Identification of Piecewise Linear Models of Complex Dynamical Systems

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Abstract: The paper addresses the realization and identification problem for a subclass of piecewise-affine hybrid systems. The paper provides necessary and sufficient conditions for existence of a realization by a piecewise-affine hybrid system. The paper also proposes an identification algorithm for this subclass of hybrid systems. The considered system class and the identification problem are motivated by applications in systems biology.

Keywords: Realization theory, identification, hybrid systems, network topology, gene-protein networks.

1. INTRODUCTION

In this paper we address the realization and identification problem for a subclass of piecewise-affine hybrid systems.

Contribution of the paper We define the class of piecewise-linear systems (abbreviated by PWL). PWLs are a subclass of piecewise-affine hybrid systems. The continuous dynamics of a PWL is determined by a finite collection of affine subsystems. However, in contrast to traditional piecewise-affine systems, we allow any change of the continuous state during a discrete-state transition, as long as the new state belongs to the set of designated initial states. In addition, we do not impose any specific mechanism for triggering discrete-state transitions.

We formulate the realization problem for this system class and partially solve it by providing necessary and sufficient conditions for existence of a realization. We show that the outputs of any PWL can also be described by a switched AR model. The main conclusion is that the outputs of any PWL can be transformed to a PWL with one discrete state while preserving input-output behavior. This means that without further restrictions, the identification problem for such systems is not necessarily interesting. Note that the conclusion above is not valid for other classes of hybrid systems, see Petreczky and van Schuppen (2010).

In addition, we present an identification algorithm for systems with full observations. This algorithm is illustrated by examples of physical and biological relevance.

Motivation The motivation for studying realization theory for PWLs is that it provides the theoretical foundation for systems identification. The motivation for identification of PWLs is that the problem of estimating the network dynamics of complex biological systems can be reduced to identification of PWLs.

Numerous biological processes in nature involve complex signaling networks, Westra et al. (2007); Gera and Srivastava (2006); Heijman et al. (2009). The underlying biological problem is to describe these flexible dynamical networks, based on (partial) observations of their internal states.

Mathematically, the processes above can be viewed as nonlinear dynamical systems with unknown parameters. Such systems often have several equilibria, and hence their behavior can be seen as a mixture of certain affine systems, where each affine system is obtained by linearizing the original system around one of the equilibria. That is, such a system can be approximated by a PWL. In addition to their simplicity, the advantage of PWL approximations is that they neatly capture the interaction among various state components around various equilibria. More precisely, interaction between the $i$th and $j$th state components can be viewed as the property that the $(i,j)$th entry of the Jacobian of the system around an equilibrium is non-zero. Since we are often interested exactly in interactions rather than the detailed dynamics, it makes sense to recast the problem of identifying such interactions into the problem of identifying PWL approximations.

Related work The results of this paper are new, to the best of our knowledge. Identification of piecewise-linear (-affine) hybrid systems has been subject of intensive research, Paoletti et al. (2007); Vidal (2008); Bako et al. (2009); Roll et al. (2004); Fox (2009). Realization theory for hybrid systems was investigated in Weiland et al. (2006); Paoletti et al. (2010); Petreczky (2006); Petreczky and van Schuppen (2010); Grossman and Larson (1995). Hybrid systems are widely used in systems biology, see de Jong (2002); Porreca et al. (2010); Koutroumpas et al. (2007); Cinquemani et al. (2008).

Outline of the paper §2 presents the formal definition of the system class of interest. §3 presents the results on realization theory, and §4 presents the identification algorithm and the results of the numerical experiments.

Notation We use the standard notation. We denote by $T = [0, +\infty)$ the time-axis. We denote by $I_n$ the $n \times n$ identity matrix. We denote the set of natural numbers including zero by $\mathbb{N}$.
2. PIECEWISE-LINEAR SYSTEMS (PWL)

The aim of the section is to define the class of piecewise-linear systems formally.

