Hamilton-Jacobi inequalities for optimal impulsive control problems

Vladimir A. Dykhta

Abstract: Sufficient and necessary global optimality conditions for nonlinear impulsive dynamic optimization problems with endpoint constraints are obtained. Proofs of these results are based on Hamilton-Jacobi canonical optimality theory. As consequence, a Maximum Principle reverse into sufficient optimality conditions is proposed.

1. INTRODUCTION

This paper is devoted to global optimality conditions for nonlinear optimal control problems with trajectories of bounded variation and impulsive controls of the regular vector measure type.

These optimality conditions corresponding to a generalization of the so-called Hamilton-Jacobi canonical optimality theory are developed for the classical optimal control problems in Arguchintsev et al. [2009], Dykhta [1990, 2004, 2010], Dykhta and Samsonyuk [2010a], Milyutin [2000], Milyutin and Osmolovskii [1998]. The key idea of this approach is operating with some solution sets of one or another Hamilton-Jacobi inequality. It was proved in Arguchintsev et al. [2009], Dykhta and Samsonyuk [2010a] that the canonical optimality theory is finer than the alternative Caratheodory and Krotov approaches Bardi and Capuzzo-Dolcetta [1997], Clarke et al., [1998], Clarke and Nour [2005], Ioffe and Tikhomirov [1979], Vinter [2000], Young [1969], Krotov [1996], Krotov and Gurman [1973].


In Section 3 the definitions of strongly increasing \( L \)-functions (Lyapunov type functions) w.r.t. the corresponding impulsive control system are introduced together with infinitesimal tests in a form of proximal Hamilton-Jacobi inequalities. Applications of another kinds of monotone \( L \)-functions are shortly discussed.

General sufficient and necessary global optimality conditions for the optimization problem in the class of impulsive processes of system \((D_0)\) are formulated in Section 4. The proof of the sufficiency follows to the canonical Hamilton-Jacobi theory and is based on an arbitrary set of strongly monotone \( L \)-functions.

Section 5 is devoted to sufficient optimality conditions in the form of Maximum Principle (MP) for important impulsive control problems governed by the differential system

\[
\begin{align*}
  dx(t) &= f(t, x(t)) dt + F(t, u(t)) \mu(dt) \\
  v(t) &\in K \quad \text{a.e. } t \in T,
\end{align*}
\]

with \( K \)-valued Borel control's measure and ordinary Borel measurable control \( u : T \to U \) (\( U \subset \mathbb{R}^m \) is a compact set). This conversion of necessary optimality conditions into sufficient ones is obtained by using a set of linear strongly increasing \( L \)-functions generated by solutions of the adjoint system corresponding to the investigated extremal process. In contrast to known similar results Miller and Rubinovich [2003] our sufficient conditions are obtained without normality assumptions and operate by a set of tuples of Lagrange multipliers. As corollary, their areal of applicability is wider.

2. DESCRIPTION OF GENERALIZED SOLUTIONS AND IMPULSIVE PROCESSES

Since set \( K \) is unbounded in system \((D_0)\) and this system is linear with respect to \( v \), one can expect that optimization problems for system \((D_0)\) may have no solutions in the class of ordinary processes; therefore these optimization

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problems have to be extended (see, e.g., Dykhta and Samsonyuk [2003], Gurman [1997], Miller and Rubinovich [2003], Sesekin and Zavalishichn [1997]). A natural extension is obtained by closing the set of trajectories of $(D_0)$ in the weak* topology in the space of functions of bounded variation. Indeed, under very natural assumptions (see (H1) below) any set of ordinary trajectories of $(D_0)$ has uniformly bounded total variations on the interval $T$ provided that the additional condition on the corresponding set of $v$, namely
\[
\int_{t_0}^{t_1} ||v(t)|| dt \leq M \quad \text{for some } M > 0, \tag{1}
\]
holds. Thus, if we consider a sequence of ordinary controls \{$v_k(\cdot)$\} such that \(||v_k(\cdot)||_{L^1} < \infty\), then the corresponding trajectories \(x_k(\cdot)\) have uniformly bounded total variations on $T$. Therefore we can take a subsequence \(\{x_k(\cdot)\}\) that tends to \(x(\cdot)\) as \(r \to \infty\) in the sense of weak* convergence in the space of functions of bounded variation.

