Quantization Scheme Design in Distributed Event-Triggered Networked Control Systems

Sun,Yu ∗ Xiaofeng Wang ∗

∗... are coupled together and each agent can receive information from some of other agents. Let \( N = \{1, 2, \cdots, N\} \).

Abstract: In an event-triggered networked control system (NCS), an agent broadcasts its state information to its neighbors when some local event occurs. This paper studies the design of quantization scheme in such systems. We derived conditions on quantized event-triggered NCS which, when satisfied, can guarantee the asymptotic stability of the system. We also propose an event-triggered dynamic logarithmic quantizer, which fits well into the current system framework, and incurs little communication overhead.

Keywords: Networked Control System, Event-triggered Control System, Quantization

1. INTRODUCTION

A distributed networked control system (NCS) consists of numerous coupled subsystems (also called “agents”), which are geographically distributed. In such a system, individual subsystems exchange information over a communication network. These networked systems are found throughout our national infrastructure with specific examples being the electrical power grid and transportation networks. Networking not only refers to the communication infrastructure supporting feedback control, but also refers to the fact that individual subsystems are physically interconnected in a way that can be modeled as a network.

Communication networks, in practice, broadcast data in discrete packets. Usually, the communication media is a resource that is accessed in a mutually exclusive manner by neighborhood agents. This means that the throughput capacity of such networks is limited. So one important issue in the implementation of such systems is to identify methods that more effectively use the limited network bandwidth available for transmitting state information.

To minimize the use of the communication resources, event-triggering comes to the stage. Recent work [Tabuada (2007); Wang and Lemmon (2009b)] shows that event-triggering can largely reduce the sampling rates in the single processor systems, compared with the periodic task models. This is because the system can adaptively adjust the rates in a manner dependent on what is currently happening within the system. This motivates the implementation of event-triggering in NCSs to save communication resources. Distributed event-triggering has an agent broadcast its state information only when some measure of the agent’s state error is above a specified threshold.

Preliminary results in distributed event-triggering were described in [Wang and Lemmon (2008)], where the authors showed asymptotic stability of NCS for linear and nonlinear systems, respectively. Transmission delays were considered in [Wang and Lemmon (2009a)], which provides an upper bound on delays to ensure the NCS is uniformly ultimately bounded.

These results, however, made the assumption that real numbers can be communicated among agents. With a band-limited communication channel, it is more appropriate to consider the situation when quantized states are being exchanged. Various authors has addressed the problem of stabilization over network in the presence of data-rate constraints [Wong and Brockett (2002); Elia and Mitter (2002); Nair and Evans (2003)]. In previous works on distributed event-triggered feedback schemes, the design of quantizer has not been investigated.

This paper studies the effect of quantization on event-triggered NCS. We derived constraints on the design parameters of the quantizers under the event-triggered control framework. In order to enforce these constraints, the idea in Brockett and Liberzon (2000); Liberzon and Nesic (2007) are adopted to propose an event-triggered dynamic quantizer along with zooming strategy. It fits well into the event-triggered control system framework and also features little communication overhead. The advantage of using zooming scheme is that it only has finite number of quantization levels but functions the same as a quantizer of infinite quantization levels. We show that under certain conditions, the NCS can be asymptotically stabilized using the proposed event-triggered zooming strategy. Our proposed quantization scheme can also serve as a testbed for further analysis of stabilizing data-rate in event-triggered networked control systems.

2. PROBLEM FORMULATION

Consider a distributed NCS containing \( N \) subsystems (also called “agents”). These \( N \) agents are coupled together and each agent can receive information from some of other agents. Let \( \mathcal{N} = \{1, 2, \cdots, N\} \) and
\[ \dot{x}_i(t) = f_i(x_{D_i}, u_i) \]
\[ u_i(t) = g_i(x_{Z_i}) \]
\[ x_i(t_0) = x_0 \]