**Definition 1. (PWL)** A piecewise-linear system (abbreviated as PWL) is a dynamical system determined by

\[ \begin{align*}
\dot{x}(t) &= A_q(t)x(t) + a_q(t) \\
y(t) &= C_q(t)x(t) + c_q(t)
\end{align*} \]  

\[ x(t^+) \in X_{q(t^+),0} \tag{1} \]

Here \( Q = \{1, \ldots, D \} \) is the finite set of discrete modes, \( \mathbb{R}^p \) is the output space, \( \mathbb{R}^{n_q} \) is the state-space of the system in mode \( q \in Q \). For each \( q, a_q \in \mathbb{R}^{n \times n_q}, c_q \in \mathbb{R}^{p}, \) the parameters of the affine system in mode \( q \in Q, X_{q,0} \subseteq \mathbb{R}^n \) is the set of initial states of the affine system in mode \( q \in Q \). The state space \( \mathcal{H}_q \) is given by \( \mathcal{H}_q = \bigcup_{q \in Q} \{q\} \times \mathbb{R}^{n_q} \). We call Σ linear, if \( c_q = 0 \) and \( a_q = 0 \) for all \( q \in Q \), otherwise Σ is called affine.

We will use the following short-hand notation

\[ \Sigma = \{q, Q, \{ (q, a_q, c_q, X_{q,0}) | q \in Q \} \} \].

Informally, the evolution of Σ takes place as follows. As long as the value of the discrete state \( q(t) \) at time \( t \) does not change, the continuous state and the continuous output change according to the affine system \( \dot{x}(t) = A_q(t)x(t) + a_q(t) \) and \( y(t) = C_q(t)x(t) + c_q(t) \). The discrete state can change at any time, however, we do not allow consecutive changes of discrete states immediately one after the other. If the discrete state changes to \( q(t^+) \) at time \( t^+ \), then the new continuous state should satisfy \( x(t^+) \in X_{q(t^+),0} \). Note that we do not specify the mechanism which triggers the change of discrete states. We also do not specify the initial discrete state, it is chosen by an unspecified mechanism. Hence, the description above allows for several state- and output-trajectories. Note that contrary to the definition used in the literature, the new continuous state after a discrete-state transition does not depend on the previous continuous state.

In order to define the evolution of PWLs formally, we need to introduce the following notation and terminology.

In the remaining part of the section, Σ denotes a PWL of the form (1).

**Definition 2. (Collins (2005)).** A time event sequence is a strictly monotone sequence \( (t_n)_{n=0}^{n^*} \) such that \( n^* \in \mathbb{N} \cup \{+\infty\} \), \( t_0 = 0 \) and for all \( 0 < n < n^* \), \( 0 \leq t_n < t_{n+1} \). If \( n^* = +\infty \) then we require that \( \sup(t_n \mid n \in \mathbb{N}) = +\infty \). If \( n^* < +\infty \), then by convention \( t_{n^*+1} = +\infty \).

The role of time event sequences is to formalize the time instances at which discrete events occur. The restrictions formulated in the definition imply that no Zeno-behavior can take place.

**Definition 3. (State-trajectory).** A state-trajectory of Σ is a map \( \xi : T \rightarrow \mathcal{H}_q \) such that there exists a time event sequence \( (t_i)_{i=0}^{i^{*}=n^*} \) and a sequence of discrete modes \( (q_i \in Q)_{i=0}^{i^{*}=n^*} \) such that for all \( 0 \leq i < i^{*} \), \( i \in \mathbb{N} \), it holds that for all \( s \in [t_i, t_{i+1}) \), \( \xi(s) = (q_i, x(s-t_i)) \) and

\[ \dot{x}(t) = A_{q(t)}x(t) + a_{q(t)} \]  

\[ x(0) \in X_{q,0}. \]

The time event sequence \( (t_n)_{n=0}^{n=0} \) is called the sequence of switching times of the state-trajectory \( \xi \).

We denote by \( BS(\Sigma) \) the set of all state-trajectories of Σ.

**Definition 4. (Output-trajectory).** An output-trajectory of Σ is a map \( y : T \rightarrow \mathbb{R}^p \) such that the following holds. There exists a state-trajectory \( \xi \) of Σ, such that \( y(t) = y_{\xi}(x(t)) \) for all \( t \in T \). Here, \( y_{\xi} \) is the readout-map of Σ, defined as

\[ y_{\xi} : \mathcal{H}_q \ni (q, x) \mapsto C_q x \in \mathbb{R}^p. \]

We denote by \( B(\Sigma) \) the set of all output-trajectories of Σ.

In the sequel, unless stated otherwise, \( f \) denotes a function \( f : T \rightarrow \mathbb{R}^p \). The definition above implies that the external behavior of a PWL is exactly a function of this type.

**Definition 5. (Realization).** The function \( f \) is said to be realized by \( \Sigma \), if \( f \) is an output-trajectory of \( \Sigma \), i.e. if \( f \in B(\Sigma) \). In this case \( \Sigma \) is called a realization of \( f \).

