Stability of Feedback Error Learning for Linear Systems

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Abstract: The aim of this study is to prove validity of feedback error learning rule for a linear representation of dynamic system with unknown parameters. A simple single-layer neural network is assumed as an adaptive linear combiner and stability techniques are applied to derive the same adaptation law as feedback error learning rule.

1. INTRODUCTION

The feedback error learning (FEL) method was proposed by Kawato in (Kawato et al., 1987). The first FEL scheme is combined adaptive linear combiners as a neural network controller in feedforward path with proportional-plus-derivative (PD) controller in feedback path which feedback controller output is fed to feedforward controller as training signal to adapt the neural network in order to minimize tracking error. Sum of the feedforward and the feedback controller is fed to plant as input and desired output is fed to feed-forward controller. So it is reasonable to accept, by minimization of tracking error, feedforward controller makes inverse model of plant because in this case, the output of PD controller is lead to zero hence, the output of feed-forward controller will be plant input while feedforward controller is trying to reach plant output to self input.

Theoretical validity of FEL was investigated for stable and stably invertible linear plants by Miyamura in (Miyamura and Kimura, 2002). Neural network was assumed as an adaptive linear combiner and FEL method was formulated as a two degree of freedom adaptive control and then based on strict positive realness, stability of FEL rule was proved.

Craig proposed (Craig et al., 1987), a method for updating of adaptive linear combiner parameters which has a common idea with Kawato’s FEL method. Both of them used a filtered servo error signal as training signal but with a few difference between their algorithms. The techniques used by Craig and Kawato were investigated in (Nordgren and Meckl, 1993). By using Craig’s method, Nordgren (Nordgren and Meckl, 1993) obtained a new adaptation law which was similar to Kawato’s learning rule. He compared the effectiveness of these three methods by applying them to control a coupled compound pendulum system in (Nordgren and Meckl, 1993). Kawato’s FEL rule can be obtained from Nordgren’s adaptation law for linear system models which will be discussed in this paper.

After a brief overview on the method that was used in (Nordgren and Meckl, 1993), we highlight some differences of Kawato’s learning rule in comparison with Nordgren’s adaptation law in section 2. In section 3, we describe our modifications in Nordgren’s method in order to eliminate the weaknesses. Also, we show that FEL rule implies some restrictions on selection of Lyapunov function. Ultimately coupled compound pendulum is used to illustrate effectiveness of Kawato’s FEL method in simulation part.

2. OVERVIEW ON DERIVATION OF ADAPTIVE LAW FOR NEURAL NETWORK

This section has a brief overview on the method used by Nordgren in (Nordgren and Meckl, 1993). He employed technique used by Craig in (Craig et al., 1987) which developed the model-based adaptive controller, to derive a similar adaptation law to Kawato’s learning rule for single-layer neural network. Fig. 1 shows the structure of FEL which was proposed by Kawato in (Kawato et al., 1987) to train feedforward controller (neural network) to form inverse dynamic model of controlled object. Kawato used desired trajectory signals and their derivatives as inputs of neural network and proposed using of PD outputs as training signal to tune adjustable weights of neural network to achieve minimum tracking error. Based on this scheme conventional controller is designed to guarantee for convergence of states.
before learning, through learning and also after learning (Gomi and Kawato, 1990).

For the sake of simplicity and clarification of analysis, Nordgren used a linear representation of dynamic system in (Nordgren and Meckl, 1993). The equation of motion of system is assumed to be described as follow matrix differential equation in joint coordinates \( q = [q_1, ..., q_n]^T \in \mathbb{R}^n \):

\[
M \ddot{q} + C \dot{q} + Gq = u
\]

(1)

where \( u \in \mathbb{R}^n \) represents the applied joint torque to the system and \( M, C \) and \( G \) are time-invariant inertia, damping and stiffness matrices, respectively. He considered a fixed-gain proportional-plus-derivative (PD) controller, as a conventional feedback controller, as follow:

\[
u_{NN} (t) = M \ddot{q}_d + C \dot{q}_d + \dot{G} q_d
\]

