Sliding Dynamics Bifurcations in the Control of Boost Converters

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Abstract: Dynamical systems techniques are exploited to assess the quality of switching control strategies leading to sliding model controllers (SMC). A boost converter controlled by SMC through a washout filter is considered. The crucial role of possible bifurcations in the ideal sliding dynamics is stressed. The possible reduction of the 3D-sliding dynamics to that of certain topologically equivalent planar standard vector field is emphasized so that the existence of Hopf bifurcations in the ideal sliding dynamics can be rigorously shown. Both a deeper insight in the global behavior of the controlled system and valuable design criteria are obtained.

Keywords: Sliding mode control, qualitative analysis, converters.

1. INTRODUCTION

In the analysis of nonlinear control systems, as dynamical systems, one naturally looks first for constant solutions associated to equilibrium points, since one of these equilibria is the desired operating point. In fact, other invariant sets for the dynamics, namely limit cycles, homoclinic or heteroclinic connections, are also relevant from the point of view of robustness issues and global dynamics.

Once designed a controller, the occasional separation of parameters from their nominal values can lead to dramatic changes in the dynamical behavior through different bifurcations, see for instance Kuznetsov [2004]. Thus, the parameter ranges where the local (and global, hopefully) dynamics is satisfactory should be found out. Extremes of these parameter ranges are typically bifurcation values: the determination of such critical parameter values and the characterization of the involved bifurcations are crucial to get a perspective of the dynamical behavior to be expected. Roughly speaking, the parameter nominal values farther from their critical values, the system is more robust.

The geometric theory of dynamical systems and in particular bifurcation theory for smooth systems have achieved a rather satisfactory maturity degree in the last decades. This is not the case for discontinuous or non-smooth differential systems, even the research effort is nowadays increasing, see Kuznetsov et al. [2003], di Bernardo et al. [2008], di Bernardo et al. [2008]. In the case of sliding mode control, few practitioners are aware of the information that can be gained through the bifurcation analysis of the sliding dynamics, perhaps due to the difficulties arising in the analysis of the sliding vector field.

In this paper, we take advantage of a methodology previously developed in Pagano and Ponce [2010], to facilitate the quoted analysis and qualify the robustness of a sliding mode controller for a boost converter, using a more realistic model than those previously analyzed in the literature (see Sira-Ramírez [2006]). As usual in bifurcation analysis, we pay attention mainly to stability and robustness issues. Once defined the control strategy through a washout filter, we arrive at a 3D discontinuous dynamical system; the relevant dynamics is to be confined to the sliding surface, which is in fact a plane. Next, it is shown how to reduce the dimensional sliding dynamics to another topologically equivalent dynamics determined by a standard planar vector field, getting so practical design criteria to choose control parameters. The existence of a subcritical Hopf bifurcation in the sliding dynamics is rigorously shown, in a similar way to what happens for a different, less efficient control system, see Ponce and Pagano [2009a,b].

The paper is organized as follows. In Section 2 we give a short review of the relevant concepts in the analysis of discontinuous control systems through dynamical systems theory and bifurcation theory, fixing also the notation to be followed in the rest of the paper. Next, in Section 3 we recall the model of the boost converter and the proposed sliding mode controller. In Section 4, we show how the methodology proposed permits to arrive at useful design criteria. Some conclusions are offered in last section.

2. SMC CONCEPTS AND NOTATION

In this section, we introduce the notation followed through the paper along with some elementary concepts about discontinuous control systems, see Pagano and Ponce [2010] for more details. We start by considering affine control systems of the form

$$\dot{x} = f(x) + g(x)u$$

(1)

where $x \in \mathbb{R}^n$ and the functions $f(x)$ and $g(x) \neq 0$, are smooth and the control signal $u$ is supposed to be a scalar discontinuous function. Assume a smooth non-constant scalar function $h : \mathbb{R}^n \rightarrow \mathbb{R}$ that defines the discontinuity...
manifold $\Sigma = \{ x \in \mathbb{R}^n : h(x) = 0 \}$, supposed to be regular, that is, $\nabla h(x) \neq 0$, $\forall x \in \mathbb{R}^n$, and splitting the state space into two open regions $S^+ = \{ x \in \mathbb{R}^n : h(x) < 0 \}$ and $S^- = \{ x \in \mathbb{R}^n : h(x) > 0 \}$. Accordingly, the switching control law is taken