Next, we formulate the realization problem for PWLs.

**Problem 1. (Realization problem).** Find conditions for existence of a PWL realization of \( f \). Find algorithms for computing a PWL realization from finite data.

As was indicated before, our motivation for studying the realization problem for PWLs is to lay the theoretical foundations for identification of PWLs. We present below the formulation of the identification problem.

**Problem 2. (Identification problem).** Assume that the value of \( f \) and its derivatives up to order \( r, r \geq 0 \) is measured at time instances \( t_1 < \ldots < t_k \). Based on the (possibly noisy) data \( \{f^{(i)}(t_i)\}_{i=1,...,k} \), find a PWL realization of \( f \).

3. REALIZATION THEORY OF PWLs

Below we present the following results on realization theory of PWL.

- Every PWL can be transformed to a PWL with one discrete-state.
- We present conditions for existence of a PWL realization and a realization algorithm. We also present necessary and sufficient conditions for existence of a PWL realization with a bounded number of discrete modes and continuous state variables. The condition is formulated purely in terms of input-output data.
- We show that every PWL can be transformed to a switched AR system.

The relevance of these results is as follows. The first one demonstrates that the identification problem for PWL is not well posed. That is, it is not possible to determine the number of discrete modes based on data. This means that when formulating the identification problem, the expected number of discrete modes has to be postulated. That is, our first result has serious implications for the formulation of the identification problem.

The result of existence of a realization and the corresponding realization algorithm tells us that it is possible to find a PWL representation from finitely many data point. If this was not true, then any attempt to derive a general identification algorithm would be futile.

Finally, the equivalence of PWL and switched AR models imply that it is sufficient to concentrate on identifying switched AR models. Switched AR models are a type of...
input-output auto-regressive equations. It is a common
wisdom that it is easier to identify input-output models
than state-space models. Hence, this result is very relevant
for system identification. In fact, we already use this result
to argue that the identification algorithm of this paper
actually solves the identification problem for PWLs.
Note that the equivalence of PWL and switched AR models
is specific to the class of piecewise-linear hybrid systems
formulated in this paper. This equivalence does not hold
in general for all piecewise-linear systems, Weiland et al.

Throughout the section, \( f \) denotes a function \( f: T \to \mathbb{R}^p \).

3.1 Realization by a PWL with one discrete state

In order to present our first result, we introduce the notion
of a linear system with state-jumps.

**Definition 6.** A linear system with state jumps, abbrevi-
ated as (LSSJ) is a linear PWL \( \Sigma \) of the form (1)
with one discrete state, i.e. \( d = 1 \). We will identify the
LSSJ \( \Sigma \) with the collection of data \((n, C, A, \bar{x}_0)\), where
\( n = n_1, C = C_1, A = A_1 \) and \( x_{1,0} = \bar{x}_0 \).

The reason we call a PWL with one discrete state a linear
system with state jumps is that the system behaves as
a linear system, with the exception that its state
occasionally jumps back to one of the initial states.

**Theorem 1.** Assume that \( f \) admits a realization by a
PWL. Then \( f \) admits a realization by a LSSJ.

The theorem above says that in general, the external
behavior of any PWLs can be represented by a linear
system with several initial states. The proof of Theorem 1
relies on the following transformations.

**Definition 7. (PWL to linear PWL).** Define the linear
PWLs \( L(\Sigma) \) associated with an affine \( \Sigma \) as follows.

\[
L(\Sigma) = (n, Q, \{ \{ n_q^L, A_q^L, C_q^L, \chi_{q,0}^L \} \mid q \in Q \})
\]

where \( n_q^L = n_q + 1, \quad c_q = 0, a_q = 0 \) and

\[
A_q = \begin{bmatrix} A_q & a_q \\ 0 & 0 \end{bmatrix}, \quad C_q = [C_q \\ \chi_{q,0}^L]
\]

\[
\chi_{q,0}^L = \{ x^T, 1 \} x \in \mathbb{R}^{n_x+1} \mid x \in x_{q,0}^L \}.
\]

**Proposition 1.** The output trajectories of \( \Sigma \) and
\( L(\Sigma) \) coincide, i.e. \( B(\Sigma) = B(L(\Sigma)) \).

The proof of the proposition can be found in Westra et al.
(2011).

