satisfies the boundary condition

\[
\phi(0, t) = 0, \quad \forall t \in [0, \infty),
\]

then the following inequalities hold:

\[
\phi^2 \leq L \int_0^L |\phi'|^2 dx, \quad \forall x \in [0, L].
\]  

Property 1. Queiroz et al. [2000a]: If the kinetic energy of the system (6) - (7), given by Eq. (2) is bounded \( \forall t \in [0, \infty) \), then \( \tilde{w}(x, t), \tilde{w}'(x, t) \) and \( \tilde{w}''(x, t) \) are bounded \( \forall (x, t) \in [0, L] \times [0, \infty) \).

Property 2. Queiroz et al. [2000a]: If the potential energy of the system (6) - (7), given by Eq. (3) is bounded \( \forall t \in [0, \infty) \), then \( w'(x, t) \) and \( w''(x, t) \) are bounded \( \forall (x, t) \in [0, L] \times [0, \infty) \).

3. CONTROL DESIGN

To stabilize the system given by governing Eq. (6) and boundary conditions Eqs. (7) and (8), the following adaptive boundary control is proposed

\[
u(t) = -k[\tilde{w}(L, t) + w'(L, t)] - \hat{M}_s(t)\tilde{w}'(L, t) + \hat{T}_0(L, t)\tilde{w}(L, t) - \hat{d}(t),
\]

where \( k \) is the control gains, \( \hat{M}_s(t), \hat{T}_0(L, t) \) and \( \hat{d}(t) \) are the estimate of \( M_s(L), T_0(L) \) and \( d \) respectively. Parameter estimate errors \( \hat{M}_s(t), \hat{T}_0(L, t) \) and boundary disturbance estimate error \( \hat{d}(t) \) are defined as

\[
\hat{M}_s(t) = M_s - M_s(t),
\]

\[
\hat{T}_0(L, t) = T_0(L) - \hat{T}_0(L, t),
\]

\[
\hat{d}(t) = d - \hat{d}(t).
\]

The adaptation laws are designed as

\[
\dot{\hat{M}}_s(t) = \gamma_m \hat{w}(L, t)[\tilde{w}(L, t) + w'(L, t)] - \zeta_m \gamma_m \hat{M}_s(t),
\]

\[
\dot{\hat{T}}_0(L, t) = -\gamma_t \hat{w}(L, t)[\tilde{w}(L, t) + w'(L, t)] - \zeta_t \gamma_t \hat{T}_0(L, t),
\]

\[
\dot{\hat{d}}(t) = \beta \hat{d}(L, t)[\tilde{w}(L, t) + w'(L, t)] - \zeta_d \gamma_d \hat{d}(t),
\]

where \( \gamma_m, \zeta_m, \gamma_t, \zeta_t, \gamma_d \) and \( \zeta_d \) are positive constants.

We choose the Lyapunov function candidate as

\[
V(t) = V_1(t) + V_2(t) + \eta(t) + \frac{1}{2} \gamma_m^{-1} \hat{M}_s^2(t) + \frac{1}{2} \gamma_t^{-1} \hat{T}_0^2(L, t) + \frac{1}{2} \gamma_d^{-1} \hat{d}^2(t),
\]

where the energy term \( V_1 \), the auxiliary term \( V_2 \) and the small crossing term \( \eta \) are defined as

\[
V_1 = \frac{\beta}{2} \int_0^L \rho(x) [\tilde{w}]^2 dx + \frac{\beta}{2} \int_0^L T(x, t)[w']^2 dx,
\]

\[
V_2 = \frac{\beta}{2} \int_0^L \rho(x)[\tilde{w}]^2 dx + \frac{\beta}{2} \int_0^L T_0(x)[w']^2 dx + \frac{\beta}{2} \int_0^L \lambda(x)[w']^4 dx,
\]

\[
V_3 = \frac{\beta}{2} \int_0^L \rho(x)[\tilde{w}]^2 dx + \frac{\beta}{2} \int_0^L T_0(x)[w']^2 dx,
\]

\[
\eta = \alpha \int_0^L \rho(x) \varphi(x) \tilde{w}(x, t) w'(x, t) dx,
\]

where \( \alpha \) and \( \beta \) are two positive weighting constants, \( \varphi(x) \) is a positive scalar function bounded by a known constant, i.e., \( \varphi(x) \leq \varphi \).

