Parametric Model Reduction in the Loewner framework

Sanda Lefteriu ∗ Athanasios C. Antoulas ∗∗
Antonio C. Ionita ∗∗∗

∗ Rice University, Houston, TX, USA, (e-mail: sanda.lefteriu@rice.edu)
∗∗ Rice University, Houston, TX, USA, Jacobs University Bremen, Germany (e-mail: aca@rice.edu)
∗∗∗ Rice University, Houston, TX, USA, (e-mail: aci1@rice.edu)

Abstract: We address the problem of parametric model order reduction given measurements of the response of a device performed with respect to frequency, and also with respect to a design parameter. The proposed approach is based on a generalization of the Loewner matrix framework [1] to the case where data depend on two variables. The purpose of this note is to address issues arising in MIMO systems.

1. INTRODUCTION

Often the behaviour of dynamical systems depends on parameters, such as material properties, geometric characteristics or varying boundary conditions. Microelectromechanical systems, electronic chips and VLSI interconnects yield examples of parameterized systems; see, e.g., [4, 2].

In parametric model reduction, the aim is to preserve the parameters as symbolic variables in the reduced models. This is important, for example in process optimization, arising in the development of numerous products.

The most common parametric reduction framework currently available is interpolatory reduction ([3, 4, 5], and references therein). The reduction problem is approached assuming the existence of a high order parametrized state space model. Projections are subsequently computed so that the reduced model matches the input/output behavior at specified values of the frequency and of the parameters. Another approach [6, 7, 8, 9] consists of generalizations of vector fitting to the multivariate case.

After collecting frequency response measurements for appropriate ranges of frequencies and parameter values, we use a generalization of the Loewner framework to the two variable case to construct models which are reduced both with respect to frequency and to the parameter.

The paper is organized as follows. After presenting a summary of one- and two-variable results in the Loewner framework, we present a generalization of the two-variable case to MIMO (multi-input multi-output) systems. Various simple examples are given which illustrate our results. In the final section a large-scale example is presented, in which measurements of the frequency response of a semiconductor device are provided, as a function of frequency and one parameter.

2. OVERVIEW OF THE LOEWEBER FRAMEWORK

Our main tool, the Loewner matrix, is derived using the Lagrange bases of polynomials. We first review the one-variable case (this corresponds to systems not depending on parameters).

Let \( \mathcal{P}_n \) denote the space of all polynomials of degree at most \( n \). This space has dimension \( n+1 \). Given \( \lambda_i, i=0, \ldots, n \), \( \lambda_i \neq \lambda_j, i \neq j \), the polynomials \( q_i(s):=\Pi_{i \neq j}^{n}(x-\lambda_j) \), \( i = 0, \ldots, n \), constitute a Lagrange basis for \( \mathcal{P}_n \). For \( \alpha_i \) set, \( p(s):=\sum_{i=0}^{n} \alpha_i q_i(s) \), satisfies the interpolation conditions \( p(\lambda_i) = \alpha_i, i=0, \ldots, n \). This is the Lagrange polynomial associated to the data \( (\lambda_i, \alpha_i) \), the unique polynomial in \( \mathcal{P}_n \) satisfying these \( n+1 \) conditions.

Given constants \( \alpha_i, \beta_i, i=0, \ldots, n \), we can write any rational function \( g(s) \) as a ratio of two Lagrange polynomials:

\[
g(s) = \frac{\sum_{i=0}^{n} \beta_i q_i(s)}{\sum_{i=0}^{n} \alpha_i q_i(s)}, \alpha_i \neq 0. \tag{1}
\]

It is easy to see that \( g(\lambda_i) = \frac{\beta_i}{\alpha_i} \), which we assume is a given value \( w_i \). We can rewrite \( g(s) = \sum_{i=0}^{n} \frac{w_i}{\sum_{i=0}^{n} \alpha_i q_i(s)} \) so \( g \) satisfies

\[
\sum_{i=0}^{n} \alpha_i \frac{w_i}{\sum_{i=0}^{n} \alpha_i q_i(s)} = 0, \quad \alpha_i \neq 0. \tag{2}
\]