Let us describe the extension of the standard processes set of the ordinary system $(D_0)$ with unbounded controls.

We assume that the following conditions are fulfilled.

(H1) The functions \(f(t,x), G(t,x)\) are continuous; for any compact set \(Q \subset \mathbb{R}^n\) there exist constants \(L_{1Q}, L_{2Q} > 0\) such that
\[
\begin{align*}
|f(t,x_1) - f(t,x_2)| &\leq L_{1Q}|x_1 - x_2|, \\
|G(t,x_1) - G(t,x_2)| &\leq L_{2Q}|x_1 - x_2|
\end{align*}
\]
whenever \((t,x_1), (t,x_2) \in T \times Q\); moreover, there exist constants \(c_1, c_2 > 0\) such that
\[
\begin{align*}
|f(t,x)| &\leq c_1(1 + |x|), \\
|G(t,x)| &\leq c_2(1 + |x|)
\end{align*}
\]
whenever \((t,x) \in T \times \mathbb{R}^n\). Here, \(|\cdot|\) denotes a vector or consistent matrix norm.

Let \(K\) be the set \(|v \in K | ||v|| = 1\), where \(||\cdot||\) is the norm \(||v|| = \sum_{j=1}^{r} |v_j|\); \(BV^+ (T, \mathbb{R}^k)\) be the Banach space of right continuous on \((t_0,t_1]\) \(k\)-vector functions of bounded variation; \(\mu\) be a \(K\)-valued bounded Borel measure on \(T\); that is,
\[
\mu(B) \in K \forall B \in \mathcal{B}_T,
\]
where \(\mathcal{B}_T\) is the set of all Borel subset of \(T\). Given \(\mu\), let \(S_{\mu}\) be the set \(|s \in T | \mu\{s\} \neq 0\) and \(\mu_c\) be the continuous component in the Lebesgue decomposition of \(\mu\). We denote by \(\gamma(\mu)\) any family \(\{d_s, \omega_s(\cdot)\}_{s \in S}\) for which the following conditions hold:
\[
\begin{align*}
&\text{(a)} \ S \text{ is at most denumerable set of points of } T \text{ such that } S_{\mu} \subseteq S; \\
&\text{(b)} \forall s \in S \ d_s \in \mathbb{R}_+ \text{ and } \omega_s : [0,d_s) \to co K_1 \text{ is an } L\text{-measurable function: } \\
&\quad d_s \geq ||\mu\{s\}||, \quad \int_0^{d_s} \omega_s(\tau) d\tau = \mu\{s\}; \\
&\text{(c)} \sum_{s \in S} d_s < \infty.
\end{align*}
\]

Here, \(co A\) is the convex hull of a set \(A\). It is clear that there exists a nonunique family \(\gamma(\mu)\) corresponding to \(\mu\). We denote by \(\pi(\mu)\) any pair \((\mu, \gamma(\mu))\) in which \(\mu\) is a \(K\)-valued bounded Borel measure on \(T\) and \(\gamma(\mu)\) is some corresponding family described above. Let \(\Pi(T,K)\) be the set of all such \(\pi(\mu)\). We shall say that any element of the set \(\Pi(T,K)\) is an impulsive control.

For \(\pi(\mu) \in \Pi(T,K)\), we define the function \(V(\cdot) = V_{\pi(\mu)}(\cdot) \in BV^+ (T, \mathbb{R})\) by the rule
\[
V(t_0) = 0, \quad V(t) = |\mu_c|([t_0,t]) + \sum_{s \leq t, s \in S} d_s, \quad t \in (t_0,t_1], \tag{2}
\]
where \(|\mu_c|\) is a total variation of the measure \(\mu_c\).

Let us consider the next impulsive control system:
\[
dx(t) = f(t,x(t)) dt + G(t,x(t)) \pi(\mu), \]
\[
\pi(\mu) \in \Pi(T,K), \tag{D}
\]
where \(x(\cdot) \in BV^+ (T, \mathbb{R}^n)\). The solutions of (D) are understood in the sense of the following definition.