Lemma 3. The Lyapunov function candidate given by Eq. (23) is bounded as

\[
0 \leq \mu_1(V_1 + V_2 + \hat{M}_s^2(t) + \hat{T}_0^2(L, t) + \hat{d}^2(t)) \leq V \leq \mu_2(V_1 + V_2 + \hat{M}_s^2(t) + \hat{T}_0^2(L, t) + \hat{d}^2(t)),
\]

where \( \mu_1 \) and \( \mu_2 \) are two positive constants defined as

\[
\mu_1 = 1 - \frac{2 \alpha L \rho \varphi}{\min(\beta \rho, \beta T_0)} > 0,
\]

\[
\mu_2 = 1 + \frac{2 \alpha L \rho \varphi}{\min(\beta \rho, \beta T_0)} > 0,
\]

provided

\[
0 < \alpha < \frac{\min(\beta \rho, \beta T_0)}{2 L \rho \varphi}.
\]

Proof: Since \( x \leq L \), using Ineq. (9) and substituting of Ineq. (12) into Eq. (26) yields

\[
[\eta(t)] \leq \alpha L \rho \varphi \int_0^L (\tilde{w}^2 + w') dx \leq \alpha_1 V_1(t),
\]

where

\[
\alpha_1 = \frac{2 \alpha L \rho \varphi}{\min(\beta \rho, \beta T_0)}.
\]

Then, we obtain

\[
-\alpha_1 V_1 \leq \eta \leq \alpha_1 V_1.
\]

Considering a positive constant, \( 0 < \alpha < \frac{\min(\beta \rho, \beta T_0)}{2 L \rho \varphi} \), we have \( 0 < \alpha_1 < 1 \), and

\[
0 < \alpha_2 = 1 - \alpha_1 = 1 - \frac{2 \alpha L \rho \varphi}{\min(\beta \rho, \beta T_0)} < 1,
\]

\[
1 < \alpha_3 = 1 + \alpha_1 = 1 + \frac{2 \alpha L \rho \varphi}{\min(\beta \rho, \beta T_0)} < 1.
\]

We further obtain

\[
0 \leq \alpha_2(V_1 + V_2) \leq V_1 + V_2 + \eta \leq \alpha_3(V_1 + V_2).
\]

Given the Lyapunov function candidate Eq. (23), we obtain

\[
0 \leq \mu_1(V_1 + V_2 + \hat{M}_s^2(t) + \hat{T}_0^2(L, t) + \hat{d}^2(t)) \leq V \leq \mu_2(V_1 + V_2 + \hat{M}_s^2(t) + \hat{T}_0^2(L, t) + \hat{d}^2(t)),
\]
\[
\mu_1 = \min(\alpha_2, \frac{1}{2\gamma_m}, \frac{1}{2\gamma_t}, \frac{1}{2\gamma_d}), \quad (38)
\]
\[
\mu_2 = \max(\alpha_3, \frac{1}{2\gamma_m}, \frac{1}{2\gamma_t}, \frac{1}{2\gamma_d}), \quad (39)
\]
are two positive constants. \[ \]

Lemma 4. The time derivative of the Lyapunov function candidate Eq. (23) can be upper bounded with

\[
\dot{V} \leq -\mu V + \psi, \quad (40)
\]
where \( \mu > 0 \) and \( \psi > 0 \).

