Task-Level Control of Motion and Constraint Forces in Holonomically Constrained Robotic Systems

Vincent De Sapio

Sandia National Laboratories, Livermore, CA 94550 USA
(e-mail: vdesap@sandia.gov)

Abstract: Holonomically constrained multibody systems constitute an important class of robotic systems. Under holonomic constraints motion is restricted to a constrained motion manifold within configuration space. The task-level control scheme presented here provides an effective approach to executing motion control in the presence of constraints. This scheme also allows for the simultaneous specification of desired constraint forces, given sufficient actuation, by exposing both motion coordinates and constraint forces within the control formalism. This allows for substantial flexibility in control synthesis and, thus, this methodology can be extensively applied to a wide range of holonomically constrained systems. An example is presented that demonstrates the efficacy of the analytical framework and its ease of implementation in practical robotic control problems involving constraints.

Keywords: Task-level control, dynamics, holonomic constraints, Lagrange multipliers

1. INTRODUCTION

The synthesis of control for holonomically constrained multibody systems is of significant importance, particularly with respect to applications involving parallel robotic devices, De Sapio et al. (2006a) and De Sapio et al. (2006b) (see Fig. 1 Left). Additionally, there are myriad other applications for which the control of constrained multibody systems is relevant. For example, holonomic constraints characterize a large array of biological systems, at both the macro and micro-scales. Such systems include the human shoulder complex, De Sapio et al. (2006a) and Holzbaur et al. (2005) (see Fig. 1 Right), as well as molecular motors and motor proteins like myosin V, Parker et al. (2009).

In the case that the system constraints are expressed as functions solely of the generalized coordinates (holonomic), rather than the generalized velocities (nonholonomic), the constraints restrict the motion of the system to a constrained motion manifold within configuration space. The specified motion must, therefore, be consistent with this restricted subset of configuration space. Simple joint space control is problematic since the entire configuration space is assumed accessible and a particular set of arbitrarily chosen joint space coordinates will likely violate the system constraints. A task space control scheme, Khatib (1987) and Khatib (1995), avoids this problem since, for redundant systems, a point in task space maps to a self-motion manifold, Burdick (1989), in configuration space. As long as the constrained motion and self-motion manifolds intersect, valid constraint consistent and task consistent solutions exist.

In addition to controlling motion, subject to constraints, it may be desirable to control constraint forces. Given sufficient actuation this can be achieved simultaneously to the execution of motion control. To this end, this paper presents a task-level approach to formulating control for both motion objectives and constraint forces in holonomically constrained robotic systems. The motion objectives are specified as a set of task conditions, or holonomic and rheonomic (explicitly time-dependent) servo constraints, De Sapio et al. (2006b) and Papastavridis (2002), while system constraints are imposed. In a complementary manner, desired constraint forces are specified and accounted for, along with the motion objectives, in the overall compensation.

Due to the symmetry between tasks and constraints, the system constraints can be incorporated naturally into the overall task-level dynamics. Building on earlier work that exploits this symmetry, De Sapio and Park (2010), a task-level formulation is presented that exposes both motion coordinates and constraint forces within the control formalism. This allows for substantial flexibility in the synthesis of a particular controller and, thus, this methodology can be extensively applied to a wide range of holonomically constrained systems. An example is presented to illustrate the practical application of this methodology to parallel robots.

2. TASK SPACE DESCRIPTION

A task space description is used as the basis for the results to be developed throughout this paper. A brief review will be presented based on De Sapio and Park (2010). For more
Fig. 1. Multibody systems consisting of holonomic constraints. (Left) Bilateral representation of a parallel-serial robotic shoulder mechanism, De Sapio et al. (2006a) and Lenarčič and Stanišić (2003). The parallel mechanism shoulder girdle is attached to a fixed torso. The humerus link is attached to the shoulder girdle via a spherical glenohumeral joint. (Right) In the human shoulder complex the shoulder girdle (scapula and clavicle) is kinematically coupled to the glenohumeral joint, De Sapio et al. (2006a) and Holzbaur et al. (2005).