The parameters \( \alpha_i \), free up to now, can be specified so that \( g \) satisfies additional interpolation conditions:

\[
g(\mu_j) = v_j, j = 0, \ldots, r, \tag{3}
\]

where \( (\mu_j, v_j) \) are pairs of complex numbers \( \mu_i \neq \mu_j, i \neq j, \lambda_i \neq \mu_j, \forall i, j \). Substituting (3) in (2) we obtain the following condition for \( \alpha_i \): \( L c = 0 \), where

\[
L = \begin{bmatrix}
\frac{v_0 - w_0}{\mu_0 - \lambda_0} & \frac{v_0 - w_n}{\mu_0 - \lambda_n} \\
\frac{v_0 - w_0}{\mu_0 - \lambda_0} & \frac{v_0 - w_n}{\mu_0 - \lambda_n} \\
\vdots & \vdots \\
\frac{v_r - w_0}{\mu_r - \lambda_0} & \frac{v_r - w_n}{\mu_r - \lambda_n} \\
\end{bmatrix}, \quad c = \begin{bmatrix}
\alpha_0 \\
\vdots \\
\alpha_n
\end{bmatrix}. \tag{4}
\]

The matrix \( L \) is the Loewner matrix [10, 11, 1, 12] associated with the row array \( (\mu_j, v_j) \), and the column array \( (\lambda_i, w_i) \). Thus \( g \) satisfies the additional interpolation.
constraints if \( c \) is in the right kernel of \( L \) (and \( \alpha_i \neq 0 \)). A key fact is that the rank of \( L \) is equal to the McMillan degree of the rational function, defined as the maximum between the degrees of the numerator and denominator.

### 2.1 Loewner-based state-space realizations

In this section we review generalized state-space (descriptor) realizations in the Loewner matrix framework.

We define the \( k \times (k+1) \) polynomial matrix in \( x \) built using the measurements \( x_i \):

\[
J(x; x_i, k) = \begin{bmatrix} x^k - x_0 x_1 & x^k - x_0 x_2 & \cdots & x^k - x_0 x_2 \cdots x^k - x_0 x_{k-1} \\ x^{k-1} - x_0 & x^{k-1} - x_2 & \cdots & x^{k-1} - x_2 \cdots x^{k-1} - x_{k-2} \\ \vdots & \vdots & \ddots & \vdots \\ x - x_0 & x - x_2 & \cdots & x - x_{k-1} \end{bmatrix},
\]

with unspecified entries, 0. Let \( a = (a_0, a_1, \ldots, a_n)^T \) and \( b = [\beta_0, \beta_1, \ldots, \beta_m] \) (where, as before, \( \beta_i = \alpha_i w_i \)).

**Lemma 1.** The following is a descriptor realization of \( g \):

\[
\tilde{C} = b, \quad \Phi(s) = \begin{bmatrix} J(s; \lambda_1, n) \\ a \end{bmatrix}, \quad \tilde{B} = e_{n+1},
\]

of dimension \( n+1 \) which can represent arbitrary rational functions. Also, it is R-controllable and R-observable, that is, \( [\Phi(s), \tilde{B}] \) and \( [\Phi^*(s), \tilde{C}^*] \) have full rank for all \( s \in \mathbb{C} \).

The following corollary is useful for the two-variable case.

**Corollary 2.** With the notation as in Lemma 1,

\[
C = \begin{bmatrix} 0 & 1 \end{bmatrix}, \quad \Phi(s) = \begin{bmatrix} \Phi(s) \\ C \end{bmatrix}, \quad B = \begin{bmatrix} \tilde{B} \\ 0 \end{bmatrix},
\]

is a R-controllable and R-observable realization of size \( n+2 \).