$$
u = u(x) = \begin{cases} 
  u^-(x), & \text{if } h(x) < 0, \text{ i.e. } x \in S^-, \\
  u^+(x), & \text{if } h(x) > 0, \text{ i.e. } x \in S^+. 
\end{cases} \tag{2}$$

where $u^{(\pm)}$ are scalar smooth functions of $x$ (typically constant ones) to be later specified. System (1) endowed with the control law (2) defines the non-smooth vector field $f^{(\pm)}(x) = f(x) + g(x)u^{(\pm)}(x)$ for $x \in S^\pm$.

As usual, we look for a stable operating point $\bar{x}$, belonging to the discontinuity manifold $\Sigma$. We will need the derivative of $h$ along the different orbits when extended continuously to the boundary of the open regions $S^\pm$, that is, the orbital derivative of $h$ or Lie derivative given by $L_{\bar{x}}h(x) = \langle \nabla h(x), f(x) \rangle$, for all $x \in S^\pm = S^- \cup S^+$ and the corresponding one for all $x \in S^\pm = S^+ \cup S^-$. Then it is natural to define the crossing part of $\Sigma$ as the $\Sigma$-open set $\Sigma_c = \{ x \in S^+ : L_{\bar{x}}h(x) \cdot L_{\bar{x}}h(x) > 0 \}$, and its complement in $\Sigma$, that is the $\Sigma$-closed set $\Sigma = \{ x \in \Sigma : L_{\bar{x}}h(x) \cdot L_{\bar{x}}h(x) \leq 0 \}$.

which is normally called the sliding part $\Sigma_s$. Of course, we are mainly interested in its attractive part, namely the $\Sigma$-open set

$$\Sigma_{as} = \{ x \in \Sigma : L_{\bar{x}}h(x) > 0, \text{ and } L_{\bar{x}}h(x) < 0 \}, \tag{4}$$

where the two vector fields from both sides out of $\Sigma$ push orbits towards $\Sigma$. Clearly, points in $\Sigma_{as}$ satisfy

$$-L_{\bar{x}}g(x)u^-(x) < L_{\bar{x}}h(x) < -L_{\bar{x}}g(x)u^+(x). \tag{5}$$

A first goal is to define the controller for the desired operating point $\bar{x}$ to belong to $\Sigma_{as}$. The two conditions in (4) are not sufficient for stability purposes however, since there appears in $\Sigma_s$ a sliding dynamics induced by the interaction of the two vector fields. For robustness purposes, as stressed in the introduction, we must know as deeper as possible this sliding dynamics around the point $x \in \Sigma_s$ and its possible bifurcations.

According to (Filippov [1988]) the sliding dynamics induced by the discontinuous vector field (1)-(2), see also Sedghi [2003], is given by the vector field

$$f_s(x) = \lambda f^-(x) + (1 - \lambda)f^+(x), \tag{6}$$

where for each $x \in \Sigma_s$ the value of $\lambda$ should be selected such that $L_{\bar{x}}h(x) = \langle \nabla h(x), f_s(x) \rangle = 0$. Imposing such condition, the sliding vector field becomes

$$f_s(x) = \frac{L_{\bar{x}}h(x) f^-(x) - L_{\bar{x}}h(x) f^+(x)}{L_{\bar{x}}h(x) - L_{\bar{x}}h(x) \cdot L_{\bar{x}}h(x)}. \tag{7}$$

Note that $f_s(x) = f(x) + g(x)u_{eq}$ where

$$u_{eq} = -\frac{L_{\bar{x}}h(x)}{L_{\bar{x}}h(x)} \cdot \frac{\langle \nabla h(x), f(x) \rangle}{L_{\bar{x}}h(x)}, \tag{8}$$

is the so called equivalent control, see Utkin [1992]. We note that the transversality condition $L_{\bar{x}}h(x) \neq 0$ is a necessary condition for the existence of $u_{eq}$.