This term is given by,
\[ \bar{J} = M^{-1}J^T \Lambda. \] (8)

The null space projection matrix, \( N^T \), is given by,
\[ N^T = 1 - J^T \bar{J} J^T. \] (9)

Fig. 2. A branching chain with task, \( x = (x_1^T \ x_2^T)^T \). The task space vectors, \( x_1 \) and \( x_2 \), describe the Cartesian positions of the two terminal points. The task force, \( f = (f_1^T \ f_2^T)^T \), is applied at the task points.

detailed expositions of task space dynamics the reader is referred to Khatib (1987) and Khatib (1995).

The operational space framework addresses the dynamics and control of branching chain robots. Given a branching chain system the initial step involves defining a set of \( m \) task, or operational space, coordinates, \( x \in \mathbb{R}^m \). The function \( x(q) \) represents a kinematic mapping from the set of \( n \) generalized coordinates, \( q \in \mathbb{R}^n \), to the set of task space coordinates. The task space coordinates can represent any function of the generalized coordinates but typically are chosen to describe the set of control coordinates (control output) associated with a motion control task. Fig. 2 illustrates a branching kinematic chain where the task space coordinates are chosen to be the coordinates associated with positioning the terminal points of the chain. Further, by taking the gradient of \( x \) we have the relationship,
\[ \dot{x} = J(q) \dot{q}, \] (1)

where \( J(q) \) is the Jacobian of \( x \).

At this point we can address the dynamics of a system in task space. In the non-redundant case any generalized force can be produced by a task space force, \( f \in \mathbb{R}^m \), acting at the task point along the task coordinates. Fig. 2 illustrates the action of the task space force for the intuitive case of a Cartesian positioning task. The generalized force (or control torque), \( \tau \in \mathbb{R}^n \), is then composed as \( J^T f \). In the redundant case an additional term needs to complement the task term in order to realize any arbitrary generalized force. We will refer to this term as the null space term and it can be composed as \( N^T \tau \), where \( \tau \) is an arbitrary generalized force and \( N^T \) is the null space projection matrix.

We now express the configuration space equation of motion,
\[ M(q) \ddot{q} + b(q, \dot{q}) + g(q) = J^T f + N^T \tau, \] (3)

where \( M(q) \in \mathbb{R}^{n \times n} \) is the configuration space mass matrix, \( b(q, \dot{q}) \in \mathbb{R}^n \) is the vector of centrifugal and Coriolis terms, and \( g(q) \in \mathbb{R}^n \) is the vector of gravity terms.

This can be manipulated to arrive at the task space (operational space) equation of motion,
\[ \Lambda(q) \ddot{q} + \mu(q, \dot{q}) + p(q) = f, \] (4)

where \( \Lambda(q) \in \mathbb{R}^{m \times m} \) is the task space mass matrix, \( \mu(q, \dot{q}) \in \mathbb{R}^m \) is the task space centrifugal and Coriolis force vector, and \( p(q) \in \mathbb{R}^m \) is the task space gravity vector. These terms are given by,
\[ \Lambda(q) = (JM^{-1}J^T)^{-1}, \] (5)
\[ \mu(q, \dot{q}) = \Lambda J M^{-1} \dot{b} - \Lambda \dot{J} \dot{q}, \] (6)
\[ p(q) = \Lambda J M^{-1} g. \] (7)

We also denote \( \bar{J} \) as the dynamically consistent inverse of \( J \), Khatib (1995). This term is given by,
\[ \bar{J} = M^{-1}J^T \Lambda. \] (8)

The null space projection matrix, \( N^T \), is given by,
\[ N^T = 1 - J^T \bar{J} J^T. \] (9)
Fig. 3. A system with holonomic constraints in the form of loop constraints. The task space vector, $x$, describes the Cartesian position of a point on one of the links. The objective is to control the system using task-level commands, in the presence of the mechanism constraints.

The overall dynamics of our system can be mapped into task space by,

$$\mathbf{M}\ddot{x} + b + g = \tau \Rightarrow \mathbf{f} = \mathbf{Λ}\dot{x} + \mu + p. \quad (10)$$

In a complementary manner the overall dynamics can be mapped into the task consistent null space (or self-motion space) using $\mathbf{N}^T\tau_n$.