### 2.2 The two-variable Loewner framework

We now consider systems depending on one parameter. That is, their transfer function depends on the complex frequency \( s \) and on the parameter in question which we denote by \( t \). Our goal is to (re)construct a state-space realization for such systems based on measurements with respect both to \( s \) and to \( t \) (for instance the frequency response as a function of \( j\omega \) and parameter \( t \)). Towards this goal we introduce the two-variable Loewner matrix.

The space \( \mathcal{P}_{n,m} \) of polynomials in two variables, \( s \) and \( t \) of degree at most \( n \) in \( s \) and at most \( m \) in \( t \), is a linear space of dimension \((n+1)(m+1)\). Given (distinct) complex numbers \( \lambda_i, i=0, \ldots, n \), and \( \pi_j, j=0, \ldots, m \), the Lagrange basis is defined by polynomials

\[
q_{i,j}(s, t) = \prod_{\ell \neq i} (s - \lambda_{\ell - i}) \prod_{l \neq j} (t - \pi_{l - j}),
\]

Given constants \( \alpha_{i,j}, \beta_{i,j} \), a two-variable rational function \( g(s, t) \) can be expressed in this Lagrange basis:

\[
g(s, t) = \sum_{i=0}^{n} \sum_{j=0}^{m} \alpha_{i,j} q_{i,j}(s, t) + \sum_{i=0}^{n} \sum_{j=0}^{m} \beta_{i,j} q_{i,j}(s, t).
\]

Assuming \( \alpha_{i,j} \neq 0, \forall i, j \), we have: \( g(s, \pi_j) = \frac{\beta_{i,j}}{\alpha_{i,j}} =: \omega_{i,j} \).

Since \( g(s) = \sum_{i=0}^{n} \sum_{j=0}^{m} \alpha_{i,j} q_{i,j}(s, t) \), then \( g \) satisfies

\[
\sum_{i=0}^{n} \sum_{j=0}^{m} \alpha_{i,j} \frac{\omega_{i,j}}{q_{i,j}(s, t)} = 0, \quad \alpha_{i,j} \neq 0.
\]

To determine \( \alpha_{i,j} \), additional conditions are imposed on \( g \):

\[
g(\mu_i, \nu_j) = v_{i,j}, \quad i = 0, \ldots, p, j = 0, \ldots, r,
\]

where \( (\nu_i, \nu_j) \), are given triples of complex numbers \( (\nu_i, \nu_j) \). Using the notation

\[
\ell_{i,j}^{k,l} = \frac{v_{k,l}}{(\mu_k - \lambda_i)(\nu_l - \pi_j)}
\]

and substituting (8) in (7), we obtain the condition \( \mathbb{L} c = 0 \) for the coefficients \( \alpha_{i,j} \), where

\[
\mathbb{L} = [\begin{array}{cccc} \ell_{0,0} & \ell_{0,1} & \cdots & \ell_{0,m} \\ \ell_{1,0} & \ell_{1,1} & \cdots & \ell_{1,m} \\ \cdots & \cdots & \cdots & \cdots \\ \ell_{n,0} & \ell_{n,1} & \cdots & \ell_{n,m} \end{array}],
\]

\[
c^{T} = [\alpha_{0,0}, \alpha_{0,1}, \ldots, \alpha_{n,0}, \alpha_{n,1}, \ldots, \alpha_{n,m}, \alpha_{0,m}, \ldots, \alpha_{n,m}].
\]

The matrix \( \mathbb{L} \) defined above is the two-variable Loewner matrix associated with the column and row data sets

\[
P_c = \{(\lambda_i, \pi_j; w_{i,j}): i = 0, \ldots, n, j = 0, \ldots, m\},
\]

\[
P_r = \{(\mu_k, \nu_l; v_{k,l}): k = 0, \ldots, p, l = 0, \ldots, r\}.
\]

Its dimension is \((p+1)(r+1)(n+1)(m+1)\). Thus, \( g \) satisfies the additional interpolation constraints if the vector \( c \), of dimension \((n+1)(m+1)\), is in the right kernel of \( \mathbb{L} \).