where \( x_i : \mathbb{R}_0^+ \to \mathbb{R}^n \) is the state of agent \( i \), \( u_i : \mathbb{R}_0^+ \to \mathbb{R}^m \) is a control input, \( g_i : \mathbb{R}^{n|Z_i|} \to \mathbb{R}^m \) is the feedback strategy of agent \( i \) satisfying \( g_i(0) = 0 \), \( f_i : \mathbb{R}^{n|D_i|} \times \mathbb{R}^m \times \mathbb{R}^k \to \mathbb{R}^n \) is continuous and locally Lipschitz satisfying \( f_i(0, 0) = 0 \), and \( x_{D_i} = \{ x_j \}_{j \in D_i}, x_{Z_i} = \{ x_j \}_{j \in Z_i} \). For notational convenience, we assume that the states/inputs of agents have the same dimension. The results in this paper can be easily extended to the case where the dimensions of agents’ states/inputs are different from each other.

Notice that \( i \notin Z_i \cup D_i \cup U_i \).

The state equation of agent \( i \) is

\[ \dot{x}_i(t) = f_i(x_{D_i}, u_i) \]
\[ u_i(t) = g_i(x_{Z_i}) \]
\[ x_i(t_0) = x_0 \]

where \( x_i : \mathbb{R}_0^+ \to \mathbb{R}^n \) is the state of agent \( i \), \( u_i : \mathbb{R}_0^+ \to \mathbb{R}^m \) is a control input, \( g_i : \mathbb{R}^{n|Z_i|} \to \mathbb{R}^m \) is the feedback strategy of agent \( i \) satisfying \( g_i(0) = 0 \), \( f_i : \mathbb{R}^{n|D_i|} \times \mathbb{R}^m \times \mathbb{R}^k \to \mathbb{R}^n \) is continuous and locally Lipschitz satisfying \( f_i(0, 0) = 0 \), and \( x_{D_i} = \{ x_j \}_{j \in D_i}, x_{Z_i} = \{ x_j \}_{j \in Z_i} \). For notational convenience, we assume that the states/inputs of agents have the same dimension. The results in this paper can be easily extended to the case where the dimensions of agents’ states/inputs are different from each other.

\[ \frac{\partial V}{\partial x} f_i(x_{D_i}, g_i(x_{Z_i})) \leq \sum_{i \in N} (\alpha_i \| x_i \|^2 + \beta_i \| e_i \|_2) \]

If for any \( i \in \mathcal{N} \), there exists \( \rho_i \in (0, 1) \) such that

\[ \beta_i \| e_i(t) \|_2 \leq \rho_i \alpha_i \| x_i(t) \|_2 \]

holds for all \( t \geq 0 \), then the NCS is asymptotically stable.

3. DISTRIBUTED BROADCAST TRIGGERING
EVENT DESIGN UNDER QUANTIZATION

An implicit assumption in Lemma 1 is that agents can exchange their states as arbitrary real numbers. In this section we consider the case when only quantized values are exchanged. Suppose agent \( i \) employs a quantizer \( Q_i : \mathbb{R}^n \to \mathbb{R}^n \) to quantize its state \( x_i \). At time \( t \), agent \( i \) broadcasts the quantized version of its state, i.e.

\[ \hat{x}_i = Q_i(x_i(\hat{b}_k)) \]

As long as inequality (3) in Lemma 1 is satisfied, the asymptotic stability result will hold. This requires the quantization error to satisfy

\[ \| x_i(\hat{b}_k^t) - Q_i(x_i(\hat{b}_k^t)) \|_2 \leq \gamma_i \| x_i(\hat{b}_k^t) \|_2 \]

where \( \gamma_i \in (0, \tilde{\rho}_i) \), and

\[ \tilde{\rho}_i = \frac{\rho_i \alpha_i}{\beta_i} \]

Theorem 2. Consider the N-agent NCS in equation (1). Assume that there exist a smooth, positive-definite function \( V : \mathbb{R}^{nN} \to \mathbb{R}_0^+ \) and positive constants \( \alpha_i, \beta_i \in \mathbb{R}^+ \) for all \( i \in \mathcal{N} \) such that equation (2) holds. If for any \( i \in \mathcal{N} \), given \( \rho_i \in (0, 1) \), the \( k \) th broadcast time, \( b_{k+1}^t \), is triggered by the violation of inequality (3), and the quantization error satisfies inequality (4) for all \( t \geq 0 \) with \( \gamma_i \in (0, \tilde{\rho}_i) \), then the NCS is asymptotically stable.