We can design control for our system in task space coordinates using (4). Additionally, we can specify the null space behavior of our system with the term $\mathbf{N}^T\tau_n$. The null space control term is guaranteed not to interfere with the task dynamics of (4) due to the condition of dynamic consistency. This allows for decoupled control design. Finally, the overall control torque applied to the system is composed as in (3), that is,

$$\tau = \mathbf{J}^T\mathbf{f} + \mathbf{N}^T\tau_n. \quad (11)$$

A controller employing (4) would be assumed to have imperfect knowledge of the system. Therefore, (4) should reflect estimates for the inertial and gravitational terms. Additionally, a control law needs to be incorporated. To this end we replace $\mathbf{f}$ in (4) with the input of the decoupled system, Khatib (1995) $\mathbf{f}^*$, to yield the dynamic compensation equation,

$$\mathbf{f} = \mathbf{Λ}\dot{x}^* + \mu + p, \quad (12)$$

where the $\ddot{x}$ represents estimates of the dynamic properties. Any suitable control law can be chosen to serve as input of the decoupled system. In particular, we can choose a linear proportional-derivative (PD) control law of the form,

$$\mathbf{f}^* = \mathbf{K}_p(x_d - x) + \mathbf{K}_\dot{x}(\dot{x}_d - \dot{x}) + \ddot{x}_d, \quad (13)$$

where $x_d$ are reference values for the task coordinates and $\mathbf{K}_p$ and $\mathbf{K}_\dot{x}$ are gain matrices.

3. TASK-LEVEL CONTROL WITH CONSTRAINTS

We now introduce a set of $m_C$ holonomic and scleronomous (not explicitly time-dependent), Lanczos (1986), constraint equations, $\phi(q) = 0$, to the system of (3). The mechanism of Fig. 3 is an example of a system where the holonomic constraints describe a kinematic loop closure. A task, $x$, is to be controlled subject to the constraints. In general, the constraint equations are satisfied on a $p = n - m_C$ dimensional manifold, $Q^p$, in configuration space, $Q = \mathbb{R}^n$. The gradient of $\phi$ yields the constraint Jacobian, $\Phi$,

$$\Phi(q) = \frac{\partial \phi}{\partial q} \in \mathbb{R}^{m_C \times n}. \quad (14)$$

Using Lagrange multipliers, $\lambda$, which represent the constraint forces, the constrained system dynamics can be expressed as,

$$\mathbf{M}\ddot{x} + b + g - \Phi^T\lambda = \tau, \quad (15)$$

subject to,

$$\phi(q) = 0. \quad (16)$$

This represents the constrained dynamics of the plant that we wish to control. Fig. 4 illustrates the plant in block diagram form. Alternately, we can express the constrained dynamics in terms of task space parameters, De Sapio and Park (2010), as this is better suited for our purposes. Expressed the constrained dynamics in this way yields,

$$\Theta^T\mathbf{J}^T(\mathbf{Λ}\dot{x}^* + \mu + p) + \Phi^T(\alpha + \rho) + \mathbf{N}^T\tau_n - \Phi^T\lambda = \tau. \quad (17)$$

Prior to defining the terms in (17) we will first define $H(q) \in \mathbb{R}^{m_C \times m_C}$ as the constraint space mass matrix which reflects the system inertia projected at the constraint. The term $\alpha(q, \dot{q}) \in \mathbb{R}^{m_C}$ is the vector of centrifugal and Coriolis forces projected at the constraint, and $\rho(q) \in \mathbb{R}^{m_C}$ is the vector of gravity forces projected at the constraint. These terms are given by,

$$H(q) = (\Phi\mathbf{M}^{-1}\Phi^T)^{-1}, \quad (18)$$

$$\alpha(q, \dot{q}) = \mathbf{H}\Phi\mathbf{M}^{-1}b - \mathbf{H}\dot{\Phi}\dot{q}, \quad (19)$$

$$\rho(q) = \mathbf{H}\Phi\mathbf{M}^{-1}g. \quad (20)$$

We also denote $\Phi$ as the dynamically consistent inverse of $\Phi$. This term is given by,

$$\Phi = \mathbf{M}^{-1}\Phi^T H. \quad (21)$$

The constraint null space projection matrix, $\Theta(q)T \in \mathbb{R}^{n \times n}$, is given by,