For applications, we are given two-variable data and we wish to construct a transfer function model which matches the measurements using the two-variable Loewner matrix. First, we determine the degree of the underlying rational function. We seek integers \((n, m)\) to write \( g \) as the quotient of two polynomials \( u, d \), both belonging to \( \mathcal{P}_{n,m} \). By fixing \( t \) and computing the rank of the Loewner matrix constructed for varying \( s \) we determine \( n \) and similarly \( m \). Second, we construct the 2D Loewner matrix \((10)\) with \( p = n \) and \( r = m \) (i.e. \( \mathbb{L} \) is square of dimension \((n+1)(m+1)\)). We recover the underlying \( g \) through Theorem 3.

### 2.3 Loewner-based two-variable-state-space realizations

We use the notation

\[
A := \begin{bmatrix} \alpha_{00} & \alpha_{01} & \cdots & \alpha_{0n} \\ \alpha_{10} & \alpha_{11} & \cdots & \alpha_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{m0} & \alpha_{m1} & \cdots & \alpha_{mn} \end{bmatrix}, \quad B := \begin{bmatrix} \beta_{00} & \beta_{01} & \cdots & \beta_{0n} \\ \beta_{10} & \beta_{11} & \cdots & \beta_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{m0} & \beta_{m1} & \cdots & \beta_{mn} \end{bmatrix},
\]

where \( \alpha_{i,j} \) are from the right null space of the two variable Loewner matrix \( \mathbb{L} \) and \( \beta_{i,j} = w_{i,j} \alpha_{i,j} \). We also need a vector \( p \in \mathbb{C}^{m+1} \) such that \([J^*(t; \pi_j, m), p] \) is a unimodular matrix in \( t \). The entries of this vector can be chosen as:

\[
p_i = \frac{1}{\prod_{i \neq j} (\pi_i - \pi_j)},
\]

in which case \( \det (J^*(t; \pi_j, m), p) = (-1)^m \).

**Theorem 3.** This is a descriptor realization of the two-variable rational function \( g \) defined by (6):

\[
\Phi(s, t) = \begin{bmatrix} A \\ J^*(t; \pi_j, m) \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ p \end{bmatrix},
\]

with \( \Phi \) partitioned as \((n, m+1, m+1) \times (n+1, m, m+1)\). It is R-controllable and R-observable, i.e. \( [\Phi(s, t), B] \) and \([\Phi^*(s, t), C^*] \) are full rank, \( \forall s, t \in \mathbb{C} \).
3. MIMO SYSTEMS

3.1 One variable case

We assume the same number of inputs and outputs, \( p \), for simplicity. We are now dealing with the MIMO Loewner matrix \( \mathbb{L}^{(M)} \) built using MIMO data \( \mathbb{L}_{i,j}^{(M)} = \frac{V_{ij} - W_{ij}}{\mu - \lambda_j} \) (instead of (4)) where \( V_i, W_j \in \mathbb{C}^{p \times p} \) and \( \mathbf{H}(\lambda_j) = W_j \), as well as \( \mathbf{H}(\mu_i) = V_i, i, j = 0, \ldots, n \). We define the MIMO analogue of (5):

\[
\mathbf{J}(s; \lambda, k) = \begin{bmatrix}
(s - \lambda_0)\mathbf{I}_p & (\lambda_1 - s)\mathbf{I}_p \\
(s - \lambda_0)\mathbf{I}_p & (\lambda_2 - s)\mathbf{I}_p \\
\vdots & \vdots \\
(s - \lambda_0)\mathbf{I}_p & (\lambda_k - s)\mathbf{I}_p
\end{bmatrix}
\]