Proof. At time instant \( b_k^t \), the quantization error satisfies inequality (4). Then inequality (3) is still valid at \( b_k^t \). It will still take an amount of time for inequality (3) to be violated since \( b_{k+1}^t \). Therefore \( b_{k+1}^t > b_k^t \). It means during \( [b_k^t, b_{k+1}^t) \), inequality (3) always holds with \( \hat{x}_i(t) = Q_i(x_i(\hat{b}_k^t)) \). Then by Lemma 1, the NCS is asymptotically stable.

Fig. 1. The structure of real-time NCS

The infrastructure of such an distributed NCS is plotted in Figure 1. In such a system, agent \( i \) can only detect its own state, \( x_i \). If the local “error” signal exceeds some given threshold, which can be detected by hardware detectors, agent \( i \) will sample and broadcast its state information to those agents in \( U_i \) through a real-time network. Meanwhile, agent \( i’ \)s control, \( u_i \), at time \( t \) is computed based on the states broadcasted by agents in \( Z_i \). These broadcasted states are denoted as \( x_{Z,i}(t) \). The control signal used by agent \( i \) is held constant by a zero-order hold (ZOH) unless one of the agents in \( Z_i \) makes a successful broadcast. This means that the state equation of agent \( i \) can be written as

\[ \dot{x}_i = f_i(x_{D_i}, u_i) \]
\[ u_i = g_i(x_{Z_i}) \]
Remark 3. It is worth pointing out that the event-triggering control framework observes the minimum rate limitation results in [Elia and Mitter (2002); Nair and Evans (2003)], in the sense that enough information about unstable states need to be provided in order to stabilize it. The proof of Theorem 2 used Lemma 1, which assumed no bandwidth limitation on communication channel. Note with quantization, Theorem 2 in fact does not require unlimited bandwidth to hold, which is an improvement from Lemma 1.

Condition (4) requires that the Euclidean 2-norm of quantization error be bounded by a constant times the same norm of the true state. This is naturally satisfied by choosing a logarithmic quantizer.

Definition 4. A 1-D logarithmic quantizer on \( \mathbb{R}^+ \) with density \( \sigma \in (0,1) \) is defined as a function \( Q^+ : \mathbb{R}^+ \rightarrow \mathbb{S} \) such that

\[
Q^+(x) = \frac{\sigma^n + \sigma^{n+1}}{2}, \quad \text{for } x \in (\sigma^{n+1}, \sigma^n), \quad n \in \mathbb{Z}.
\]

where \( \mathbb{S} = \{\frac{\sigma^m + \sigma^{m+1}}{2} \mid m \in \mathbb{Z}\} \).

That is, every \( x \) in range \( \sigma^{n+1} < x \leq \sigma^n \) is quantized as \( \frac{\sigma^m + \sigma^{m+1}}{2} \), and represented by integer \( n \) during transmit. An extension to 1-D logarithmic quantizer on the whole real line \( \mathbb{R} \) is straightforward:

Definition 5. A 1-D logarithmic quantizer on \( \mathbb{R} \) with density \( \sigma \in (0, 1) \) is defined as a function \( Q : \mathbb{R} \rightarrow \{0\} \cup \{\pm \mathbb{S}\} \), such that:

1. If \( x = 0 \), then \( Q(x) = 0 \);
2. If \( x \neq 0 \), then \( Q(x) = \text{sgn}(x)\frac{\sigma^n + \sigma^{n+1}}{2} \), for \( \sigma^{n+1} < |x| \leq \sigma^n, n \in \mathbb{Z} \).