$$\Theta(q)T = 1 - \Phi^T \Phi T. \quad (22)$$

The term $\Lambda_c(q) \in \mathbb{R}^{m \times m}$ is the task/constraint space mass matrix, $\mu_c(q, \dot{q}) \in \mathbb{R}^{m}$ is the task/constraint space
Fig. 5. The relationship between the number of generalized coordinates - \( n \), degrees of freedom - \( p \), constraints - \( m_C \),... = 0). That is, 
\[ x \equiv (q_7 \ q_8)^T. \] (40)
We will specify the reference value as, 
\[ x_d = (−0.25 \ 2.75)^T. \] (41)
centrifugal and Coriolis force vector, and \( p_c(q) \in \mathbb{R}^m \) is the task/constraint space gravity vector. These terms are given by, 
\[ \Lambda_c(q) = (JM^{-1} \Theta^T J^T)^{-1}, \] (23)
\[ \mu_c(q, \dot{q}) = \Lambda_c JM^{-1} \Theta^T b - \Lambda_c (\dot{J} - JM^{-1} \Phi^T \dot{\Phi}) q, \] (24)
\[ p_c(q) = \Lambda_c JM^{-1} \Theta^T g. \] (25)
The term \( N_c^T \tau_c \) in (17) represents the null space (with respect to both task and constraints) component of the generalized force. The task/constraint null space projection matrix \( N_c^T \in \mathbb{R}^{n \times n} \) can be formulated recursively from the individual constraint and task null spaces (see Appendix A), 
\[ N_c(q)^T = \Theta^T (1 - J^T \Lambda_c J \Theta M^{-1}). \] (26)
Equation (17) expresses the control torque as a function of the task accelerations, \( \ddot{x} \), the kinematic and dynamic properties, and the constraint forces, \( \lambda \). Employing a linear control law the control equation can be expressed as,
\[ \tau + \Phi^T \lambda = \hat{\Theta}^T \hat{J}^T (\hat{\Lambda}^T \Gamma^* + \hat{\mu}_c + \hat{\rho}_c) + \Phi^T (\hat{\alpha} + \hat{\rho}) + \hat{N}^T \tau_c, \] (27)
where,
\[ \Gamma^* = K_p (x_d - \dot{x}) + K_v (\ddot{x}_d - \ddot{x}) + \ddot{x}_d. \] (28)
These equations need to be complemented by the condition on the unactuated joints,
\[ S_p \tau = 0, \] (29)
where \( S_p \) is a selection matrix that identifies the passive (unactuated) joints.

### 3.1 Controlling Constraint Forces

Before proceeding to specify the control of constraint forces we will review the dimensionality of our problem. Given a system with \( n \) generalized coordinates and \( m_c \) holonomic constraints there are \( p = n - m_c \) degrees of freedom characterizing the constrained motion. This motion space can be controlled by specifying a task with \( m \) task coordinates. The null space dimensionality, \( N = p - m \), characterizes the remaining dimensionality of the constrained motion space not used by the task. This is illustrated in Fig. 5. Given \( k \) actuated joints we will refer to the condition that \( k \geq p \) as motion actuated, De Sapio et al. (2006b).