(13)

where \( \mathbf{J}(s; \lambda, k) \in \mathbb{R}^{bp \times (k+1)p} \) and \( \mathbf{a} \in \mathbb{R}^{p \times (k+1)p} \). Note that \( \mathbf{c} \), the right null space of the MIMO Loewner matrix \( \mathbb{L}^{(M)} \) of dimension \((k+1)p\) built using \( 2(k+1) \) points is

\[
\mathbf{c} = \begin{bmatrix}
0 \\
\vdots \\
0
\end{bmatrix}
\]

(14)

Remark. We want the dimension of this realization to be close to the dimension of the minimum realization one can construct with \( \mathbf{D} = 0 \), which is \( n + p \), where \( n \) is the McMillan degree and \( p \) is the number of ports. We choose \( k \) such that the size of our realization is as close as possible to \( n + p \). Thus, \( k \) is chosen as \( \left\lfloor \frac{n}{p} \right\rfloor \).

Example. Consider the transfer function

\[
\mathbf{H}(s) = \begin{bmatrix}
2 + 5s & 3 + 7s \\
1 + 2s & 1 + 2s \\
1 + 2s & 1 + 2s
\end{bmatrix}
\]

and the measurement points \( \lambda_0 = 1, \lambda_1 = 2, \mu_0 = 3, \mu_1 = 4 \).

We build the MIMO Loewner matrix \( \mathbb{L}^{(M)} \)

\[
\mathbb{L}^{(M)} = \begin{bmatrix}
1/21 & 1/21 & 1/35 & 1/35 \\
4/21 & 1/21 & 1/35 & 1/35 \\
1/27 & 1/27 & 1/45 & 1/45 \\
1/27 & 1/27 & 1/45 & 1/45
\end{bmatrix}
\]

It has dimension 4 and its nullspace is of dimension 2:

\[
\mathbf{c} = \begin{bmatrix}
-3/5 & 0 \\
0 & 1 \\
1 & 0 \\
0 & 1
\end{bmatrix}
\]

thus, \( \alpha_0 = -3/5 \) and \( \alpha_1 = 1 \). In fact, \( \alpha_i \) can be chosen as any vector such that \( [\alpha_0, \alpha_1]^T \) is in the right nullspace of \( \mathbb{L}^{(M)} \). In this case, the McMillan degree \( n = 2 \), so \( k = 1 \), and the minimal realization with \( \mathbf{D} = 0 \) is of order \( n + p = 2 + 2 = 4 \). This is the same as the dimension of the realization we can construct using Lemma 4:

\[
\mathbf{C} = \begin{bmatrix}
-7/5 & -2 & 12/5 & 17/5 \\
-7/5 & -13/5 & 15/5 & 22/5
\end{bmatrix}
\]

\[
\mathbf{F}(s) = \begin{bmatrix}
(s - 1)\mathbf{I}_2 & (2 - s)\mathbf{I}_2 & (3 - 5)s\mathbf{I}_2 & \mathbf{B} = \begin{bmatrix}
0 \\
1
\end{bmatrix}
\]

recovers the original MIMO transfer function.

3.2 Two variable case

Assuming the same number of inputs and outputs, \( p \), we are now using the 2D MIMO Loewner matrix \( \mathbb{L}^{(M)} \) built using MIMO data \( \mathbb{L}_{i,j}^{(M)} = \frac{V_{ij} - W_{ij}}{(\mu_i - \lambda_j)(\nu_i - \lambda_j)} \) (instead of (9)) where \( V_{k,i}, W_{i,j} \in \mathbb{C}^{p \times p} \) and \( \mathbf{H}(\lambda_j, \pi_j) = W_{i,j} \), as well as \( \mathbf{H}(\mu_i, \nu_k) = V_{k,i}, i, k = 0, \ldots, n, j, l = 0, \ldots, m \).