(2)

where \( q_d \) is the desired trajectory vector and \( M, C \) and \( \dot{G} \) are estimated inertia, damping and stiffness matrices, respectively. He considered a fixed-gain proportional-plus-derivative (PD) controller, as a conventional feedback controller, as follow:

\[
u_{PD} (t) = K_p (q_d - q) + K_d (\dot{q}_d - \dot{q})
\]

(3)

Where \( K_p \) and \( K_d \) are diagonal positive definite gain matrices. As shown in Fig. 1, the system equation is formed by combining foregoing equations by \( u = u_{NN} + u_{PD} \) which is given by

\[
M \ddot{q}_d + C \dot{q}_d + \dot{G} q_d + K_p (q_d - q) + K_d (\dot{q}_d - \dot{q}) = M \ddot{q} + C \dot{q} + Gq
\]

(4)

In order to form the governing dynamic equation in terms of position error, Nordgren added \( M \ddot{q}_d + C \dot{q}_d + \dot{G} q_d \) to both sides (4) and rearranged it, the resulting expression is given as

\[
M (\ddot{q}_d - \ddot{q}) + C (\dot{q}_d - \dot{q}) + G (q_d - q) + K_p (q_d - q) + K_d (\dot{q}_d - \dot{q}) = M \ddot{q} + C \dot{q} + Gq
\]

(5)

then by defining \( E = q_d - q \), Nordgren rewrote foregoing equation as follow:

\[
\dot{E} + M^{-1} (C + K_d) \dot{E} + M^{-1} (G + K_p) E
\]

\[
= M^{-1} \left[ M - M \ddot{q} + C \dot{q} + G \dot{q} \right] \ddot{q}_{d}
\]

(6)

where \( W_d \in \mathbb{R}^{n \times n} \) is a function of the desired trajectory and its derivatives which can be consider as inputs of neural network, and \( \Phi = W \dot{W} \dot{q} \) where \( W \in \mathbb{R}^v \) and \( \dot{W} \in \mathbb{R}^v \) are vectors containing actual and estimated parameters of system, respectively. Actually \( \Phi \in \mathbb{R}^v \) is a vector of parameter errors and \( \dot{W} \) can be considered as neural network weights which its desired value are system parameters given by \( W \in \mathbb{R}^v \). The right-hand-side of (6) can be treated as forcing function which depends on actual inverse inertia matrix, desired trajectory and parameter error. The left-hand-side of foregoing equation depends on controller gains and actual system parameters.

Nordgren obtained an adaptation law by using Lyapunov stability analysis on (6) in similar way to Craig’s method in (Craig et al., 1987), which yields to

\[
\Phi = -\Gamma W_{d}^T M^{-1} E_f
\]

(7)

where \( \Gamma = \text{diag} (\gamma_1, \gamma_2, ..., \gamma_s) \) and \( \gamma_i > 0 \) is non-negative adaptation rate gain and \( E_f \) represents a "filter error" in (Nordgren and Meckl, 1993) and (Craig et al., 1987) given by

\[
E_f = \dot{E} + \Psi E
\]

(8)

where \( \Psi = \text{diag} (\psi_1, \psi_2, ..., \psi_n) \) and \( \psi_i > 0 \). \( E_f \) is used to convert adaptive closed loop transfer function to a strictly positive real (SPR) transfer function, in order to be usable by SPR lemma in (Narendra and Annaswamy, 1989) to prove stability of adaptive system. Note that there is a difference between Craig’s adaptation law and Nordgren’s adaptation law. Because of difference in schemes, \( M^{-1} \) is replaced by \( M^{-1} \) in Craig’s adaptation law.

Now, we want to highlight some differences between Kawato’s learning rule and Nordgren’s adaptation law.

**Remark 1.** One of the most important properties of Kawato’s learning rule is its independency upon any knowledge of system dynamic parameters, whereas (7) shows adaptation law need to identification of inertia matrix.

**Remark 2.** In Kawato’s strategy the output of the PD controller is directly fed to the feedforward as the training signal whereas (7) shows another one.