Given a sufficient number of actuated joints, \( k \), some of the constraint forces, \( \lambda \), can be controlled. For example, if \( k > p \), then \( k - p \) constraint forces can be controlled. Let us introduce a selection matrix, \( S_c \in \mathbb{R}^{(k-p) \times m_c} \), to select the controlled constraint forces and a selection matrix, \( S_u \in \mathbb{R}^{(n-k) \times m_c} \), to select the uncontrolled constraint forces. That is,
\[ \lambda_c = S_c \lambda \quad \text{and} \quad \lambda_u = S_u \lambda, \] (30)
or,
\[ \begin{pmatrix} \lambda_c \\ \lambda_u \end{pmatrix} = \begin{pmatrix} S_c \\ S_u \end{pmatrix} \lambda, \] (31)
where \( \lambda_c \) and \( \lambda_u \) are the vectors of controlled and uncontrolled constraint forces, respectively, selected out of the full vector of constraint forces, \( \lambda \). Inverting (31) we have,
\[ \lambda = \begin{pmatrix} S_c^T \\ S_u^T \end{pmatrix} \begin{pmatrix} \lambda_c \\ \lambda_u \end{pmatrix}. \] (32)
Substituting this into (27) we have,
\[ \tau + \Phi^T S_c^T \lambda_u = \hat{\Theta}^T \hat{J}^T (\hat{\Lambda}^T \Gamma^* + \hat{\mu}_c + \hat{\rho}_c) + \Phi^T (\hat{\alpha} + \hat{\rho} - S_c^T \lambda_c) + \hat{N}^T \tau_c. \] (33)
Thus, (27) and (29) can be represented as the following system of equations,
\[ \begin{pmatrix} \Phi^T S_c^T \\ \Phi^T S_u^T \end{pmatrix} \begin{pmatrix} \tau \\ \lambda_u \end{pmatrix} = \begin{pmatrix} h(q, \dot{q}) \\ 0 \end{pmatrix}. \] (35)
Given the inverse,
\[ \begin{pmatrix} \Phi^T S_c^T \\ \Phi^T S_u^T \end{pmatrix} \begin{pmatrix} \Phi^T S_c^T \end{pmatrix}^{-1} = \begin{pmatrix} 1 - \Phi^T S_c^T (S_c \Phi^T S_c^T)^{-1} S_c \Phi^T S_c^T (S_c \Phi^T S_c^T)^{-1} \end{pmatrix}^{-1}, \] (36)
we have the following solution for the control torque,
\[ \tau = (1 - \Phi^T S_u^T (S_c \Phi^T S_c^T)^{-1} S_p) h(q, \dot{q}). \] (37)
A block diagram of this control scheme is shown in Fig. 6. Appendix B validates that this control torque produces the desired motion and constraint forces. As an illustrative example of this control scheme we consider the parallel mechanism depicted in Fig. 7. The constraint equations describe the loop closures and are given by,
\[ \phi(q) = \begin{pmatrix} r_{p_1} - r_{l_1} \\ r_{p_2} - r_{l_2} \\ r_{p_3} - r_{l_3} \end{pmatrix}. \] (38)
Considering two of the base joints, \( q_1, q_3 \), as well as the elbow joints, \( q_2, q_4 \), and \( q_6 \), to be actuated we have,
\[ S_p = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}. \] (39)
We will define the task to control the position of the platform (see Fig. 7) while its orientation is uncontrolled \( (\tau_u = 0) \). That is,
\[ x \triangleq (q_7 \ q_8)^T. \] (40)
We will specify the reference value as,
\[ x_d = (-0.25 \ 2.75)^T. \] (41)
Fig. 6. A task space tracking controller for the constrained plant of Fig. 4. The desired task motion and constraint forces are tracked using appropriate dynamic compensation which accounts for the constraints.

Additionally, we wish to specify the constraint forces at the interface of \( r_p \) and \( r_1 \). These correspond to,

\[
\lambda_c \triangleq (\lambda_1, \lambda_2)^T .
\] (42)

We will specify the reference value as,

\[
\lambda_{cd} = (25 \sin(t/50) \ 150 \cos(t/200))^T .
\] (43)

The remaining constraint forces will be unspecified. Thus,

\[
S_c = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix},
\] (44)

and,

\[
S_u = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.
\] (45)

The linear (PD) control law of (28) is used as the input of the decoupled system. The gains are chosen so as to achieve critically damped behavior of the task motion. Equation (37) is used to compute the control torque. The system dynamics were computed symbolically using a Lagrangian formulation and solved numerically. Fig. 8 shows simulation plots for the system under goal position commands on the task coordinates, \( x \), and sinusoidal tracking commands on the constraint force coordinates, \( \lambda_c \). The time response of the platform orientation shows undamped oscillation due to the uncontrolled null space.

Fig. 9 shows simulation plots of some of the control torques generated for this motion. It is noted that zero control torque is produced at the passive joint, \( \tau_5 \), due to the condition of (29). The last plot shows the time response of one of the uncontrolled constraint forces.