For any \( x \in \mathbb{R} \), denote quantization error \( \epsilon = x - Q(x) \). It’s straightforward to verify the following property for the above log-quantizer:

\[
|\epsilon| \leq \frac{1 - \sigma}{2\sigma} |x|, \quad (7)
\]

i.e. the ratio of (maximum) quantization error \( |\epsilon| \) to state \( |x| \) is bounded by a constant. For any \( x \in \mathbb{R}^n \) with each dimension of \( x \) quantized by a 1-D logarithmic quantizer of density \( \sigma \), inequality (7) implies:

\[
||\epsilon||_2 \leq \frac{1 - \sigma}{2\sigma} ||x||_2. \quad (8)
\]

Therefore, condition (4) can be satisfied by simply choosing \( \sigma_i \) for agent \( i \) such that \( \frac{1 - \sigma_i}{2\sigma_i} \leq \gamma_i \) is satisfied for all agents.

However, in realistic situations such a static quantizer cannot be implemented since to quantize the entire state space \( \mathbb{R}^n \), it needs infinite number of quantization levels. A more practical scheme is the so-called dynamic quantization developed by Brockett and Liberzon (2000); Liberzon and Nesic (2007), which employs a zooming strategy to adjust the range of the quantizer on-line. For example in 1-D case, a dynamic logarithmic quantizer \( Q(x) \) with \( B \) bits can encode \( 2^B \) quantization levels. With quantization density \( \sigma \), this could correspond to a range \( [a, a(\frac{1}{\sigma})^B] \), where \( a > 0 \) is called ‘zooming parameter’ and can be adjusted. Upon proper zooming, the whole real line \( \mathbb{R} \) can be quantized with finite bit-length.

In the next section, we describe a dynamic logarithmic quantizer and derive the corresponding event-triggered zoom-in/zoom-out strategy such that constraint (4) is enforced over the whole state space \( \mathbb{R}^n \).

4. EVENT-TRIGGERED DYNAMIC QUANTIZATION

Definition 6. A 1-D dynamic logarithmic quantizer with bit-length \( B \), density \( \sigma \) and zooming parameter \( a > 0 \) is defined as a piecewise constant function \( Q(a, \cdot) \) that maps any \( x \in (a, a(\frac{1}{\sigma})^n) \) into the following set

\[
\left\{ \frac{a^{\frac{1}{\sigma}}^m + a^{\frac{1}{\sigma}}^{m+1}}{2} \mid m = 0, 1, \ldots, 2^B - 1 \right\}
\]

such that

\[
Q(a, x) = a^{(\frac{1}{\sigma})^{m} + (\frac{1}{\sigma})^{m+1}}, \quad \text{for } x \in (a(\frac{1}{\sigma})^n, a(\frac{1}{\sigma})^{n+1}].
\]

Such quantizer has the property that for any \( a > 0 \) and \( x \in [a, a(\frac{1}{\sigma})^2] \), the inequality \( |\epsilon| \leq \frac{1 - \sigma}{2\sigma} |x| \) is guaranteed.

For the \( N \)-agent event-triggered NCS (1), we want to develop a dynamic logarithmic quantizer and an event-triggered zooming scheme satisfying (4) for any \( x \in \mathbb{R}^n \). The zooming scheme adjusts current quantization range to contain state \( x(t) \). It is implemented in a distributed fashion: whenever agent \( i \) takes a zooming action, it needs to also broadcast a ‘zooming event’ to inform all agents \( j \in U_i \) so that they can correctly interpret messages broadcasted thereafter by agent \( i \). Note that each agent \( i \) has two type of events that need to be broadcasted to the whole system: the violation of inequality (4), and the zooming actions of the dynamic quantizer.

A poor design of quantization scheme may lead to unnecessary heavy traffic in the network. For example, in cartesian coordinates, a straightforward implementation is for any agent \( i \) to quantize every dimension of \( x_i(t) \in \mathbb{R}^n \) independently using a 1-D dynamic logarithmic quantizer, as defined above. Such an implementation requires that agent \( i \) inform all agents \( j \in U_i \) whenever a zooming action in any of the \( n \) quantizers happens.