We need a matrix \( \mathbf{p} \in \mathbb{C}^{(k+1)p \times p} \) such that \( \mathbf{J}^T(t; \pi_j, k_2) \mathbf{p} \) is a unimodular matrix in \( t \). We can choose

\[
p_i = \frac{1}{\Pi_{j=0,j\neq l}(\pi_i - \pi_j)},
\]

so \( \mathbf{p} = \begin{bmatrix}
p_0 \cdot \mathbf{I}_p \\
\vdots \\
p_{k+1} \cdot \mathbf{I}_p
\end{bmatrix} \) in which case

\[
\det \mathbf{J}(t; \pi_j, k_2) = (-1)^{k_2}.
\]

Moreover, \( \mathbf{A} \) and \( \mathbf{B} \) are now

\[
\mathbf{A}_i := \begin{bmatrix}
\alpha_{00} & \alpha_{01} & \alpha_{02} & \cdots & \alpha_{k_1,0} \\
\alpha_{01} & \alpha_{11} & \alpha_{12} & \cdots & \alpha_{k_1,1} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\alpha_{0k_2} & \alpha_{1k_2} & \alpha_{2k_2} & \cdots & \alpha_{k_1,k_2}
\end{bmatrix}
\]

with \( \alpha_{i,j} \) of dimension \( p \times p \); \( \alpha_{i,j} = \begin{bmatrix}
\alpha_{i_1} & \cdots & \alpha_{i_{p+1}} \\
\alpha_{i_{p+1}} & \cdots & \alpha_{i_{2p+1}} \\
\vdots & \ddots & \vdots \\
\alpha_{i_{2p+1}} & \cdots & \alpha_{i_{3p+1}}
\end{bmatrix}
\]

the entries from the right null space of the two variable Loewner matrix \( \mathbb{L}^{(M)} \) constructed using MIMO data:

\[
\mathbb{L}^{(M)} = \begin{bmatrix}
\alpha_{00} & \cdots & \alpha_{0k_2} & \cdots & \alpha_{k_1,0} & \cdots & \alpha_{k_1,k_2}
\end{bmatrix}^T = \mathbf{0}
\]

and, finally,

\[
\mathbf{B}_i := \begin{bmatrix}
\beta_{00} & \beta_{01} & \cdots & \beta_{0k_1,0} \\
\beta_{01} & \beta_{11} & \cdots & \beta_{0k_2,0} \\
\vdots & \vdots & \ddots & \vdots \\
\beta_{0k_2,0} & \beta_{1k_2,0} & \cdots & \beta_{k_1,k_2}
\end{bmatrix}
\]

where

\[
\beta_{i,j}(1:p+1) = W_{i,j}^T \alpha_{i,j}
\]

Theorem 5. This is a descriptor realization of the two-variable rational function \( \mathbf{H} \)

\[
\mathbf{C} = \begin{bmatrix}
[0 & 0 & e_{k+1}^T \cdot \mathbf{I}_p] & \mathbf{B} = [0 & 0 & \mathbf{p}]
\end{bmatrix}
\]

\[
\Phi(s, t) = \begin{bmatrix}
\mathbf{A} & \mathbf{J}(t; \pi_j, k_2) \\
\mathbf{B} & \mathbf{0}
\end{bmatrix}
\]

of dimension \((k_1+2k_2+2)p\).

Remark. Proofs of these results were omitted due to space limitations.