4. CONCLUSION

We have presented a novel approach to formulating task-level motion control for holonomically constrained robotic systems that allows for the simultaneous specification of desired constraint forces. The control equations underlying this approach have been derived here, building on our previous work. The necessary conditions relating the number of actuators, task coordinates, and constraint coordinates have also been defined. The approach presented leverages the symmetry between constrained dynamics and task space dynamics. It provides a natural scheme for control synthesis by exposing the coordinates, both motion and constraint, of interest. An example was presented to demonstrate the efficacy of this approach in simulation. As a practical matter it is assumed that the controller has access to the system state (via a forward dynamics solver in the simulated case or via sensors in the physical case) and estimates of the dynamic properties of the physical system. The results indicate that the analytical framework presented can be implemented in practical constrained robotic control problems.

It is believed that the approach presented here can be applied to a broader extent than engineered robotic systems. Certain aspects of skeletal physiology (e.g., the human shoulder complex) can be modeled and simulated using this approach. Additionally, there are potential applications in structural biology. Specifically, task-level decomposition and control synthesis has potential relevance to problems in protein folding and the behavior of molecular motors (see Fig. 10 for a representation of myosin V). Both of these problems involve understanding the behavior of collections of polypeptide chains with large numbers of conformational degrees-of-freedom. The control synthesis framework presented here possesses potential efficacy in dealing with such systems and could be useful, at a coarse-grained level, for investigating conformational changes in molecular motors and proteins.

REFERENCES


Fig. 7. (Left) Parallel mechanism consisting of serial chains with loop closures. Two base joints, $q_1$, $q_3$, as well as the elbow joints, $q_2$, $q_4$, and $q_6$, are actuated while the remaining joints are passive. (Right) The position of the platform is commanded to move to a target while its orientation is uncontrolled. In this case $n = 9$, $m_C = 6$, $p = 3$, $m = 2$, $N = 1$, and $k = 5$.

Fig. 8. The position of the platform is commanded to move to a target while its orientation is uncontrolled and the constraint forces at the interface of $r_p$ and $r_1$ are specified. (Left) Time response of the platform position showing linear critically damped motion to the target. Time response of the platform orientation shows undamped null space oscillation due to the uncontrolled null space. The control gains are $K_p = 100$ and $K_v = 20$. (Right) Time response of the constraint forces, $\lambda_1$ and $\lambda_2$, showing tracking of the reference command.

Fig. 10. A representation of the motor protein, myosin V, which can be described as a multibody chain with loop closures formed when the myosin heads attach to actin binding sites. It undergoes conformational changes as it walks along the actin filament.
Fig. 9. (Top) Time response of the control torques $\tau_1$ and $\tau_2$ during goal movement. (Bottom Left) Zero control torque (numerical error at the order of $10^{-12}$) is produced at the passive joint $\tau_5$, due to the imposition of the passivity requirement in the controller. (Bottom Right) Time response of one of the uncontrolled constraint forces, $\lambda_4$.


Appendix A. TASK/CONSTRAINT NULL SPACE

The task/constraint null space matrix can be formulated using the recursive approach of De Sapio and Park (2010). We have,

$$N_c^T = \Theta^T \left[ 1 - \Theta^T J^T ( J \Theta M^{-1} \Theta^T J^T )^{-1} J \Theta M^{-1} \right].$$  \hspace{1cm} (A.1)

This can be simplified by noting two useful identities. First,

$$\Theta^T \Theta^T = (1 - \Phi^T \Phi)(1 - \Phi^T \Phi) = 1 - 2 \Phi^T \Phi^T$$

$$+ \Phi^T \Phi^T \Phi^T \Phi^T = 1 - \Phi^T \Phi^T = \Theta^T. \hspace{1cm} (A.2)$$

Next,

$$\Theta M^{-1} \Theta^T = (1 - \Phi \Phi)M^{-1}(1 - \Phi^T \Phi^T), \hspace{1cm} (A.3)$$

or,

$$\Theta M^{-1} \Theta^T = M^{-1} - M^{-1} \Phi^T \Phi^T - \Phi \Phi M^{-1}$$

$$+ \Phi \Phi M^{-1} \Phi^T \Phi^T, \hspace{1cm} (A.4)$$

where,

$$\frac{\Phi \Phi M^{-1} \Phi^T \Phi^T}{H^{-1}} = M^{-1} \Phi^T \Phi M^{-1} \Phi^T \Phi^T$$

$$= \Phi \Phi M^{-1} = M^{-1} \Phi^T \Phi^T. \hspace{1cm} (A.5)$$

So,

$$\Theta M^{-1} \Theta^T = M^{-1} - M^{-1} \Phi^T \Phi^T = M^{-1} \Theta^T$$

$$= M^{-1} - \Phi \Phi M^{-1} = \Theta M^{-1}. \hspace{1cm} (A.6)$$

Thus,

$$J \Theta M^{-1} \Theta^T J^T = J M^{-1} \Theta^T J^T = J \Theta M^{-1} J^T = \Lambda_c^{-1}. \hspace{1cm} (A.7)$$