Example 7. Consider an agent with its state \( x(t) = (x_1(t), x_2(t)) \in \mathbb{R}^2 \). Suppose at time \( b_k \), the state is sampled and quantized, \( Q(x(b_k)) = (1, \sqrt{99}) \). Let \( \gamma = 0.25 \), then the triggering condition (4) is equivalent to

\[
||x(t) - Q(x(b_k))||_2^2 \leq 0.25 ||x(t)||_2^2. \quad (9)
\]

Consider a ball \( B \) of center \( (1, \sqrt{99}) \) and radius 2. For any \( x(t) \in B \), triggering condition (9) is satisfied. Now fix \( x_2(t) = \sqrt{99} \) and let \( x_1(t) \to 0 \), the 1-D dynamic logarithmic quantizer on \( x_1 \) needs to progressively zoom-in, but the event is never triggered since \( x(t) \in B \) hold.

This example shows that many zooming events are not ‘critical’ to stability in the sense that event (4) is not triggered. But if we choose not to broadcast these events, failure to interpret the subsequent quantized data correctly will lead to future instability.

This issue can be resolved under our proposed quantization scheme, where we require the zooming events to be ‘embedded’ into regular broadcast events. That is, the zooming events form a sub-sequence of regular broadcast event sequence. This requirement suppressed the frequent
‘uncritical’ zoomings and broadcasting, thereby greatly reduced communication expense. Note that Example 7 implies that such embedding is not possible for the n-
dynamic logarithmic quantizer in cartesian coordinates.
We use a quantization scheme in n-dimensional spherical coordinates.
A vector \( z = (z_1, \ldots, z_n) \) in Cartesian coordinates can be expressed as \( u = (\varphi_1, \ldots, \varphi_{n-1}, r) \). Let \( j \) be the imaginary unit, \( \arg(j) \) be the function returning the angle of complex number. The \( n-1 \) angles \( \varphi_i \) and radius \( r \) are defined by the following equations

\[
\begin{align*}
\varphi_1 &= \arg(z_1 + j z_2) \in [-\pi, \pi], \\
\varphi_i &= \arg\left(\sum_{l=1}^{i} z_l^2 + j z_{i+1}\right) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], \\
& \quad i \in \{2, \ldots, n-1\}, \\
\varphi_n &= \arg(z_n, r).
\end{align*}
\]

The transformation from spherical back to cartesian coordinates is given by

\[
\begin{align*}
z_i &= r \cdot b_{i-1} \sin(\varphi_{i-1}), \quad i \in \{n, \ldots, 2\}, \\
z_1 &= r \cdot b_1 \cos(\varphi_1),
\end{align*}
\]

where the \( b_i \) are defined as

\[
b_i = \prod_{l=1}^{n-1} \cos(\varphi_l), \quad i \in \{n-2, \ldots, 0\}, \\
b_{n-1} = 1.
\]

In spherical coordinates, \( x(t) \in \mathbb{R}^n \) can be quantized using the so-called spherical logarithmic quantization (SLQ) Huber and Matschkl (2004). SLQ performs usual logarithmic quantization for the radius \( r \), and independent uniform quantization for angle variables \( \varphi_i \). Uniform quantization for angles corresponds to partitioning \( n-1 \) dimensional sphere into \( n-1 \) dimensional hyper-cubes (n-1 dimensional sphere is the surface of n dimensional ball).

It can be shown that there exists a \( \delta \in \left[\frac{1}{\sqrt{n}}, 1\right] \) such that as long as the partition of angles \( \varphi_i \) are fine enough, quantization error of SLQ always satisfies

\[
||e||_2 \leq \delta ||x||_2, \quad \forall x \in \mathbb{R}^n
\]

where \( \sigma \) is the density of logarithmic quantizer. In particular, we can fixed \( \delta := \frac{1}{\sqrt{n}} \) and a fine partition of \( \varphi_i \) will guarantee

\[
||e||_2 \leq \frac{1 - \sigma}{2 \sigma} ||x||_2, \quad \forall x \in \mathbb{R}^n
\]

Definition 8. A n-dimensional dynamic spherical logarithmic quantizer (DSLQ) is defined as the composition of a 1-D dynamic logarithmic quantizer for radius \( r \), and uniform quantizer for angles of the \( n-1 \) dimensional hyper-sphere.