**Definition 8. (Linear PWL to LSSJ).** Let \( \Sigma \) be a linear
PWL of the form (1). Define the LSSJ \( LS(\Sigma) \) associated
with \( \Sigma \) as follows.

\[
LS(\Sigma) = (n, C, A, \bar{x}_0),
\]

where \( n = \sum_{q \in Q} n_q \) and

\[
A = \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_D \end{bmatrix} \quad \text{and} \quad C = [C_1 \cdots C_D]
\]

\[
\bar{x}_0 = \bigcup_{q \in Q} \chi_{q,0}^L
\]

\[
\chi_{q,0}^L = \{ (0, \ldots, 0, x, 0, \ldots, 0)^T \mid x \in x_{q,0}^L \}, \forall q \in Q
\]

**Proposition 2.** The output trajectories of \( \Sigma \) and \( LS(\Sigma) \) are
the same, i.e. \( B(\Sigma) = B(LS(\Sigma)) \).

The proof of the proposition can be found in Westra et al.
(2011).

**Proof.** [Proof of Theorem 1] Assume that \( \Sigma \) is a PWL
realization of \( f \). If \( \Sigma \) is affine, then replace \( \Sigma \) with the
associated linear PWL \( L(\Sigma) \), which is also a realization
of \( f \). Hence, we can assume that \( \Sigma \) is already a linear
PWL. Construct then the LSSJ \( LS(\Sigma) \) associated with
\( \Sigma \). It then follows that \( LS(\Sigma) \) is a realization of \( f \).

3.2 Existence of a realization

The conditions for existence of a realization will be formul-
ated using the rank of the Hankel-matrix of \( f \). In order to
define the Hankel-matrix of \( f \), we have to define the notion of
Markov-parameters. To that end, we need the notion of
piecewise-analytic functions.

**Definition 9.** (Piecewise-analytic). The map \( f \) is called
piecewise-analytic, if there exist a finite or infinite number
of time instances, \( t_i \in T, \quad t_i < t_{i+1}, \quad i \leq N_f, \quad i \in \mathbb{N} \)
for some \( N_f \in \mathbb{N} \cup \{ \infty \} \), such that the following holds.
If \( N_f < \infty \), let \( t_{N_f+1} = +\infty \). Then we require that
\( t_0 = 0 \) and \( \bigcup_{i=0}^{N_f} [t_i, t_{i+1}) = T \) and for each \( i \in \mathbb{N}, \quad i \leq N_f \),
\( f \) is analytic on \([t_i, t_{i+1})\), but \( f \) is not analytic on any
neighborhood of \( t_i \) in \( \mathbb{R} \). We call the points \( \{ t_i \}_{i=0}^{N_f} \) the
points of non-analyticity. We define the set

\[
I_f = \{ i \in \mathbb{N} : i \leq N_f \}
\]

of indices of points of non-analyticity.

The intuition behind the definition is as follows. If \( f \) has
a realization by a PWL \( \Sigma \), then the only points where
\( f \) is not analytic are the points where the corresponding
state-trajectory of \( \Sigma \) switches from one discrete mode
to another. In fact, one can show that there always exists
a state-trajectory of \( \Sigma \) which yields \( f \) as output trajectory
and which switches only at time instances at which \( f \) is
not analytic.

**Definition 10.** (Markov-parameters). Assume that \( f \)
is piecewise-analytic and let \( \{ t_i \}_{i=0}^{N_f} \) be the points of
non-analyticity of \( f \). For each \( i \in I_f \), define the ith Markov-
parameter \( M^f_i \) of \( f \) as a sequence \( M^f_i : \mathbb{N} \to \mathbb{R}^p \)

\[
\forall k \in \mathbb{N} : M^f_i (k) = \frac{d^k}{dt^k} f(t_i + s)|_{s=0}.
\]

It is easy to see that the collection of Markov-parameters
\( \{ M^f_i \}_{i=0}^{N_f} \) determines the map \( f \) uniquely. We use the
Markov-parameters to define the Hankel-matrix of \( f \).