Definition 1. A function \(x(\cdot) \in BV^+ (T, \mathbb{R}^n)\) is said to be the solution of (D) corresponding to an impulsive control \(\pi(\mu) \in \Pi(T,K)\) if the following equality
\[
x(t) = x(t_0) + \int_{t_0}^{t} f(t,x(t)) dt + \int_{t_0}^{t} G(t,x(t)) \mu_c(dt) + \sum_{s \leq t, s \in S} \left(z_s(d_s;x(-)) - x(s-)\right), \quad t \in (t_0,t_1]
\]
is fulfilled, where for each \(s \in S\) the function \(z_s(\cdot;x)\) satisfies the limiting system
\[
\frac{dz_s(\tau)}{d\tau} = G(s,z_s(\tau)), \quad z_s(0) = x, \quad \tau \in [0,d_s].
\]

A pair \((x(\cdot), \pi(\cdot))\) consisting of an impulsive control \(\pi(\mu)\) and the corresponding trajectory \(x(\cdot)\) is called an impulsive process and denoted by \(x\). The function \(V_{\pi(\mu)}(\cdot)\) corresponding to \(\pi(\mu)\) and given by (2) will be also denoted as \(V_{\pi(\cdot)}(\cdot)\) or even \(V(\cdot)\) if \(\pi(\cdot)\) is easy for the context. We shall use the similar notation for \(s\)'s trajectory, that is, \(x_s(\cdot)\) or \(x(\cdot)\). Let us note that, for a given \(\pi(\mu)\), the function \(V(\cdot)\) has jumps only at points of the set \(S\), moreover, \(d_s = [V(s)] := V(s) - V(s-) \forall s \in S\); therefore we shall identify the set \(S\) with \(S_{\pi(\cdot)}(V) := \{s \in T | [V(s)] > 0\}\).

We shall use the following notation:
\[
\begin{align*}
&\Sigma_T \text{ is the set of all impulsive processes defined on the interval } T; \\
&T E_T = \{ (x_\sigma(\cdot), V_\sigma(\cdot)) | \sigma \in \Sigma_T \}; \\
&T E_{[t_0,t]} \text{ is the restriction of } T E_T \text{ to an interval } [t_0,t] \subseteq T.
\end{align*}
\]

3. STRONGLY INCREASING L-FUNCTIONS FOR IMPULSIVE CONTROL SYSTEM (D)

Following to Dykhta and Samsonyuk [2010a,b] let us define some property of strong monotonicity for a continuous Lyapunov type function \((t,x,V) \to \varphi(t,x,V)\) relative to the impulsive system (D). Note that, besides the variables \(t\) and \(x\), the function \(\varphi\) may depend on the variable \(V\), which is responsible for the impulsive dynamics and has properties of the state and time variables simultaneously.

Definition 2. A function \(\varphi\) is said to be strongly increasing if for any pair \((x_0(), V_0()) \in T E_T\) the function \(t \to \)
\( \varphi(t, x(t), V(t)) \) does not decrease on \( T \), and the following
inequalities hold
\[
\varphi(t_0, x(t_0), 0) \leq \varphi(t, x(t), V(t)) \leq \varphi(t, x(t), V(t)) \quad \forall t \in (t_0, t_1].
\]

Let \( \Phi_s \) be the set of all continuous strongly increasing \( L \)-functions related to system \( (D) \).

Now we shall establish infinitesimal condition for strongly increasing \( L \)-functions. Before stating this condition, let us recall some concepts of the nonsmooth analysis Clarke et al. [1998], Vinter [2000].

A vector \( p \in \mathbb{R}^k \) is called a proximal subgradient of \( \varphi : \mathbb{R}^k \to \mathbb{R} \) at \( y \in \mathbb{R}^k \) if there exist a neighborhood \( O(y) \) of \( y \) and a positive number \( c \) such that
\[
\varphi(z) \geq \varphi(y) + \langle p, z - y \rangle - c|z - y|^2 \quad \forall z \in O(y).
\]
The set of all proximal subgradients at \( y \) is called the proximal subdifferential of \( \varphi \) at \( y \) and denoted by \( \partial P \varphi(y) \). Note that the set \( \partial P \varphi(y) \) may be empty (then the corresponding proximal Hamilton–Jacobi inequalities below are considered as fulfilled at the point \( y \)). Note also that if \( \varphi \) is a differentiable function, then \( \partial P \varphi(y) \subseteq \{ \nabla \varphi(y) \} \); moreover, the equality takes place if \( \varphi \) is a twice continually differentiable function. The proximal superdifferential of \( \varphi \) at \( y \), written \( \partial P^+ \varphi(y) \), is defined as the set \(- \partial P(- \varphi(y))\).