Nordgren and Meckl (Nordgren and Meckl, 1993) selected a beautiful way to drive a similar adaptation law to Kawato’s learning rule for FEL scheme, this is the strength of their method but their adaptation law is not same law as Kawato’s learning rule. These weaknesses will be removed in the next section.

3. DERIVATION FEEDBACK ERROR LEARNING RULE FROM STABILITY TECHNIQUES

In this section, stability techniques will be employed for the governing dynamic equation in terms of position error represented in (6), in order to eliminate Nordgren’s method.
weaknesses to reach more similar adaptation law to Kawato’s learning rule.

In Kawato’s scheme and schemes like that, adaptive part (feedforward controller) is used to provide required input for following the desired trajectory and the feedback controller then stabilizes the tracking error dynamics, as you can see in (Craig et al., 1987) and (Kawato et al., 1987). Hence, it is not necessary fixed-gains of feedback controller be optimally tuned.

After a perfect learning which leads to providing required input for following the desired trajectory, (6) can be written as follow:

\[ \dot{E} + M^{-1}(C + K_d)E + M^{-1}(G + K_p)E = 0 \]  

(9)

Although coefficients of error equation are unknown constant matrices, but we assume these matrices lie in some bounded intervals. The uncertainty in these matrices or parameters of system leads to error in controller gains selection, but we know that in Kawato’s scheme, it is not necessary fixed-gains be optimally tuned. Since, if we consider that \( M, C \) and \( G \) lie in some bounded intervals, we can choose appropriate \( K_p \) and \( K_d \) for the PD controller such that, the error equation represented in (9) be stabilized. So in the rest of paper, we assume that the PD controller stabilized unstable poles of (9).

Now by taking the Laplace, error transfer function in (6), can be written as

\[ E(s) = [s^2 + M^{-1}(C + K_d)s + M^{-1}(G + K_p)]^{-1}F(s) \]  

(10)

where \( F(s) \) is the transform of the term on the right hand of (6). The transfer function has two more poles than zeros, so it is not a strictly positive transfer function since in order to eliminate causality problem, the output of error equation can be chosen such that give us a strictly positive transfer function, or from other angle, following the techniques used by Craig (Craig et al., 1987) and Nordgren (Nordgren and Meckl, 1993), it can be cascaded with a non-causal filter, to turn transfer function to strictly positive one. By defining output as

\[ Y = K_pE + K_d\dot{E} \]  

(11)

The output transfer function can be written as

\[ Y(s) = [K_p + K_d][s^2 + M^{-1}(C + K_d)s + M^{-1}(G + K_p)]^{-1}F(s) \]  

(12)

where \( K_p \) and \( K_d \) are diagonal matrices with positive elements. This transfer function is strictly positive real (SPR), then by the SPR lemma (Narendra and Annaswamy, 1989) we are assured of existence of symmetric positive definite matrices \( P \) and \( Q \) such that satisfy following equations:

\[ A^TP + PA = -Q \]  

(13)

\[ PB = C^T \]

where \( A, B \) and \( C \) are derived from (6) to form state space equations of error as (14). Note that, \( A \) was stabilized by PD controller as we presented.

\[ \dot{X} = AX + BM^{-1}W_d\Phi \]  

(14)

where \( X = [E^T \dot{E}^T]^T \) represents error state vector and by assumption \([\bar{K}_p \ \ ar{K}_d] = C \) which \( C \) is derived from (13).

Lyapunov stability theory will be used to derive a globally asymptotically stable (GAS) adaptation law. Using the same choice as Craig in (Craig et al., 1987) and Nordgren in (Nordgren and Meckl, 1993), we write a Lyapunov function of the form:

\[ V(X, \Phi) = X^TPX + \Phi^T\Gamma^{-1}\Phi \]  

(15)

with \( \Gamma = \text{diag}(\gamma_1, \ldots, \gamma_n) \) and \( \gamma_i > 0 \) is adaptation rate gain and \( P \) is derived from (13) which is positive definite matrix. Differentiation with respect to time and by substituting \( \dot{X} \) from (14), \( Q \) and \( C \) from (13) and \( Y \) from (14), in (16) we have

\[ \dot{V}(X, \Phi) = -X^TQX + 2\Phi^T(W_d^T M^{-1}Y + \Gamma^{-1}\Phi) \]  

(16)

By choosing adaptation law as

\[ \Phi = -\frac{d\hat{W}}{dt} = -\Gamma W_d^T M^{-1}Y \]  

(17)

we have

\[ \dot{V}(X, \Phi) = -X^TQX \]  

(18)

which is non-positive. Using Lyapunov stability theory for (16) and (18) imply that \( X \) asymptotically converge to zero if we use (17) as adaptation law.