Discontinuous system (1) inherits the equilibria of each vector field $f^\pm(x)$, which can be real or virtual equilibria. In particular, we call (i) admissible or real equilibrium points to both the solutions of $f^-(x) = 0$ that belong to $S^-$ and the solutions of $f^+(x) = 0$ that belong to $S^+$; (ii) virtual equilibrium points are both the solutions of $f^-(x) = 0$ that belong to $S^-$, and the solutions of $f^+(x) = 0$ that belong to $S^+$. Virtual equilibria are not true equilibrium points, but they can play a role in the dynamics for the corresponding region.

Regarding now the dynamical system corresponding to the vector field $f_s(x)$ induced on the sliding set $\Sigma_s$, and following Kuznetsov et al. [2003] we call pseudo-equilibrium points to the solutions of $f_s(x) = 0$, with $x \in \Sigma_s$. Pseudo-equilibrium points are, in some sense, almost true equilibria for system (1).

If $L_{\bar{x}}h(x) = 0$ for some $x \in \Sigma_s$, then we have $L_{\bar{x}}h(x) = L_{\bar{x}}h(x)$, and from (3) this common value vanishes. Thus the point belongs to the boundary of $\Sigma_{as}$, where the two tangency sub-manifolds of $\Sigma$ intersect, and is called a singular sliding point. See Colombo et al. [2009], Jeffrey and Colombo [2009] for details about possible intricate dynamics around these points.

Another interesting possibility from the point of view of bifurcation analysis is the appearance of boundary equilibrium points, that is, solutions of $f^-(x) = 0$ or $f^+(x) = 0$ that simultaneously satisfy $h(x) = 0$. Such boundary equilibrium bifurcations give rise to changes in the sliding dynamics, and can be responsible for the appearance or disappearance of pseudo-equilibrium points, see for instance di Bernardo et al. [2008]. When boundary equilibrium bifurcations are not possible, pseudo-equilibrium points can also enter or escape from the attractive sliding set $\Sigma_{as}$ through singular sliding points, see Section 3.

The ideal sliding dynamics is governed by the vector field (7) and typically involves fractional expressions with non-constant denominators. We note that from (5)

$$L_{\bar{x}}h(x) - L_{\bar{x}}h(x) = L_{\bar{x}}h(x)(u^+ - u^-)(x) < 0, \tag{9}$$

so that, as usually $u^+(x) > u^-(x)$, we can assume that $L_{\bar{x}}h(x) < 0$ in $\Sigma_{as}$, the attractive part of $\Sigma$. Then, the following result easily follows.

Proposition 1. Under the assumption $L_{\bar{x}}h(x) < 0$ for all $x \in \Sigma_{as}$, the de-singularized sliding vector field

$$f_{ds}(x) = L_{\bar{x}}h(x)g(x) - L_{\bar{x}}h(x)f(x), \tag{10}$$

and the sliding vector field (7) are topologically equivalent in $\Sigma_{as}$, that is they have identical orbits and the systems are distinguished only by the time parametrization along the orbits. Therefore, pseudo-equilibria of (7) are also equilibria for (10) and the discontinuity manifold $\Sigma$ remains invariant under the flow generated by $f_{ds}$.

Furthermore, if $x$ is a singular sliding point of the vector field (7), that is, $L_{\bar{x}}h(x) = L_{\bar{x}}h(x) = 0$, then it is a standard equilibrium point for the de-singularized sliding vector field, so that $f_{ds}(x) = 0$.

Proof. See Pagano and Ponce [2010].