Let us define the functions
\[
h_0(t, x, \psi) = \langle \psi, f(t, x) \rangle, \\
h_1(t, x, \psi) = \min_{\omega \in \bar{K}_1} \langle \psi, G(t, x, x) \rangle,
\]
and the next condition relative to a function \( \varphi(t, x, V) \):

**Condition (H-J)\_s.** The function \( \varphi \) is a solution of the following system of proximal Hamilton–Jacobi inequalities:

\( (A1) \)
\[
\forall p = (p_x, p_v) \in \partial P \varphi(t, x, V), \\
\forall (t, x, V) \in (0, t_1) \times \mathbb{R}^n \times (0, +\infty).
\]

\( (A2) \)
\[
\forall (p_x, p_v) \in \partial P \varphi(t, x, V), \\
\forall (t, x, V) \in (0, t_1) \times \mathbb{R}^n \times (0, +\infty).
\]

**Theorem 3.** \( \varphi \in \Phi_s \) iff condition \( (H-J)_s \) holds.

**Proof.** We shall use a description of the impulsive control system \( (D) \) via an auxiliary control system obtained by a certain discontinuous time transformation (namely, the so-called space-time systems in the terminology of Motta and Rampazzo [1995]). The auxiliary control system corresponding to \( (D) \) has the following form
\[
\begin{align*}
\dot{t}(\tau) &= t_0, \\
\dot{t}(\tau) &= t_1, \\
\dot{x}(\tau) &= f(t(\tau), x(\tau)) \varphi_0(\tau) + G(t(\tau), x(\tau)) \psi(\tau), \\
\dot{V}(\tau) &= 1 - \varphi_0(\tau), \\
(\omega(\tau), \omega(\tau)) \in \mathbb{R}^n \times \mathbb{R}^n
\end{align*}
\]
\( \text{L-a.e. } \tau \in [0, t_1] \).

Here, \( t(\cdot), x(\cdot), V(\cdot) \) are absolutely continuous functions, \( \omega(\cdot), \omega(\cdot) \) are \( L \)-measurable bounded functions, \( K_1 = \{ (\varphi_0, \varphi_0) : \varphi_0 \geq 0, \varphi_0 = 0 \} \), and \( (\tau_1, \tau_2) = t_1 \). Then \( \varphi \) is a nonfixed terminal instant of time (by definition, \( \tau_1 = t_1 - t_0 + V(\tau_1) \)), and the prime denotes differentiation with respect to the new time \( \tau \).

Let \( G \) be the set of all solutions \( (\tau_1, t(\cdot), x(\cdot), V(\cdot)) \) of the auxiliary control system \( (3), (4) \). Then there is the one-to-one correspondence between \( \Sigma_T \) and \( G \). Moreover, any given \( \{ \tau, t(\cdot), x(\cdot), V(\cdot) \} \in T \) corresponding to each other satisfy the following conditions
\[
t = t(\eta(t)), \quad x(t) = x(\eta(t)), \quad V(t) = V(\eta(t)) \quad \forall t \in T,
\]
where \( \eta(t) := \inf \{ \tau : t(\tau) > t \} \). It is readily seen that there may be a nonunique \( g \in \mathcal{G} \) corresponding to a given \( \{ \tau, t(\cdot), x(\cdot), V(\cdot) \} \in T \). However, for every \( g \in \mathcal{G} \) there exists a sequence \( \{ g_k \} \in \mathcal{G} \) such that, for each \( k, g_k \) corresponds to a pair of absolutely continuous functions \( \{ x_k(\cdot), V_k(\cdot) \} \in T \) and, moreover, \( \{ g_k \} \) converges uniformly to \( g \) as \( k \to \infty \).