Proof: Differentiating Eq. (23) with respect to time leads to

\[
\dot{V} = \dot{V}_1 + \dot{V}_2 + \dot{\eta} + \gamma^{-1}_m \dot{M}_s(t) \dot{M}_s(t) + \gamma^{-1}_d \dot{d}(t) \dot{d}(t)
\]
\[
+ \gamma^{-1}_t \dot{T}_0(L, t) \dot{T}_0(L, t), \quad (41)
\]
The first term of the Eq. (41) is equal to

\[
\dot{V}_1(t) = A_1 + A_2 + A_3, \quad (42)
\]
where

\[
A_1 = \beta \int_0^L \rho(x) \dot{w} \dot{w} dx, \quad (43)
\]
\[
A_2 = \frac{\beta}{2} \int_0^L \dot{T}(x, t) [\dot{w}']^2 dx, \quad (44)
\]
\[
A_3 = \beta \int_0^L T(x, t) \dot{w} \dot{w} dx. \quad (45)
\]

Substituting the governing equation (6) into \( A_1 \), and using integration by parts and substituting the boundary condition Eq. (8), we obtain

\[
A_1 = \beta T(L, t) \dot{w}(L, t) \dot{w}'(L, t) - \beta \int_0^L T(x, t) \dot{w} \dot{w} dx
\]
\[
+ \beta \lambda(L) \dot{w}(L, t) \dot{w}'(L, t)^3 - \beta \int_0^L \lambda(x) \dot{w}'(L, t)^3 \dot{w} dx
\]
\[
+ \beta \int_0^L \dot{w} \dot{w} dx. \quad (46)
\]

Since \( \dot{T}(x, t) = 2\lambda(x) \dot{w}'(x, t) \dot{w}'(x, t) \), then we have

\[
A_2 = \beta \int_0^L \lambda(x) \dot{w}'(L, t)^3 \dot{w} dx. \quad (47)
\]

Substituting Eqs. (46), (47) and (45) into Eq. (42), we obtain

\[
\dot{V}_1 = \beta T(L, t) \dot{w}(L, t) \dot{w}'(L, t) + \beta \lambda(L) \dot{w}(L, t) \dot{w}'(L, t)^3
\]
\[
+ \beta \int_0^L \dot{w} \dot{w} dx. \quad (48)
\]

Substituting Eq.(4) and using Ineq. (13) in Lemma 1, we obtain

\[
\dot{V}_1(t) \leq \frac{\beta T_0(L)}{2} [\dot{w}(L, l) + \dot{w}'(L, t)]^2 - \frac{\beta T_0(L)}{2} [\dot{w}(L, t)]^2
\]
\[
- \beta \int_0^L \dot{w} \dot{w} dx + \beta \int_0^L \dot{w} \dot{w} dx, \quad (49)
\]
where \( \delta_1 \) is a positive constant.

Substituting boundary condition Eq. (7) and the control law Eq. (16) into the second term of Eq. (41), we have

\[
\dot{V}_2 \leq \beta [\dot{w}(L, t) + \dot{w}'(L, t)]
\]
\[
+ M_s \dot{w}'(L, t) \dot{w}'(L, t) + \dot{T}_0(L) \dot{w}'(L, t)
\]
\[
- \dot{a}(t) \dot{d}(t) - \dot{T}_0(L) \dot{w}(L, t) - 2\lambda(L) \dot{w}'(L, t)^3
\]
\[
= -k\beta [\dot{w}(L, t) + \dot{w}'(L, t)]^2 + \beta \dot{M}_s(t) \dot{w}'(L, t) \dot{w}(L, t)
\]
\[
+ \dot{w}'(L, t) + \beta \dot{a}(t) \dot{d}(L, t) + \dot{w}'(L, t)
\]
\[
- \beta \dot{T}_0(L) \dot{w}'(L, t) \dot{w}(L, t) + \dot{w} \dot{w} dx,
\]
\[
- 2\beta \lambda(L) \dot{w}'(L, t) \dot{w}'(L, t)^3 \leq \beta \lambda(L) \dot{w}'(L, t)^4. \quad (50)
\]