Since $\Theta^T \Theta^T = \Theta^T$ and $J \Theta M^{-1} \Theta^T J^T = \Lambda_c^{-1}$, we have,

$$N_c(q)^T = \Theta^T (1 - J^T \Lambda_c J \Theta M^{-1}). \hspace{1cm} (A.8)$$
Appendix B. VALIDATION OF CONTROL TORQUE

We wish to validate that the control torque expressed in (37) generates the desired motion and constraint forces when fed into the plant. The constrained dynamics problem can be solved to yield explicit results for the generalized acceleration and constraint forces, De Sapio et al. (2006b). Given the constrained equations of motion,

\[
M \ddot{q} + b + g - \Phi T \gamma = \tau, \quad (B.1)
\]

and the constraint equations,

\[
\Phi \ddot{q} + \dot{\Phi} q = 0, \quad (B.2)
\]

the generalized acceleration is,

\[
\ddot{q} = \Theta M^{-1}(\tau - b - g) - \dot{\Phi} \Phi q, \quad (B.3)
\]

and the constraint force is,

\[
\gamma = -\dot{\Phi}^T (\tau - b - g) - H \dot{\Phi} \Phi q. \quad (B.4)
\]

The task acceleration can then be expressed as,

\[
\ddot{x} = J \ddot{q} + \dot{J} v = J \Theta M^{-1} \tau - J \Theta M^{-1} (b + g) - J \dot{\Phi} \Phi q + \dot{J} \dot{q} \quad (B.5)
\]

and the controlled constraint force can be expressed as,

\[
\lambda_c = S_c \lambda = -S_c \Phi^T \tau + S_c \Phi^T (b + g) - S_c H \dot{\Phi} \Phi q. \quad (B.6)
\]

The control torque is given by (37) and can be expressed as,

\[
\tau = \Theta^T J^T (A_c \Gamma_c + \mu_c + \rho_c) + \Phi^T (\alpha + \rho - S_c^T \lambda_c) + N^T \tau - \Phi^T S_b^T (S_b \Phi^T S_b^T)^{-1} S_b \Theta^T J^T (A_c \Gamma_c + \mu_c + \rho_c) - \Phi^T S_b^T (S_b \Phi^T S_b^T)^{-1} S_b \Phi^T (\alpha + \rho - S_c^T \lambda_c) - \Phi^T S_b^T (S_b \Phi^T S_b^T)^{-1} S_b N^T \tau, \quad (B.7)
\]

where we have made the assumption that the estimates of all dynamic parameters are perfect.

We would now like to compute the resulting task acceleration associated with this control torque. First we note,

\[
\Theta M^{-1} \Phi^T = (1 - \Phi^T) M^{-1} \Phi^T = M^{-1} \Phi^T - \Phi^T \Phi M^{-1} \Phi^T = M^{-1} \Phi^T - M^{-1} \Phi^T H \Phi M^{-1} \Phi^T = 0. \quad (B.8)
\]

We also note,

\[
J \Theta M^{-1} N^T_c = J \Theta M^{-1} \Phi^T (1 - J^T A_c J \Theta M^{-1}) = J \Theta M^{-1} \Theta^T - J \Theta M^{-1} = 0. \quad (B.9)
\]