The mathematical expression of the de-singularized sliding vector field (10) is normally much easier to deal with than (7). Furthermore, the invariance of $\Sigma$ under the flow determined by $f_{ds}$ allows us to reduce the dimension of the problem by one, to be usually done by projecting the dynamics in $\Sigma$ onto one of the coordinate planes, see Section 4.
We take the inductor current is given by \( E \). Ponce and Pagano [2009b]. The alternation of the so called Discontinuous Conduction Mode (DCM), we will not study the last two possibilities related with \( E \) possible: thus, the possible states for (resistive load and \( v \) represented by discrete variables. The value of \( v \) depends both on \( E \) across \( C \), \( y \), \( ˙x = bx - y \). The equilibrium points for the vector fields \( f^+(x) = \left[ \frac{1 - bx - y}{x - ay} \right] \), \( f^-(x) = \left[ \frac{1 - bx}{w(x - z)} \right] \), according to the sign of \( h(x) \). Note that \( f(x) = f^-(x) \) (to be admissible or virtual, depending on their position with respect to \( \Sigma \)) are 

\[
\mathbf{e}^+ = \begin{pmatrix} \frac{a}{1 + ab} & \frac{1}{1 + ab} & \frac{1}{1 + ab} \\ \frac{a}{1 + ab} & \frac{1}{1 + ab} & \frac{1}{1 + ab} \end{pmatrix}, \quad \mathbf{e}^- = \begin{pmatrix} 1 & 0 & 1 \\ b & 0 & b \end{pmatrix}
\]

respectively. Obviously the last point cannot be considered in the case \( b = 0 \). From \( 1 + ab > 1 \) and \( y_k > 1 \), we have \( h(e^-) < 0 \), so that \( \mathbf{e}^+ \) is a virtual equilibrium point. However \( h(e^-) = -y_k < 0 \), and then \( \mathbf{e}^- \) is a real or admissible stable equilibrium point, with negative eigenvalues \( -b, -a \) and \( -w \). In principle this is not good for our purposes but, since \( b \) is rather small, it is far from the desired operating point, and so the SMC strategy can lead to another attractor with a sufficiently big basin of attraction, as it will be seen below.

3.1 The sliding vector field and its desingularization

Equation (7) shows that the sliding vector field is

\[
f_s(x) = \frac{1}{x - ky} \left[ x - bx^2 - kwxy - ay^2 + kwyz \right], \quad (12)
\]

to be used only within the attractive set \( \Sigma_{as} \), where the two involved vector fields push orbits towards \( \Sigma \). Here,

\[
G_F(s) = X_F(s) = \frac{s}{s + w} = 1 - \frac{w}{s + w},
\]

where \( w \) denotes the reciprocal of the filter constant and \( x \) is the system output which should be driven to a regulated desired value \( v_{out} > E \).
The equivalent planar vector field

\[ \frac{dx}{dt} = \begin{pmatrix} \frac{bx^2 - x + ay^2 + kwy(x - z)}{-k(bx^2 - x + ay^2 + kwy(x - z))} \end{pmatrix}, \]

which possesses the same equilibrium points as \((12)\), but also some others out of \(\Sigma_{as}\), satisfying the condition \(x - ky = 0\), to be considered later.

4. THE EQUIVALENT PLANAR VECTOR FIELD

The analysis is much easier, as stressed before, if we project the orbits of \((14)\) onto the plane \(z = 0\), by considering only the first two component of the vector field and eliminating \(z\) from the condition \(h(x) = 0\). We arrive so to the planar vector field

\[ \left( \begin{array}{c} \frac{dx}{dt} \\ \frac{dy}{dt} \end{array} \right) = \left( \begin{array}{c} \frac{bx^2 - x + ay^2 + kwy(y - y)}{-k(bx^2 - x + ay^2 + kwy(y - y))} \end{array} \right), \]

to be significative only in \(\Sigma_{pas}\), projection of \(\Sigma_{as}\), namely \(\{(x, y) : 0 < k(1 - bx) + w(y - y) - ay < ky - x\}\) (16).