Let us note that \( \varphi \) is a strongly increasing \( L \)-function relative to \( (D) \) if \( \varphi \) is the same for \( (3), (4) \); that is, the function \( \tau \to \varphi(t(\tau), x(\tau), V(\tau)) \) does not decrease on \( [0, \tau_1] \) for all \( (\tau_1, t(\cdot), x(\cdot), V(\cdot)) \in \mathcal{G} \). Therefore it is sufficient to study condition \( (H-J)_s \) with respect to \( (3), (4) \). It is easy to see that \( (A1) \) is necessary and sufficient for the following proximal inequality
\[
\min_{(\omega_0, \omega_0) \in \bar{K}_1} \left\{ p x \varphi_0 + \langle p_x, f(t, x) \varphi_0 + G(t, x, x) \rangle \right\} + \varphi_0(1 - \varphi_0) > 0 \\
\forall p = (p_x, p_v) \in \partial P \varphi(t, x, V), \\
\forall (t, x, V) \in (0, t_1) \times \mathbb{R}^n \times (0, +\infty).
\]

Indeed, let us represent the left-hand side of \( (5) \) in the equivalent form
\[
\min_{(\omega_0, \omega_0) \in \bar{K}_1} \left\{ p x \varphi_0 + \langle p_x, f(t, x) \varphi_0 + G(t, x, x) \rangle \right\} + \varphi_0(1 - \varphi_0) > 0 \\
\forall p = (p_x, p_v) \in \partial P \varphi(t, x, V), \\
\forall (t, x, V) \in (0, t_1) \times \mathbb{R}^n \times (0, +\infty).
\]

By using \( (A1) \), hence the inequality \( (5) \) follows. Now, by merging \( (A1) \) with \( (A2) \) and following Clarke et al. [1998], Vinter [2000], we see that \( \varphi \) is a strongly increasing \( L \)-function relative to \( (3), (4) \) iff condition \( (H-J)_s \) holds. Theorem is proved.

Strongly monotone \( L \)-functions are very convenient for obtaining an outer estimates of the reachable sets and sufficient (necessary) global optimality conditions of Carathéodory and Krotov types (in particular, canonical optimality conditions; see below). In Dykhta and Samsonyuk [2010], another kinds of monotone \( L \)-functions for impulsive system, which are useful for the control theory, are defined.

**4. GLOBAL OPTIMALITY CONDITIONS FOR OPTIMIZATION PROBLEM OF IMPULSIVE SYSTEM \( (D) \)**

Consider the following optimal control problem (we label it \( (P) \) ) in which dynamics is governed by impulsive system \( (D) \):
\[
\begin{align*}
\text{minimize} & \quad J(\sigma) := l(q) \\
\text{subject to} & \quad q \in Q, \quad \sigma \in \Sigma_T, \\
q &= q_0 = (x(t_0), x(t_1), V(t_1)), \quad V(\cdot) = V_\varphi(\cdot),
\end{align*}
\]
Denote by $\Sigma$ the set of feasible processes in problem (P) and let
\[ \bar{q} := \sigma(t_0), x(t_1), V(t_1). \]

**Definition 4.** Denote by $\mathcal{R}$ a set of triples $q = (x_0, x_1, V_1)$ such that, for $q \in \mathcal{R}$, there exists $(x(t), V(t)) \in TET$ such that $x(t_0) = x_0$, $x(t_1) = x_1$, $V(t_1) = V_1$. We call $\mathcal{R}$ the conjoined set of system (D).

It is evidently that
\[ \min\{l(q) \mid q \in \mathcal{R} \cap Q\}. \]

Let $\Phi \subseteq \Phi_s$ be a certain set of strongly increasing $L$-functions for impulsive system (D). Introduce the set
\[ E(\Phi) = \{ q \in \mathbb{R}^{2n+1} \mid \varphi(t_1, x_1, V_1) - \varphi(x_0, x_0, 0) \geq 0 \quad \forall \varphi \in \Phi \} \]
and consider the following finite-dimensional extremal problem (EP(\Phi)):
\[ \text{minimize } l(q) \text{ subject to } q \in E(\Phi) \cap Q. \]

**Theorem 5.** (a) Every set $\Phi \subseteq \Phi_s$ gives the estimate
\[ \min\{l(q) \mid q \in E(\Phi) \cap Q\}. \]

(b) Suppose that there exists a set $\Phi \subseteq \Phi_s$ such that the vector $q$ is a global minimum point in problem (EP(\Phi)), i.e.
\[ J(\bar{q}) = l(q) = \min\{l(E(\Phi)) \}. \]

Then $\bar{q}$ is a global minimizer in problem (P).