Substituting Eq. (6) to the third term of the Eq. (41), we have

\[
\dot{\eta} = \alpha \int_0^L x \varphi(x) \left\{ \frac{1}{2} \frac{\partial T(x, t) [\dot{w}']^2}{\partial x} + \frac{1}{2} \frac{T'(x, t) [\dot{w}']^2}{\partial x} \right\} dx
\]
\[
+ \alpha \int_0^L x \varphi(x) \left\{ \frac{3}{4} \frac{\partial \lambda(x) [\dot{w}'(L, t)]^4}{\partial x} + \frac{1}{4} \frac{\lambda'(x) [\dot{w}'(L, t)]^4}{\partial x} \right\} dx
\]
\[
+ \alpha \int_0^L x \varphi(x) \dot{w} \dot{w} dx + \alpha \int_0^L \rho(x) \varphi(x) x \frac{\partial [\dot{w}']^2}{\partial x} dx.
\]

Using integration by parts and Ineq. (13), substituting boundary condition Eq. (8) and the tension expression (4), we have

\[
\dot{\eta} \leq \frac{\alpha}{2} L \varphi(L) T_0(L) [\dot{w}'(L, t)]^2 + \frac{\alpha}{2} L \varphi(L) [\dot{w}(L, t)]^2
\]
\[
+ \frac{3\alpha}{2} L \varphi(L) \lambda(L) [\dot{w}'(L, t)]^4 + \frac{\alpha L}{\delta_2} \int_0^L f^2 dx
\]
\[
+ \alpha \int_0^L \left[ \frac{\partial \varphi(x) x}{\partial x} T_0(x) - x \varphi(T_0(x)) [\dot{w}'(L, t)]^2 \right] dx
\]
\[
- \frac{\alpha}{2} \int_0^L \left[ \frac{\partial \varphi(x) x}{\partial x} \lambda(x) + \frac{\partial \varphi(x) \lambda(x)}{\partial x} - 3x \varphi(x) \lambda(x) \right] [\dot{w}'(L, t)]^4 dx
\]
\[
- \alpha \int_0^L \left[ \frac{5}{4} \frac{\partial \varphi(x)}{\partial x} \lambda(x) + \frac{\partial \varphi(x) \lambda(x)}{\partial x} - 3x \varphi(x) \lambda(x) \right] [\dot{w}'(L, t)]^4 dx.
\]

where \( \delta_2 \) is a positive constant. Substituting Eqs. (49), (50) and (52) into Eq. (41), we obtain

\[
\dot{V} \leq - \left( k - \frac{T_0}{2} \right) [\dot{w}(L, t) + \dot{w}'(L, t)]^2
\]
\[
- \frac{1}{2} \int_0^L \left[ \frac{\partial \varphi(x) x}{\partial x} - \frac{\partial \varphi(x) x}{\partial x} \right] \frac{\varphi^2(x)}{\varphi^2(x)} dx
\]
\[
- \frac{\alpha}{4} \int_0^L \left[ \frac{5}{4} \frac{\partial \varphi(x) x}{\partial x} \lambda(x) + \frac{\partial \varphi(x) \lambda(x)}{\partial x} - 3x \varphi(x) \lambda(x) \right] [\dot{w}'(L, t)]^4 dx
\]
\[
- \frac{\alpha}{2} \int_0^L \left[ \frac{\partial \varphi(x) x}{\partial x} T_0(x) - x \varphi(T_0(x)) - 2\delta_2 L \varphi^2 \right] [\dot{w}'(L, t)]^2 dx
\]
\[
- \left[ \frac{\beta}{2} \dot{T}_0(L) - \frac{\alpha}{2} L \varphi(L) T_0(L) \right] [\dot{w}'(L, t)]^2
\]
\[
- \left[ 2\beta \lambda(L) - \frac{3\alpha}{2} L \varphi(L) \lambda(L) \right] [\dot{w}'(L, t)]^4
\]
where the parameters $\alpha$, $\beta$, $k$, $\varphi(x)$, $\delta_1$ and $\delta_2$ are chosen to satisfy the following conditions:

$$0 < \alpha < \frac{\min(\beta \mu, \beta T_0)}{L \rho \varphi},$$

$$\beta \frac{T_0(L)}{2} - \frac{\alpha}{2} L \varphi(L) T_0(L) \geq 0,$$

$$2 \beta \lambda(L) - \frac{3 \alpha}{2} L \varphi(L) \lambda(L) \geq 0,$$

$$\beta \frac{T_0(L)}{2} - \frac{\alpha}{2} L \varphi(L) \rho(L) \geq 0,$$

$$\sigma_1 = \int_0^L \left( \frac{\alpha \partial^2 p x}{2 \partial x^2} - \beta \delta_1 \right) \mathrm{d}x > 0,$$

$$\sigma_2 = \frac{\alpha}{4} \int_0^L \left[ 5 \frac{\partial^2 p x}{\partial x^2} \lambda + \frac{\partial^2 \varphi x}{\partial x^2} - 3 \varphi x \lambda \right] \mathrm{d}x > 0,$$

$$\sigma_3 = \frac{\alpha}{2} \int_0^L \left[ \frac{\partial^2 p x}{\partial x^2} - \frac{\partial^2 \varphi x}{\partial x^2} - 2 \beta \frac{T_0(L)}{2} \varphi \right] \mathrm{d}x > 0,$$

$$\sigma_4 = \beta \left( k - \frac{\rho}{2} \right) > 0.$$

Applying Eqs. (20) - (22) to Eq. (53) and let

$$\mu_4 = \min \left\{ \frac{2 \sigma_3}{\beta \lambda}, \frac{2 \sigma_1}{\beta T_0}, \frac{2 \sigma_3}{\beta M_s} \right\} > 0,$$

$$\varepsilon = \left( \frac{\beta}{\delta_2} \alpha L \right) \int_0^L \beta^2 \mathrm{d}x \in \mathbb{L}_\infty.$$

we have

$$\hat{V} \leq -\mu_3 \left[ V_1 + V_2 \right] + \zeta_n M_s(t) \bar{M}_s(t) + \zeta_m \bar{d}(t) \bar{d}(t)$$

$$+ \zeta_n \frac{T_0(L)}{2} \bar{T}_0(L, t) + \varepsilon$$

$$\leq -\mu_3 \left[ V_1 + V_2 \right] - \frac{\zeta_n}{2} M_s^2(t) + \frac{\zeta_m}{2} M_s^2 - \frac{\zeta_t}{2} \bar{T}_0^2(L, t)$$

$$+ \zeta_n \frac{T_0(L)}{2} \bar{T}_0(L) - \frac{\zeta_n}{2} \bar{d}(t)^2 + \frac{\zeta_m}{2} \bar{d}(t)^2 + \varepsilon$$

$$\leq -\mu_4 \left[ V_1 + V_2 + \bar{M}_s^2(t) + \bar{T}_0^2(L, t) + \bar{d}(t)^2(t) \right] + \psi,$$

where $\mu_4 = \min(\mu_3, \frac{2 \sigma_3}{\beta \lambda}, \frac{2 \sigma_1}{\beta T_0}, \frac{2 \sigma_3}{\beta M_s})$ is a positive constant and

$$\psi = \frac{\zeta_t}{2} M_s^2 + \frac{\zeta_n}{2} T_0^2(L, t) + \frac{\zeta_m}{2} \bar{d}(t)^2 + \varepsilon.$$

Combining Inqs. (37) and (64), we have

$$\hat{V}(t) \leq -\mu V(t) + \psi,$$

where $\mu = \mu_4 / \mu_2 > 0$ and $\psi > 0.$

With the above lemmas, we are ready to present the following stability theorem of the closed-loop nonuniform string system.

**Theorem 1.** For the system dynamics described by (6) and boundary conditions (7), (8), under Assumptions 1, 2 and the boundary control Eq. (16), given that the initial conditions are bounded, we can conclude that: the system state $w(x, t)$ will eventually converge to the compact set $\Omega$ defined by

$$\Omega : \left\{ (x, t) \in \mathbb{R} \big| \lim_{t \to \infty} |w(x, t)| \leq D_2, \quad \forall x \in [0, L] \right\},$$

where constant $D_2 = \sqrt{\frac{2 \lambda}{\beta \mu T_0 \mu_1}}$.