System (15) has not only the equilibrium points corresponding with the projections of those for (12), but also there appear new ones through the desingularization process, namely the origin \(q_0 = (x_d, y_d)\), where \(y_d = (wy_d - k)/(w - a - bk^2)\) and \(x_d = ky_d\). The last point comes from the sliding singular point at the corner of the boundary of \(\Sigma_{as}\), as predicted by Proposition 1. Note that both new equilibria are located on the line \(x - ky = 0\). We will not analyze these two points in detail because they are out of \(\Sigma_{pas}\). It should be noticed however, that these additional equilibria help to understand the global dynamics in \(\Sigma_{pas}\), as shown below.
(1.1832, 2, 1.1832) becomes stable after a Hopf bifurcation for \( k > k_c \approx 4.1548 \). A value of \( k \) around 6 would be much better, since the basin of attraction bounded by the unstable limit cycle is bigger. We could take even greater values of \( k \) to the right of HC curve. In any case we should first analyze the effect in the dynamics of possible excursions out of \( \Sigma \), by entering in the Discontinuous Current Mode, see Ponce and Pagano [2009b].

#### 4.2 Real inductance case \((b > 0)\)

Now the design conditions (13) should be assured in order to get the possibility of pseudo-equilibria. The two equilibria \( p^\pm \) which are born for \( a = a_{SN} = 1/(4by_r^2) \), that is for \( \gamma = 0 \), are equal for \( a = a_{SN} \) to separate for \( a < a_{SN} \). At the critical value both have \( x = 1/(2b) \) and after substituting in (16) we have

\[
0 < \frac{k}{2} - \frac{1}{4by_r} < k_y - \frac{1}{2b} = 2y_r \left( \frac{k}{2} - \frac{1}{4by_r} \right),
\]

so that the equilibria are born within \( \Sigma_{pas} \) only if \( k > 1/(2by_r) \). Thus we have a fold bifurcation of pseudo-equilibria in \( \Sigma_{pas} \) at \( a = a_{SN} \) whenever \( k > 1/(2by_r) \).

We now check the possibility of entering or escaping the set \( \Sigma_{pas} \) for each equilibrium point separately. For both points we have \( x(1 - bx) = ay_r^2 \), and since inequalities in (16) lead to \( 0 < ay_r(ky_r - x)/x < ky_r - x \), each projected pseudo-equilibrium point belongs to \( \Sigma_{pas} \) if and only if \( ay_r < x < ky_r \).

The first inequality is always satisfied for both points \( p^{(\pm)} \), after simple algebraic manipulations leading to the equivalent condition \( 1 + \alpha < 2y_r \). Regarding the second inequality, it should be noticed that when satisfied the equilibrium corresponding with \( p^- \) has positive determinant, being the determinant negative for \( p^+ \). Thus the last point cannot be stable within \( \Sigma_{pas} \) and will not be considered anymore. For the operating point \( p^- \), easy computations show that it belongs to \( \Sigma_{pas} \) for any \( a < a_{SN} \) whenever \( k > 2ay_r \), or, if \( k < 2ay_r \), when \( by_r^2 - k + ay_r < 0 \), that is for parameter values to the right of the half-parabola in Fig. 3. At the points of this curve we have transcritical bifurcations: the operating point enters \( \Sigma_{pas} \) through its corner \( q_d \).

Recalling the condition for the trace, the local stability of \( p^- \) requires \( k > k_1 \), where

\[
k_1 = \frac{(2b + w)x - 1}{2ay_r} = \frac{w - (2b + w)\gamma}{4aby_r}.
\]  

For moderate values of \( a \), this stability condition is again fulfilled through a subcritical Hopf bifurcation, as shown in the Appendix. Thus, when \( p^- \) becomes stable there is one unstable limit cycle surrounding the point in \( \Sigma_y \). This limit cycle defines within \( \Sigma_y \) the basin of attraction of the pseudo-equilibrium point so that its stability is only local. Similar to the \( b = 0 \) case, the \( H_{sub} \) curve intersect the T-curve in two different non-transversal Bogdanov-Takens co-dimension-two points.