These assertions immediately follow from the evident inclusion $E(\Phi) \supset \mathcal{R}$.

**Definition 6.** A set $\Phi \subseteq \Phi_s$ is said to be a lower support set of $L$-functions for problem (P) if
\[ \min\{l(q) \mid q \in E(\Phi) \cap Q\}. \]

It is notable that Theorem 5 admits the conversion for the problem (P_0) with fixed initial condition $x(t_0) = x_0$. Moreover, for a such problem there exists a lower support set $\Phi \subseteq C^1(\mathbb{R}^{n+1})$, i.e. it consists of smooth $L$-functions. The proof of this conversion is based on the connections between problem (P_0) and the corresponding standard optimal control problem (P_0) governed by spacetime system (3), (4). Besides, the proof of the necessary conditions essentially uses the known results on a smooth duality for a conventional optimal control problem Clarke and Nour [2005], Vinter [1993], Dykhta [2010], Dykhta and Samsonyuk [2010a].
\((MH_1)\) \(\forall\) pair of generalized control \((u(\cdot), \mu(dt))\) such that
\[u(t) \in U_L \quad \text{and} \quad \mu(B) \in K \quad \forall \ B \in B_T\]
the next maximality condition holds:
\[
\int_T H_1\left(t, \dot{x}(t), \psi(t), \frac{d\mu(t)}{dt}\right) \, d\mu(t) \\
\leq \int_T H_1\left(t, \dot{x}(t), \psi(t), \vartheta(t)\right) \, dV(t) = 0,
\]
where \(\mu(t) = \text{Var} \mu(\tau)\).

Note that MP for problem \((P_{u, \mu})\) includes condition \((MH_1)\) but here we consider the functions \(\psi(\cdot)\) without any transversality conditions.

Assume that \(\Psi(\bar{\sigma}) \neq \emptyset\). This is the main condition strengthening MP. Then it is easy to prove that any linear function
\[\varphi^\sigma(t, x, V) = \varphi_\sigma(t) (\dot{x}(t) - x) + \psi_\sigma(V(t) - V), \quad \psi \in \Psi(\bar{\sigma})\]
is strongly increasing w.r.t. system \((D_{u, \mu})\). Thus we may take the family of \(L\)-functions
\[\Phi = \{\varphi^\sigma(t, x, V) \mid \psi \in \Psi(\bar{\sigma})\}\]
and consider the finite-dimensional extremal problem \((EP(\Phi))\) that is similar to problem \((EP(\Phi))\). By this way we shall obtain an analog of Theorem 5, i.e. the sufficient optimality conditions in the form of MP. We omit the evident details of corresponding arguments.

Notably, the presented converse of MP is true without normality assumptions for process \(\sigma\) and condition \((MH_0)\) is weaker than the concavity condition of the function \(x \to H_0(t, x, \varphi_\sigma(t))\) that used in well-known variants of MP sufficientity (see, for example, [Miller and Rubinovich, 2003, Chapter 6]).

6. CONCLUSION

We have obtained Hamilton-Jacobi canonical sufficient and necessary optimality conditions for impulsive optimal control problems under common nonsmooth endpoint constraints. These conditions are formulated in terms of a set of nonsmooth strongly monotone \(L\)-functions, i.e. supersolutions of the special proximal Hamilton-Jacobi equation. In addition, general sufficient optimality conditions in the form of MP for the nonconvex problems of impulsive controls have been proposed. This MP reverse uses a set of coextremals (i.e. solutions of the adjoint system corresponding to an examined extremal of the impulsive control system).

Some other issues are assumed to be represented in the report. In particular, necessary optimality conditions in terms of supersolutions of the Hamilton-Jacobi equation, a construction of suboptimal impulsive feedback control (an analog of methods developed in Clarke et al. [1997], Dykhta [2010], Subbotin [1995]), and applications.

REFERENCES