Remark. Similar to the one variable case, \( k_1 \) can be chosen as \( \left\lfloor \frac{n}{p} \right\rfloor \), while \( k_2 \) can be chosen as \( \left\lfloor \frac{m}{p} \right\rfloor \).
Example. Consider the transfer function of McMillan degree
\( n = 2 \) in \( s \) and \( m = 2 \) in \( t \),
\[
H(s) = \begin{bmatrix}
s + 5t + 9st & 2s + 6t + 10st - 2 \\
2s + 3t + 9st - 1 & 2s + 3t + 9st - 1 \\
3s + 7t + 11st - 3 & 2s + 3t + 9st - 1 \\
2s + 3t + 9st - 1 & 2s + 3t + 9st - 1
\end{bmatrix},
\]
as well as the measurement points \( \lambda_0 = 2, \lambda_1 = 1/2, \mu_0 = 3/2, \mu_1 = 3, \tau_0 = -1/2, \tau_1 = -3/2, \alpha_0 = -1, \alpha_1 = -2 \). Note that, indeed, we have \( 2 = k_1 + 1 = \left\lceil \frac{n}{p} \right\rceil + 1 \). We build the 2D MIMO Loewner matrix of dimension \( 8 = 2 \cdot 2 \cdot (k_1 + 1)(k_2 + 1)p \) with nullspace of dimension 2:
\[
c = \left\{ \begin{bmatrix} -2/21 & 0 & -6/7 & 0 & -1/3 & 0 & 1 & 0 \end{bmatrix}^T, \begin{bmatrix} 0 & -2/21 & 0 & -6/7 & 0 & -1/3 & 0 & 1 \end{bmatrix}^T \right\},
\]
thus, after a reasonable scaling, \( \alpha_{00} = -2I_2, \alpha_{01} = -18I_2, \alpha_{10} = -7I_2 \) and \( \alpha_{11} = 21I_2 \). The matrix \( B \) is
\[
\begin{bmatrix}
\beta_{00} & \beta_{01} \\
\beta_{01} & \beta_{11}
\end{bmatrix} = \begin{bmatrix}
42 & 44 & -21 & -26 \\
46 & 48 & -31 & -36 \\
-134 & -148 & 59 & 70 \\
-162 & -176 & 81 & 92
\end{bmatrix},
\]
Using the previous formulas we have
\[
C = \begin{bmatrix} 0_2 & 0_2 & 0_2 & -I_2 \end{bmatrix}, \quad B = \begin{bmatrix} 0_2 & -I_2 & I_2 & 0_2 \end{bmatrix}^T
\]
\[
\Phi(s,t) = \begin{bmatrix} (s-2)I_2 & 0_2 & 0_2 & 0_2 \\
(1/2-s)I_2 & 0_2 & 0_2 & 0_2 \\
(s+1/2)I_2 & 0_2 & 0_2 & 0_2 \\
(s-3/2)I_2 & 0_2 & 0_2 & 0_2
\end{bmatrix}
\]
which recovers the original MIMO transfer function.

4. MIMO REDUCTION FROM MEASUREMENTS

We analyze an example from [14] consisting of two microstrip lines, which idealizes a 5cm interconnect link loaded by a device. The design parameter, the microstrip’s width \( w \), was considered for 15 values between 60 and 130\( \mu m \) in steps of 5\( \mu m \). For each parameter value, the 2 port scattering matrix was computed for 100 frequencies between 10 MHz and 10 GHz. To avoid numerical difficulties, frequencies were scaled by \( 10^{-6} \). To ensure a real system, we assumed the value of the transfer function evaluated at \( -j2\pi f_i \) to be the complex conjugate of the measurement provided at \( j2\pi f_i \). Thus, for each of the total 200 values for \( s \) and 15 values for \( w \), we are provided with a \( 2 \times 2 \) matrix which represents the S-parameters measured for that particular frequency and width parameter. The goal is to construct a low order model which incorporates the dependence on both frequency and parameter and matches all measurements to a reasonable extent.

The computation consists of the following steps:

- Using the theory developed in [15, 1], we construct the Loewner matrix for each \( w \) and varying \( \omega = j2\pi f_i \), to determine the order \( n \). We compute the singular values of each Loewner matrix (shown in Fig. 1(a)). Due to the noise in the data, the rank of the matrix, and consequently, the degree \( n \), is not obvious, but we decide to truncate at \( 24 = n + p \), where \( p = 2 \), so \( n = 22 \) and \( k_1 = \left\lceil \frac{n}{p} \right\rceil = 11 \).