A significative bifurcation set in the parameter plane \((k, a)\) is shown in Fig. 3. Only the bifurcation curves relevant for \( p^- \) are drawn. There the straight line (dashed line) \( \gamma = 0 \) corresponds with the saddle-node bifurcation (SN) at \( a = 1/(4by_r^2) \). The stability parameter region lies to the right of the \( H_{sub} \) curve (in blue) and below the transcritical bifurcation semi-parabola (in red).

A qualitative reduction in the stability parameter region for the case \( b > 0 \), when comparing the two bifurcation sets of Figs. 2 and 3, should be noticed. In particular, such region turns out to be bounded with respect to admissible values of the parameter \( a \). In any case, a robust design requires to fix the operating point within the parameter stability region and sufficiently far from the bifurcation boundaries of this region. While the parameter \( a \) has a nominal value subjected to the possible variations that appear from external perturbations (load changes, mainly), the parameter \( k \) is to be chosen in advance by the designer. For this task, the above bifurcation sets constitute essential tools.

#### 5. CONCLUSION

A useful methodology based on bifurcation theory to determine the stability and robustness of sliding mode controllers has been illustrated through a realistic problem. New concepts have been exploited: in particular, the de-singularized sliding vector field and its dimensional reduction via projection have been shown to be crucial for alleviate the computational effort for the analysis. The determination of the corresponding bifurcation sets leads to practical rules for choosing parameters in order to achieve a robust control design. In looking for both global stability conditions and performance in transient responses, we are not done: chattering avoiding devices, for instance, are needed. Anyway, the bifurcation analysis must not be underestimated since leads to a considerable insight about the dynamics of sliding mode controllers and how to modify it.

#### REFERENCES


**Appendix A. SLIDING HOPF BIFURCATIONS**

Here, we show that sliding Hopf bifurcations are always subcritical, that is, one unstable limit cycle bifurcates for \( k > k_c \), bounding the attraction basin of the corresponding pseudo-equilibrium point. For the notation and more details, see Section 3.5 of Kuznetsov [2004].

Recall that for \( b = 0 \) the pseudo-equilibrium point is \((\bar{x}, \bar{y}) = (ay^*_y, y_*)\) on the set \( \Sigma_{ax} \) and \( k_c \) is given in (17), while for \( b > 0 \) the relevant pseudo-equilibrium is \( p^- \) with \( \bar{x} = (1 - \gamma)/(2b) \) and the critical value of \( k_c \) is given in (18). First, we apply in any case the change of variables \( \bar{x} = x - \bar{x}, \bar{y} = y - y_* \) to put the equilibrium at the origin. The system can be expressed, dropping the tildes, as

\[
\begin{aligned}
\begin{bmatrix}
\dot{x} \\
\dot{y}
\end{bmatrix} &= 
\begin{bmatrix}
-\gamma (2a - w)y_y \\
-k\gamma x - 2akyr
\end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + 
\begin{bmatrix}
b x^2 + (a - w)y^2 \\
xwy - k(bx^2 + ay^2)
\end{bmatrix},
\end{aligned}
\]

(A.1)

where we separate the linear part at the new origin, from the nonlinear, quadratic part of the vector field, and from now on we assume that if \( b = 0 \) then \( \gamma = 1 \). Let us define as the bifurcation parameter \( \alpha = k_c - k \), so that \( \alpha > 0 \) when \( k < k_c \). The eigenvalues for the linearization at the origin of system (A.1) are \( \lambda(\alpha) = \mu(\alpha) \pm i\omega(\alpha) \), where \( \mu(\alpha) = ay_y \alpha \) and \( \omega(\alpha)^2 = \omega_y^2 (k_c y_y - ay_y - \alpha) - \alpha^2 \gamma^2 b^2 \). We see that \( \mu(0) = 0 \) so that \( \lambda(0) = \pm i\omega_0 = \pm i\omega(0) \) with \( \omega_0^2 = \omega_y^2 (k_c - ay_y) \), denoting a possible Hopf bifurcation at \( k = 0 \) (\( k = k_c \)).