On the other hand, in a general manner, the learning rule of the FEL scheme is represented in (Gomi and Kawato, 1990) as follow:

\[ \frac{d\hat{w}}{dt} = \eta \left( \frac{\partial u_{PD}}{\partial w} \right)^T u_{PD} \]  

(19)

where \( \eta \) is a positive definite learning rate matrix and \( \hat{w} \) represent adaptive parameters of neural network. Equation (19) can be rewritten for our single-layer neural network as

\[ \frac{d\hat{W}}{dt} = \eta W_d^T u_{PD} \]  

(20)
where \( \eta \) can be replaced by \( \Gamma \). When the FEL learning rule in (20) is compared with GAS adaptation law in (17), yields if \( M^{-1} \dot{y} = u_{PD} \), from stability of GAS adaptation law can reach to stability of FEL rule. The \( M^{-1} \dot{y} = u_{PD} \) implies following equation:

\[
B^T P = \begin{bmatrix} MK_p & MK_d \end{bmatrix}
\]

(21)

Equation (21) implies a restriction on \( P \), therefore we have to investigate a symmetric positive definite \( P \) such that satisfy (21). Suppose that \( P \) is as follow:

\[
P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}
\]

(22)

Symmetric property of \( P \) implies \( P_{11} = P_{11}^T, P_{12} = P_{21}^T \) and \( P_{22} = P_{22}^T \). For such a \( P \), (21) implies following equation:

\[
P_{22} = MK_d, \quad P_{21} = MK_p
\]

(23)

Foregoing equation and symmetric property of \( P \), force \( M \) be a diagonal matrix because of \( P_{22} = P_{22}^T \) be satisfied, also symmetric property of \( P \) leads to \( P_{12} = K_p M \). So in this case \( (M \) be a diagonal matrix), based on symmetric positive definite matrix properties, by choosing a proper \( P \) which satisfies following conditions, \( P \) will be turned to a symmetric positive definite matrix and stability of FEL can be proven.

1. \( P_{11} > 0 \)
2. \( P_{22} - P_{21} P_{11}^{-1} P_{21}^T > 0 \)

Above conditions can be satisfied by choosing

\[
P_{11} = \epsilon P_{21}^T P_{21}
\]

(24)

where \( \epsilon \) is a positive constant. We know that \( M \) is a symmetric positive definite matrix and \( K_d \) is a diagonal positive definite matrix. Based on foregoing properties, \( P_{22} \) is a positive definite matrix which if \( M \) be a diagonal matrix then \( P_{22} \) will be too. In a similar way, we can drive the same properties for \( P_{21} \). It’s clear \( P_{11} \), represented in (25) is a symmetric positive definite matrix because it can be rewritten as \( P_{11} = \epsilon K_p M^2 K_p \). By substituting \( P_{11} \) from (25), the second part of (24) gives

\[
P_{22} - \epsilon I > 0
\]

(26)

Since \( P_{22} \) is a real positive definite matrix, there will be a \( \epsilon \) such that satisfy (26). Therefore (24) will be satisfied by choosing (25) for \( P_{11} \) and then \( P \) will be a symmetric positive definite matrix.