Note that for DSLQ, zooming actions are performed only in radial direction. For a DSLQ with bit length \( B \), since the quantization of radius \( r \) and \( n-1 \) dimensional hyper-sphere are separated, we can split the \( B \) bits into two parts: \( B_r \) bits allocated to radius quantization and \( B_\varphi \) bits for angle quantization, \( B = B_r + B_\varphi \). Denote \( M_r = 2^{B_r} \), \( M_\varphi = 2^{B_\varphi} \), the quantization levels for radius and hyper-sphere, respectively. The total quantization levels \( M = M_r M_\varphi = 2^B \).

We assume that the DSLQ on all the agents have the same \( M_r \) and \( M_\varphi \). Extension to more general cases where \( M_r \) and \( M_\varphi \) are different between agents is straightforward.

Assumption 9. We make the following assumptions about each agent \( i \in \mathcal{N} \)

1. At \( t = t_0 \), agent \( i \) knows the following information of any agent \( j \in Z_i \): radial zooming parameter \( a_j \), radial quantization density \( \sigma_j \), and how the \( n-1 \) spherical angles are quantized. This is to ensure that agent \( i \) can interpret the message sent by agent \( j \).
2. The messages sent at any time include a zooming flag indicating one of the following three commands: ‘zoom-in’, ‘zoom-out’, and ‘no action’.

Remark 10. The first assumption can be satisfied by initialization in deployment stage. The second assumption requires several more bits in the broadcast packet to indicate the zooming flag.

Now we are ready to describe the zooming strategy:

1. Whenever event (4) is triggered, agent \( i \) checks its state \( x_i(t) \):
   a. If \( ||x_i(t)||_2 \in [a_i, a_i(\frac{1}{\sigma})^{M_r}] \), it take quantized measurement, set action flag to ‘no action’, and send message to \( j \in U_i \).
   b. If \( ||x_i(t)||_2 < a_i \), it zoom-in by setting \( a_i = (\sigma_i)^{M_r} a_i \). Then it take quantized measurement, set action flag to ‘zoom-in’, and send message to \( j \in U_i \).
   c. If \( ||x_i(t)||_2 > a_i(\frac{1}{\sigma})^{M_r} \), it zoom-out by setting \( a_i = (\sigma_i)^{M_r} a_i \). Then it take quantized measurement, set action flag to ‘zoom-out’, and send message to \( j \in U_i \).

2. When a message coming from \( j \in Z_i \) is received, agent \( i \) first check the action flag
   a. If it is ‘no action’, decode the data based on \( a_j \).
   b. If it is ‘zoom-in’, then set \( a_j = (\sigma_j)^{M_r} a_j \), and decode the date based on new \( a_j \).
   c. If it is ‘zoom-out’, then set \( a_j = (\sigma_j)^{M_r} a_j \), and decode the date based on new \( a_j \).

Based on this event-triggered zooming scheme, we now derive constraints on design parameters \( \sigma_i, M_r \) and \( \rho_i \), so that the asymptotic stability of NCS is achieved. First, since DSLQ quantizer for agent \( i \) satisfies inequality \( ||e_i||_2 \leq \delta ||x_i||_2 \), we require that

\[
0 < \delta_i < \gamma_i < 1
\]

are satisfied for each agent \( i \). Recall that \( \gamma_i \in (0, \tilde{\rho}_i) \). This is to guarantee a small enough quantization error so that triggering condition is not immediately violated at each sampling instance, as stated in Theorem 2.

Note that the quantized measurements are not taken until (3) is violated. So when the state \( x(t) \) goes out of current quantization range, the quantizer will be held from zooming until event (5) is triggered. We want to guarantee that by zooming in/out once, the state \( x(t) \) will always fall
into the new quantization range. So we need the second condition:
\[(\sigma_i)_{Mr} < 1 - \tilde{\rho}_i\]  
(11)
where \(\tilde{\rho}_i = \frac{a_i \alpha_i}{\beta_i}\).