**Definition 11.** (Hankel-matrix). We define the Hankel-
matrix \( H_f \) of \( f \) as the infinite matrix, rows of which
are indexed by \( \mathbb{N} \times \{ 1, \ldots, p \} \), and columns of which
are indexed by \( \mathbb{N} \times I_f \). The entry of \( H_f \), indexed by row index
(\( i, r \)) and by column index (\( j, l \)) equals

\[
[H_f]_{(i,r),(j,l)} = (M^f_i (i + j))_r,
\]

where \( (M^f_i (i + j))_r \) denotes the \( r \)th entry of Markov-
parameter \( M^f_i (i + j) \). The rank of \( H_f \), denoted by rank \( H_f \),
is the dimension of the linear space spanned by the
columns of \( H_f \).
Theorem 2. (Existence of a PWL realization). The map $f$ can be realized by a PWL if and only if it is piecewise-analytic and rank $H_f < +\infty$. Moreover, a LSSJ realization of $f$ can be constructed from a suitably large finite sub-matrix of $H_f$.

The proof of the theorem can be found in Westra et al. (2011). The main idea behind the proof of Theorem 2 is that existence of a PWL realization is equivalent to the existence of a LSSJ realization. The latter is just a linear system with several initial states and its existence can be characterized by the finite rank condition of the Hankel-matrix.

It was already mentioned that any PWL can be transformed to a PWL with one discrete state, by increasing the number of continuous states. One way to avoid this is to restrict the number of continuous states. To this end, we introduce the notion of $K-N$ realization.

Definition 12. $(K-N)$. A PWL $\Sigma$ is said to be a $K-N$ PWL if $\Sigma$ is of the form $(1)$ and $|Q| \leq K$ and for all $q \in Q$, $n_q \leq N$.

Below we present conditions for existence of a $K-N$ PWL realization of $f$.

Definition 13. (partitioned Hankel-matrices). Consider the partitioning $C = (C_q)_{q=1}^{K} \subseteq \mathbb{C}$ of the set of $I_f$ by indices of points of non-analiticity of $f$. For each $q = 1, \ldots, K$, let $H^q_{f,C}$ be the sub-matrix of Hankel-matrix $H_f$ which is formed by the columns of $H_f$ indexed by indices of the form $(j,i)$, $j \in \mathbb{N}$ and $i \in C_q$.

Theorem 3. (Existence of $K-N$ realizations). The map $f$ has a realization by a $K-N$ PWL if and only if there exists a partitioning $C = (C_q)_{q=1}^{K}$ of $I_f$ with $D \leq K$ such that for all $i = 1, \ldots, D$, rank $H^q_{f,C} \leq N$. If the latter condition holds, then a linear $K-N$ PWL realization of $f$ can be computed from a suitable finite sub-matrix of $H_f$.

The proof of the theorem can be found in Westra et al. (2011). The main idea behind the proof of Theorem 3 is as follows. The column spaces of the matrices $H^q_{f,C}$ represent the Hankel-matrices of the linear subsystems. The requirement that $H^q_{f,C}$ is of finite rank is then analogous to the classical results.

The result above demonstrates that the modification of the identification problem for PWLs, whereby we restrict attention to PWLs with a fixed upper bound for the number of continuous and discrete states can in principle be solved.

3.3 Equivalence of PWL and switched AR models

Below we show that $f$ is realizable by a PWL if and only if $f$ satisfies a switched AR model.

Definition 14. (SARS models). A switched AR system (abbreviated as SARS) is a collection of matrices $T = \{p, n, Q, \{A_q\}_{q \in Q}, i = 1, \ldots, n\}$ (2) where $Q = \{1, \ldots, D\}$ is the set of discrete modes, $D > 0$, $n_q > 0$ and $A_q \in \mathbb{R}^{n \times n}$, $i = 1, \ldots, n_q$ are the matrices of $T$. The SARS $T$ is a realization of $f$, if the following holds.

Let $(t_i)_{i=0}^{N_f}$ be the points of non-analiticity of $f$. For any $i \in I_f$, and for any $t \in [t_i, t_{i+1})$, denote by $f^{(k)}(t)$ the $k$th order right-hand derivative of $f$ at $t$. Then we require that for any $i \in I_f$ there exist $q(i) \in Q$, such that $\forall t \in [t_i, t_{i+1}), f^{(k)}(t) = \sum_{k=1}^{n-1} A_{q(i),k} f^{(n-k)}(t)$.

Theorem 4. The function $f$ has a realization by a PWL with $D$ discrete states if and only if there exists a SARS realization of $f$ with $D$ discrete modes.

The proof of the theorem can be found in Westra et al. (2011). The main idea behind the proof of the theorem is as follows. Clearly, any SARS can be converted to a PWL realization of $f$. Conversely, if $f$ has a PWL realization, then $f$ has a linear PWL realization. By rewriting the linear subsystems of the latter realization to AR systems, we obtain a SARS realization of $f$.