Our more ambitious objective is to finding a symmetric positive definite matrix \( (P) \) for any symmetric positive definite inertia matrixes. In order to achieve this target we rewrite (23) as follow:

\[
B^T P = \begin{bmatrix} M \Delta & M \end{bmatrix} K_d
\]

(27)

where \( \Delta = K_p K_d^{-1} \) is a diagonal symmetric positive definite matrix. The \( K_d \) term can be eliminated by a proportional gain \( (K_d^{-1}) \) which will be cascaded with PD controller. From other side the gain \( (K_d^{-1}) \) is a specific diagonal constant matrix, so this seems reasonable \( K_d^{-1} \) be lumped in with \( \Gamma \), and \( \Gamma \) be assigned to do the task of the proportional gain. Note that \( \Gamma \) is an arbitrary diagonal matrix with positive elements therefore we can assert, our assumption is true for any \( K_d^{-1} \). Hence we may write (27) as

\[
B^T P = \begin{bmatrix} M \Delta & M \end{bmatrix}
\]

(28)

Foregoing equation leads to

\[
P_{22} = M, \quad P_{21} = M \Delta
\]

(29)

From (29), \( P_{22} = P_{22}^T \) and with respect to symmetric property of \( P \), we select \( P_{12} = P_{12}^T \). Now based on previous experience, we choose \( P_{11} = \epsilon P_{21}^T P_{21} \) such that (24) and symmetric property of \( P \) be satisfied. This \( P \) can be used for any symmetric positive definite inertia matrix.

Now, by finding a positive definite Lyapunov function with respect to FEL rule restrictions on Lyapunov function, we can say FEL method is a globally stable learning algorithm. Because of, this positive definite Lyapunov function can be a Lyapunov function for unforced system (9), and (9) was stabilized by PD controller therefore, the rate of Lyapunov function change cannot be a positive definite function. So this learning algorithm is globally stable.

Note that, we proved globally stability of FEL for unknown parameters but in order to achieve globally asymptotically stability, we must find out \( P_{11} \) such that \( \theta \) be a symmetric positive definite matrix and then the rate of Lyapunov function change will be non-positive. With respect to FEL rule restrictions on Lyapunov function (which leads to (29)) and symmetric property of \( P \), from (13) and (18), the rate of Lyapunov function change can be written as follow:

\[
\dot{V}(X) = -2(u_{PD})^T K_d^{-1}(u_{PD}) - X^T \begin{bmatrix} \Delta G + C^T \Delta & -P_{11} + \Delta C + G^T \\ -P_{11} + C^T \Delta + G & -\Delta M - M \Delta + C^T + C \end{bmatrix} X
\]

(30)

Foregoing equation shows that with this formulation, for specific PD controller gains, we need to known parameters in order to design \( P_{11} \) such that globally asymptotically stability of FEL method be satisfied. But with an exact view to (30), it
can be found that the right hand side of foregoing equation is consist of two parts, the first part depends on PD controller output and the second part depends on parameters of system, \( p_{11} \) and \( \Delta \) which all of them are arbitrary variable except parameters of system. Hence, we can always adapt controller gains to satisfy globally asymptotically stability of FEL method. By choosing sufficiently large controller gains with a proper \( \Delta \) \( (\Gamma, \Gamma (p, K, M, C, G) \approx \Gamma) \) globally asymptotically stability can be obtained for FEL method.

Fig. 2. Coupled compound pendulum system (Nordgren and Meckl, 1993)

4. SIMULATION

A coupled compound pendulum shown in Fig. 2 is used to illustrate effectiveness of Kawato’s FEL scheme. The linearized dynamics of the coupled compound pendulum can be written in form (1) which \( q = [\theta_1, \theta_2]^T \) , \( u = [\tau_1, \tau_2]^T \) and the inertia, damping and stiffness matrixes can be written as:

\[
\begin{bmatrix}
  I_1 & 0 \\
  0 & I_2
\end{bmatrix}
\]

\[
C = \begin{bmatrix}
  bL^2 & -bL^2 \\
  -bL^2 & bL^2
\end{bmatrix}
\]

\[
G = \begin{bmatrix}
  mgL + kl^2 & -kl^2 \\
  -kl^2 & mgL + kl^2
\end{bmatrix}
\]