The reasoning goes as follows: assume the DSLQ with density \(\sigma\) has original radial range \([a_i, a_i(\frac{1}{\sigma_i})^{Mr}]\). Then we have \(\|Q(x_t(b_k^i))\| \in [a_i, a_i(\frac{1}{\sigma_i})^{Mr}]\). Suppose at time \(b_k^i + 1\) a new event is triggered, this implies that the inequality (3) is violated, i.e. \(\frac{\|Q(x_t(b_k^i))\|}{\|x_t(b_k^i)\|} = \frac{a_i \alpha_i}{\sigma_i} = \tilde{\rho}_i\) at \(t = b_k^i + 1\). If the agent \(i\) finds its state \(x_t(b_k^i + 1)\) out of current quantization range \([a_i, a_i(\frac{1}{\sigma_i})^{Mr}]\), then a zooming action is also triggered.

Case I: When \(\|x_t(b_k^i + 1)\| < a\), the violation of condition (3) implies
\[\frac{\|Q(x_t(b_k^i))\| - \|x_t(b_k^i)\|}{\|x_t(b_k^i)\|} < \tilde{\rho}_i\].

This means \(\|x_t(t)\| \geq \frac{1}{1 + \tilde{\rho}_i}\|Q(x_t(b_k^i))\| > \frac{1}{1 + \tilde{\rho}_i}\). So we only need to make sure that \(\frac{a_i \alpha_i}{(1 + \tilde{\rho}_i)}\) falls in the new quantization range. Let \(a'_i\) be the new parameter satisfying \(a'_i(\frac{1}{\sigma_i})^{Mr} \geq a_i\) and \(a'_i \leq \frac{\alpha_i}{(1 + \tilde{\rho}_i)}\). This condition translates to \((\sigma_i)_{Mr} < 1 - \tilde{\rho}_i\).

Case II: When \(\|x_t(t)\| > a_i(\frac{1}{\sigma_i})^{Mr}\), a zoom-out condition can be derived similarly. We get
\[\|x_t(t)\| \leq \frac{\|Q(x_t(b_k^i))\|}{\|x_t(b_k^i)\|} < a_i(\frac{1}{\sigma_i})^{Mr}\]

To ensure that we can find \(a'_i\) such that the interval \([a_i(\frac{1}{\sigma_i})^{Mr}, a_i(\frac{1}{\sigma_i})^{Mr}]\) falls into \([a'_i, a_i(\frac{1}{\sigma_i})^{Mr}]\), we need \((\sigma_i)_{Mr} < 1 - \tilde{\rho}_i\).

Summarizing case I and II, for a given \(\tilde{\rho}_i\), we need to choose parameters \(M_r, \sigma_i\) to satisfy
\[(\sigma_i)_{Mr} < \min\{1, \frac{1}{1 + \tilde{\rho}_i}, 1 - \tilde{\rho}_i\} = 1 - \tilde{\rho}_i\],
which gives condition (11).

The event-triggered zooming strategy along with condition (10), (11) guarantee that DSLQ can satisfy (4) on \(\mathbb{R}^n\). We hence can invoke Theorem 2 to obtain the following stability result:

**Theorem 11.** Consider the N-agent NCS in equation (1). Assume that there exist a smooth, positive-definite function \(V: \mathbb{R}^{nN} \rightarrow \mathbb{R}_+^n\) and positive constants \(\alpha_i, \beta_i \in \mathbb{R}_+^n\) for all \(i \in N\) such that equation (2) holds. For given \(\rho_i \in (0, 1)\), if agent \(i\) broadcasts its DSLQ-quantized state on violation of inequality (3), and the parameters \(M_r, M_s, \tilde{\delta}_i, \sigma\), satisfy (10) and (11), then the NCS is asymptotically stable.