Since \( \mu'(0) = ay_y > 0 \), the transversality condition for the eigenvalues is clearly satisfied. If \( \alpha(0) \) is the linear part of system (A.1) in terms of the bifurcation parameter, then its critical value is

\[
A(0) = \begin{bmatrix} -\gamma (2a - w)y_y \\ k_c \gamma - \gamma \end{bmatrix}.
\]

Now, we select some normalized right- and left-eigenvectors \( q, p \) of \( A(0) \) for \( \lambda(0) = i\omega_0 \), so that \( A(0)q = i\omega_0 q \) and \( A(0)^T p = -i\omega_0 p \) and \( (p, q) = \bar{p}q = 1 \), where \( (\cdot, \cdot) \) stands for the standard scalar product in \( \mathbb{C}^2 \). We choose

\[
q = \frac{1}{2k_c\gamma \omega_0} \begin{bmatrix} 1 + \omega_0 \\ -ik_c \gamma \end{bmatrix}, \quad p = \begin{bmatrix} k_c \gamma \\ \gamma - i\omega_0 \end{bmatrix},
\]

and so \( (p, q) = 1 \). We now calculate the coefficient

\[
c_1(0) = \frac{i}{2\omega_0} \left( g_{20}g_{11} - 2 |g_{11}|^2 \frac{1}{3} |g_{02}|^2 \right) + \frac{g_{21}}{2}.
\]

The sign of its real part determines the character of the Hopf bifurcation. Here \( g_{21} \) stands for the coefficient of the term \( z^2 \bar{z}^2 \) in the scalar equation which results from the use of the complexified variable \( Z = k_c \gamma x + (\gamma + i\omega_0) y \).

Our system is quadratic, so that \( g_{21} = 0 \). We only need

\[
g_{20} = (p, B(q, q)) = k_c \gamma B_1(q, q) + (\gamma + i\omega_0) B_2(q, q),
\]

and

\[
g_{11} = (p, B(q, q)) = k_c \gamma B_1(q, q) + (\gamma + i\omega_0) B_2(q, q),
\]

where \( B \) stands for the bilinear form associated to the quadratic terms of the system for the critical value \( k = k_c \).

Computations give rise to

\[
B_1(q, q) = q^T \begin{bmatrix}
2b & 0 \\
0 & 2(a - w)
\end{bmatrix} q = \frac{w - a}{2\omega_0^2} + \frac{(\omega_y + i\gamma)^2}{2k_c \gamma^2 \omega_0^2},
\]

\[
B_1(q, q) = q^T \begin{bmatrix}
2b & 0 \\
0 & 2(a - w)
\end{bmatrix} q = \frac{a - w}{2\omega_0^2} + \frac{\gamma^2 + \omega_0^2}{2k_c \gamma^2 \omega_0^2},
\]

\[
B_2(q, q) = q^T \begin{bmatrix}
-2bk_c & w \\
w & -2ak_c
\end{bmatrix} q = \frac{b + ak_c^2 + w}{2k_c \omega_0^2} - \frac{b}{k_c \gamma^2},
\]

and

\[
B_2(q, q) = \frac{ak_c^2}{2\omega_0^2} + \frac{\gamma - i\omega_0}{2k_c \gamma \omega_0} - \frac{(\omega_y + i\gamma)^2}{2k_c \gamma^2 \omega_0^2}.
\]

Thus, after standard computations, we have for \( \Re(c_1(0)) \) the expression

\[
\frac{1}{8k_c^2 \omega_0^2 \gamma} \left( (2b + w)\omega_0^2 + (2b^2 + 3bw + w^2)\omega_0^2 \gamma^2 + (2b + w)ak_c^2 \omega_0^2 \gamma^2 + (2b + 2ak_c^2 + w)(1 + k_c^2 \gamma^2 w) \right).
\]

Since all the coefficients are positive, we conclude that \( \Re(c_1(0)) > 0 \), and the subcritical character of the Hopf bifurcation is shown.