- We decide on the truncation order \( 8 = m+p \) (so \( m = 6 \) and \( k_2 = \left\lceil \frac{m}{p} \right\rceil = 3 \)) after checking the SVD drop (Fig. 1(b)) of the Loewner matrix built from all parameter values, for each frequency. We select \( 2(k_2+1) \) parameter samples adaptively for each frequency and, in the end, use the ones which appear most often.

- Fig. 3 shows the singular value drop of the 2D Loewner matrix of dimension \( (k_1+1)(k_2+1)p = 124 \cdot 2 = 96 \). The last singular values are small (not precisely 0 due to the noise in the data), so we consider the null space as the last 2 right singular vectors.
To check the accuracy, we computed the errors for all parameter and frequency values (displayed in Fig. 4). Fig. 4(a) shows the absolute value of the errors on a logarithmic scale, namely \( \log_{10}(\| \hat{S}_{1,1}^{(k,i)} - S_{1,1}^{(k,i)} \|) \), where \( \hat{S}_{1,1}^{(k,i)} \) is the value of the \( S_{1,1} \) entry of the rational model evaluated for frequency \( j2\pi f_k \) (shown on the y-axis) and parameter \( w_i \) (shown on the x-axis), while \( S_{1,1}^{(k,i)} \) is the corresponding measured \( S_{1,1} \). Similarly, the errors in the other entries are shown in Fig. 4(b), 4(c) and 4(d). The color bar shown on the right of each figure suggests that in the areas with dark red, the errors are in the \( 10^{-8} \) range, while in those with orange, the errors are in the \( 10^{-6} \) range. Nevertheless, all errors are below 50dB (the largest error is in fact 2.4e−3, while the largest relative error is 9.3e−3).

**Remark** The dimension of the model is \((k_1 + 2k_2 + 2)p = 2(11+2.3+2) = 38\), and it incorporates information about the frequency, as well as the width parameter. Thus, this reduced model is able to predict the S-parameter values within an error of 50dB for all 200 measured values of \( s \) and 15 values of the parameter width \( w \).

We also compared the poles with respect to the frequency, obtained by substituting each parameter value in our realization, to those obtained after the modeling step (shown initially in Fig. 2). After removing poles with large magnitude which are, in fact, poles at infinity, but due to numerical issues appear as 5 orders of magnitude higher than the rest, we obtain the plot in Fig. 5. We notice that all recovered poles (obtained from the final realization) are stable, even though they are not precisely in the same location as the original ones (obtained after the modeling step). This is due to the fact that we are dealing with measurement noise and the two variable MIMO Loewner matrix is not precisely singular.

Instead of looking at each entry, an alternative way to checking the accuracy of the model is to look at the sigma plot. Fig. 6 shows the 2 singular values of each measured S-parameter, as well as the 2 singular values of the matrix we obtain by evaluating our model at \( j2\pi f \) for each frequency (this is the same as one would obtain by using the function `sigma` in Matlab and focus on the desired frequency range). We are only presenting the plots for parameter values \( w_1 = 60\mu m \) and \( w_{15} = 130\mu m \) since they yield the largest errors.

5. CONCLUSION

We summarized a two-variable generalization of the Loewner framework as the main tool for parametric model order reduction from measurements. We also described a generalized state-space (or descriptor) form realization of low dimension. For details on SISO systems, we refer the reader to [16]. In the present paper we presented generalizations to the MIMO case and applied them to a large-scale example involving S-parameter measurements. Future work includes extending the Loewner framework to multi-parameter systems.

REFERENCES


Fig. 4. Errors for all frequencies and parameters.
Fig. 5. Poles after modeling compared to the recovered poles - 'o': original, '+': recovered

Fig. 6. Sigma plots