**Proof:** Multiplying Eq. (40) by $e^{\mu t}$ yields

$$\frac{\partial}{\partial t} (V(t)e^{\mu t}) \leq \psi e^{\mu t}.$$  

Integrating the above inequality, we obtain

$$V(t) \leq \left( V(0) - \frac{\psi}{\mu} \right) e^{-\mu t} + \psi \mu \in \mathbb{L}_\infty,$$

which implies $V(t)$ is bounded. Utilizing Ineq. (15) and Eq. (24), we have

$$\frac{\beta}{2 L} T_0 w(x, t) \leq \int_0^L T_0(x) |w'(x, t)|^2 \mathrm{d}x \leq V_1(t)$$

$$\leq V_1(t) + V_2(t) \leq \frac{1}{\mu_1} V(t) \in \mathbb{L}_\infty.$$  

Appropriately rearranging the terms of the above inequality, we obtain

$$w(x, t) \leq \sqrt{\frac{2 L}{\beta T_0 \mu_1}} \left( V(0)e^{\mu t} + \frac{\psi}{\mu} \right).$$

$$\forall (x, t) \in [0, L] \times [0, \infty).$$  

Furthermore, we have

$$\lim_{t \to \infty} |w(x, t)| \leq \sqrt{\frac{2 L \psi}{\beta T_0 \mu_1}}, \quad \forall x \in [0, L].$$

4. NUMERICAL SIMULATIONS

Consider a nonuniform string excited by the disturbances $f(x, t)$ and $d(t)$, the initial conditions are $w(x, 0) = x$.

$$\dot{w}(x, 0) = 0.$$  

Parameters of the nonuniform string are listed in Table 1.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Description</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L$</td>
<td>Length of string</td>
<td>1m</td>
</tr>
<tr>
<td>$\rho(x)$</td>
<td>Mass per unit length</td>
<td>0.1(x + 5)kg/m</td>
</tr>
<tr>
<td>$M_s$</td>
<td>Mass of the tip payload</td>
<td>1kg</td>
</tr>
<tr>
<td>$T_0(x)$</td>
<td>Initial tension</td>
<td>10(x + 1)N</td>
</tr>
<tr>
<td>$\lambda(x)$</td>
<td>Elastic modulus</td>
<td>0.1(x + 1)</td>
</tr>
</tbody>
</table>

**Table 1:** Parameters of the nonuniform string
The effectiveness of the proposed control has been successfully verified by numerical simulations.

REFERENCES


Fig. 2. Displacement of the nonuniform string without control.

Fig. 3. Displacement of the nonuniform string with robust adaptive control.

Fig. 4. Adaptive control input \( u(t) \).

Fig. 2 shows displacement of the nonuniform string under disturbances without control input, i.e., \( u(t) = 0 \). Displacement of the nonuniform string with adaptive boundary control (16) under disturbances, by choosing \( k = 50, \beta = 1, \zeta_m = \zeta_d = \zeta_f = 1, \gamma_m = \gamma_f = \gamma_d = 1 \), is shown in Fig. 3. Fig. 3 illustrates that the proposed adaptive boundary control is able to stabilize the string at the small neighborhood of its equilibrium position. The corresponding adaptive boundary control input \( u(t) \) is shown in Fig. 4.

5. CONCLUSION

In this paper, the control problem of a nonuniform string system under unknown spatiotemporally varying tension and disturbance has been investigated. The dynamics of the string system is represented by a nonlinear nonhomogeneous PDE and two ODEs. In order to suppress the vibration and compensate the system uncertainties, robust boundary control has been developed at the right boundary of string. With the proposed control, the state of the nonuniform string system has been proven to be uniformly ultimately bounded and converge to a small neighborhood of zero by appropriately choosing design parameters.