Given (A.7), (B.8), and (B.9) we can evaluate \( J \Theta M^{-1} \tau \),

\[
J \Theta M^{-1} \tau = J \Theta M^{-1} \Theta^T J^T (A_c \Gamma_c + \mu_c + \rho_c) - J \Theta M^{-1} \Phi^T (\alpha + \rho - S_c^T \lambda_c) + J \Theta M^{-1} N^T \tau_c - J \Theta M^{-1} \Phi^T S_b^T (S_b \Phi^T S_b^T)^{-1} S_b \Theta^T J^T (A_c \Gamma_c + \mu_c + \rho_c) - J \Theta M^{-1} \Phi^T S_b^T (S_b \Phi^T S_b^T)^{-1} S_b \Phi^T (\alpha + \rho - S_c^T \lambda_c) - J \Theta M^{-1} \Phi^T S_b^T (S_b \Phi^T S_b^T)^{-1} S_b N^T \tau_c. \quad (B.10)
\]

So,

\[
J \Theta M^{-1} \tau = f^* + J M^{-1} \Theta^T (b + g) - (J \dot{J} M^{-1} \Phi^T H \dot{\Phi}) q - J \dot{\Phi} \Phi q. \quad (B.11)
\]

Thus, the resulting task acceleration associated with the control torque is,

\[
\ddot{x} = J \Theta M^{-1} \tau - J \Theta M^{-1} (b + g) - J \dot{\Phi} \Phi q + \dot{J} \dot{q} \quad (B.12)
\]

or,

\[
\ddot{x} = f^* + J M^{-1} \Theta^T (b + g) - \dot{J} \dot{q} + J \dot{\Phi} \Phi q - J \Theta M^{-1} (b + g) - J \dot{\Phi} \Phi q + \dot{J} \dot{q} = \Gamma^*. \quad (B.13)
\]

This expresses the feedback linearized system dynamics,

\[
\ddot{x} = f^* = K_p (x_d - x) + K_v (\dot{x}_d - \dot{x}) + \ddot{x}_d. \quad (B.14)
\]

We would now like to compute the resulting constraint force associated with the control torque. First we note the identity,

\[
\Phi^T \Theta^T = \Phi^T (1 - \Phi^T \Phi^T) = \Phi^T - \Phi^T \Phi^T \Phi^T = 0. \quad (B.15)
\]

We can now evaluate \( S_c \Phi^T \tau \),

\[
S_c \Phi^T \tau = S_c \Phi^T \Theta^T J^T (A_c \Gamma_c + \mu_c + \rho_c) - S_c \Phi^T \Theta^T (1 - J^T A_c J \Theta M^{-1}) \tau_c - S_c \Phi^T S_b^T (S_b \Phi^T S_b^T)^{-1} S_b \Theta^T J^T (A_c \Gamma_c + \mu_c + \rho_c) - S_c \Phi^T S_b^T (S_b \Phi^T S_b^T)^{-1} S_b \Phi^T (\alpha + \rho - S_c^T \lambda_c) - S_c \Phi^T S_b^T (S_b \Phi^T S_b^T)^{-1} S_b N^T \tau_c. \quad (B.16)
\]

So, noting that,

\[
S_c S_b^T = 0 \quad (B.17)
\]

we have,

\[
S_c \Phi^T \tau = S_c (\alpha + \rho - S_c^T \lambda_c) - S_c S_b^T (S_b \Phi^T S_b^T)^{-1} S_b \Theta^T J^T (A_c \Gamma_c + \mu_c + \rho_c) - S_c S_b^T (S_b \Phi^T S_b^T)^{-1} S_b \Phi^T (\alpha + \rho - S_c^T \lambda_c) - S_c S_b^T (S_b \Phi^T S_b^T)^{-1} S_b N^T \tau_c. \quad (B.18)
\]

and,

\[
S_c \Phi^T \tau = S_c (\alpha + \rho - S_c^T \lambda_c) = S_c (\alpha + \rho) - \lambda_c. \quad (B.19)
\]

Thus, the resulting constraint force associated with the control torque is,

\[
\lambda_c = -S_c \Phi^T \tau + S_c \Phi^T (b + g) - S_c H \dot{\Phi} \Phi q. \quad (B.20)
\]

or,

\[
\lambda_c = -S_c (\alpha + \rho) + \lambda_c + S_c [\Phi^T (b + g) - H \dot{\Phi} \Phi q] = \lambda_c. \quad (B.21)
\]

and the resulting constraint force in the controlled directions is the desired reference value,

\[
\lambda_c = \lambda_c. \quad (B.22)
\]