5. AN ILLUSTRATIVE EXAMPLE

In this section we present simulation results demonstrating the distributed event-triggered quantization scheme. The system under study is a collection of carts coupled by springs (Figure 2). The state of the \(i\)th subsystem is the vector \(x_i = [y_i, \dot{y}_i]^T\) where \(y_i\) is the \(i\)th cart’s position. We assume that at equilibrium, all springs are unstretched. We

![Fig. 2. Three carts coupled by springs](image)

![Fig. 3. State trajectories in an event-triggered NCS](image)

We choose the same \(\sigma = 0.9, M_r = 10, M_s = 20\) for all three agents, it can be shown that the ratio \(\|\delta\|_{\|x\|}^2\) is lower bounded by 0.03. Take \(\delta = 0.03\), it’s easy to check condition (10):
\[-0.0976\|x_1(t)\|_{\|x\|}^2 + \|e_1(t)\|_{\|x\|}^2 = 0\]
\[-0.1\|x_2(t)\|_{\|x\|}^2 + \|e_2(t)\|_{\|x\|}^2 = 0\]
\[-0.129\|x_3(t)\|_{\|x\|}^2 + \|e_3(t)\|_{\|x\|}^2 = 0\]

We choose the same \(\sigma = 0.9, M_r = 10, M_s = 20\) for all three agents, it can be shown that the ratio \(\|\delta\|_{\|x\|}^2\) is upper bounded by 0.03. Take \(\delta = 0.03\), it’s easy to check condition (10):
\[-0.0976\|x_1(t)\|_{\|x\|}^2 + \|e_1(t)\|_{\|x\|}^2 = 0\]
\[-0.1\|x_2(t)\|_{\|x\|}^2 + \|e_2(t)\|_{\|x\|}^2 = 0\]
\[-0.129\|x_3(t)\|_{\|x\|}^2 + \|e_3(t)\|_{\|x\|}^2 = 0\]

and condition (11):
\[(\sigma_i)_{Mr} = 0.3487 < 0.8709 = \min\{1 - \tilde{\rho}_i\}\].

So this event-triggered NCS should be asymptotically stable.
6. DISCUSSIONS

This paper studies the quantization scheme design of distributed event-triggered NCS. We provide conditions on quantizers under an event-triggered control framework. A dynamic spherical logarithmic quantizer and event-triggered zooming strategy are proposed which always satisfies these conditions. The proposed event-triggered zooming quantization scheme can be applied to more general cases: as long as agents use events of the form

\[ \|e_i(t)\|_2 \leq C\|x_i(t)\|_2 \quad \text{or} \quad \|e_i(t)\|_2 \leq C\|Q(x_i(t))\|_2 \]

to trigger their broadcast (where C is constant), it can serve as a candidate quantizer to guarantee different stability concepts. The zooming scheme can also be extended to uniform quantizer, and the triggering event becomes \( \|e_i(t)\|_2 \leq \epsilon_i \), with \( \epsilon_i \) being some constant.

Our analysis was carried out under the assumption that there are no transmission delays or packet dropout in the communication channel. When there are delays, it is possible that the message with zooming flags set to 'zoom-in' or 'zoom-out' arrives late, resulting in one or more misinterpreted broadcast packets. To rule out this situation, we can assume that these zooming requests have higher priority so that they always arrive at recipients before any message sent after them.

In the presence of packet dropout, the NCS may become unstable if zooming requests are lost. This is because all later broadcast packets will be misinterpreted. We studied the effect of stochastic packet dropout via simulation. Assume a packet loss probability of \( p \), both the broadcast events and the zooming events have probability \( 1 - p \) to be received. We set \( p = 0.5 \) and run the simulation repeatedly. It is observed that when zooming packets are

Fig. 4. Broadcast events and zooming events marked on their trajectories, as shown in Figure 4. It well demonstrates the ability of event-triggering in adjusting broadcast periods in response to variations in the system’s states. Also notice that the periods are always greater than a positive constant, even when the states are close to the equilibrium. We can see zooming events are ‘embedded’ into the sequence of broadcast events. There are not many zooming actions triggered, implying a low communication overhead incurred by our zooming scheme.

Fig. 5. Trajectories of all agents under packet loss lost, the NCS may demonstrates instability. One such unstable trajectory is plotted in Figure 5.

However, if we assume only the broadcast events are subject to packet loss, and the zooming packets are always reliably transmitted, then simulation shows that the system is stable provided the number of successive packet loss doesn’t exceed some bound. This is consistent with results reported in Wang and Lemmon (2009a).

REFERENCES