Our fully actuated system consists of two subsystems which we use a PD controller for each subsystem and just similar to Kawato’s method in (Kawato et al., 1987) we also use an adaptive linear combiner for each subsystem to form inverse dynamic model by using the PD controller output as training signal. Scheme of our simulation can be seen in Fig. 3. \( W_d \) and \( W \) can be written as follow (Nordgren and Meckl, 1993):

\[
W = \begin{bmatrix}
  I_1 & bL^2 & mgL + kl^2 & -bL^2 & I_2 & -bL^2 & -kl^2 & mgL + kl^2
\end{bmatrix}
\]

\[
W_d = \begin{bmatrix}
  \dot{\theta}_1 - \dot{\theta}_2 & \dot{\theta}_1 & \dot{\theta}_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

Note that in previous section, we discussed about error convergence between actual and desired trajectory. Parameter estimate error (\( \Phi \) ) convergence is discussed in (Morgan and Narendra, 1977), (Anderson, 1977) and (Narendra and Annaswamy, 1989). It can be driven from these researches that, if the SPR lemma be satisfied and \( (A, B) \) be a controllable pair, convergence of parameter estimate error depends on desired trajectory. Hence, we choose desired trajectory as follow:

\[
\theta_{d1}(t) = 2 \sin 4t - \sin 8t
\]

\[
\theta_{d2}(t) = 1 + \cos 4t - \cos 8t
\]

which convergence of parameter estimate errors for foregoing desired trajectory is discussed in (Nordgren and Meckl, 1993). Values of system parameters and PD controller gains and adaptation gains are given in table 1. Assume that, the gains are selected such that globally asymptotically stability will be satisfied. Initial weight values of neural network are set to zero and consider initial values of system are \( \theta_1(0) = 1.5 \), \( \theta_2(0) = -0.5 \) and \( \theta_1(0) = \theta_2(0) = 0 \).

Fig. 4 shows convergence of position errors which justify our explanations in previous section and our assumption about gains. Error in weights estimation is shown in Fig. 5. All weights converge to desired weights as shown in Fig. 5. (The difference between these simulation and Nordgren’s simulations (Nordgren and Meckl, 1993) is that, we directly use PD controller output as training signal and the initial conditions were set to in harder position).

Table 1. System and controllers parameters

<table>
<thead>
<tr>
<th>PD</th>
<th>NN</th>
<th>SYS</th>
</tr>
</thead>
<tbody>
<tr>
<td>( K_p = 42 )</td>
<td>( \Gamma_1 = 0.005 )</td>
<td>( \Gamma_2 = 0.005 )</td>
</tr>
<tr>
<td>( K_d = 30 )</td>
<td>( \Gamma_3 = 0.36 )</td>
<td>( \Gamma_4 = 0.07 )</td>
</tr>
<tr>
<td>( K_p = 17 )</td>
<td>( \Gamma_5 = 1.8 )</td>
<td>( \Gamma_6 = 1.08 )</td>
</tr>
<tr>
<td>( K_d = 7 )</td>
<td>( \Gamma_7 = 2.4 )</td>
<td>( \Gamma_8 = 0.25 )</td>
</tr>
</tbody>
</table>

Fig. 3. Block diagram of simulation
Fig. 4. Simulation results. (a) Desired trajectory and actual trajectory. (b) Error in trajectory

Fig. 5. Error in weight estimation

Here we want to highlight some notices about this study:

**Remark 1.** Because of structure of formulation, the achievements of this study can be useful for adaptive control of nonlinear system.

**Remark 2.** An interesting reflection of FEL restrictions on Lyapunov function is by choosing $P_{22} = M$ error kinetic energy will be appearing in Lyapunov function and if we consider $P_{11} = K_p + (G + G^T)/2 > 0$ error potential energy will be appearing in Lyapunov function and asymptotic stability will be obtained if error mechanical energy (sum of kinetic and potential energy) converge to zero.

5. CONCLUSIONS

In this study, we showed that under formulation of this paper, FEL scheme implies some restrictions on choose of Lyapunov function in SPR lemma and then it be shown that we are able to find a symmetric positive definite matrix ($P$) for Lyapunov function such that FEL rule be driven and stability be satisfied with respect to the restrictions and under the assumption that PD controller is able to stabilize the adaptive system after a perfect learning.

REFERENCES