4. IDENTIFICATION ALGORITHM FOR PWLS

Below we present an algorithm which computes a realization with full observations based on measurements at finely many time instances. More precisely, the algorithm solves the following problem.

Problem 3. Fix integers $D > 0$ and $n > 0$. Consider a piecewise-analytic function $f : T \rightarrow \mathbb{R}^n$ and a finite sequence of time instances $t_1 < \ldots < t_M$ and assume that $\{f(t_i), f(t_{i+1})\}_{i=1}^{M}$ are known, i.e. we know that value of $f$ and its derivative at time instances $t_1, \ldots, t_M$. Find a PWL $\Sigma$ of the form $(1)$ with full-observations such that $\Sigma$ realizes $f$, and $C_q = I_n$, $c_q = 0$, and $n_q = n$ for all $q \in Q$.

Motivation of the identification problem

The motivation for considering the identification problem with full observations is the following.

(1) Notice that the identification problem with full observations and the identification problem for SARSs are equivalent. Indeed, a PWL with full observation can be considered as a SARS with $n = 2$. Conversely, for a SARS $T$, associated PWL can be viewed as a PWL with full observations, if the high-order derivatives of $f$ can be measured. Hence, by Theorem 4, the identification problem for PWLs is equivalent to the identification problem for PWL with full observations, if we assume that high-order derivatives of the output can be measured too.

(2) The problem of identification with full observations is still a non-trivial problem. Even in this case we have problems with identifiability, see Example 1.

(3) The solution of Problem 3 enables us to prove experimentally the feasibility of approximating complex systems by PWLs.

Example 1. Consider the LSSJs $\Sigma_1 = (2, C_1, A_1, X_1^0)$ and $\Sigma_2 = (2, C_2, A_2, X_2^0)$ where $C_1 = C_2 = I_2$ is the identity matrix and $X_0 = X_0^2 = \{(1, 0)^T\}$ and the remaining parameters are as follows: $A_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ and $A_2 = \begin{bmatrix} 0 & 2 \\ 0 & 3 \end{bmatrix}$. It then follows that $f(t) = (1, 0)^T$, $t \in T$ can be realized both by $\Sigma_1$ and $\Sigma_2$, but clearly $\Sigma_1 \neq \Sigma_2$. In other worlds, realizations of $f$ with full observation need not be identifiable.

In Algorithm 1 we present a solution for Problem 3. Algorithm 1 is an iteration consisting of the following steps.
First, the algorithm starts with a random initialization of \( \{A_q, a_q\}_{q \in Q} \). Subsequently, in each iteration step, the current estimates \( \{A_q, a_q\}_{q \in Q} \) are updated by applying first step 3 for estimating the updated weights with the fixed system parameters \( \{A_q, a_q\}_{q \in Q} \), and then applying step 4 with the previously updated weights \( \{w_q, \hat{a}_q\}_{q \in Q, i=1,...,M} \) for updating the estimates of the system parameters \( \{A_q, a_q\}_{q \in Q} \). The interpretation of the weights is as follows: \( w_{q,i} \) is one if in \( t_i \), the discrete state \( q \in Q \) is active, and it is zero otherwise. The algorithm terminates when an absolute criterion \( E \) falls below a pre-specified threshold \( \epsilon \). The iteration fails if after a predefined maximum number of iterations \( T_{\text{max}} \) the criterion \( E \) has not yet reached the lower threshold \( \epsilon \). Unfortunately, the conditions under which the algorithm terminates and returns a correct realization are not known yet.

4.1 Numerical example

We conducted several numerical experiments. In line with the definitions above, we use \( K \) and \( N \) to quantify the dimension of the \( \text{PWL} \) of interest and \( M \) to denote the number of data points. Dynamical data \( \{f(t_i), \hat{f}(t_i)\}_{i=1}^{M} \) was sampled from a given dynamical model \( \dot{x} = f(x) \) at regular time intervals \( t_i = t_i \Delta, i = 1, \ldots, M \) for some \( \Delta > 0 \). Several initial states were chosen randomly, and a sampled trajectory was obtained for each of them. We then concatenated these sampled trajectories into one time series. Notice that concatenation of finite components of two output trajectories of a \( \text{PWL} \) is itself a finite component of a valid output trajectory. Hence, the combined data can be viewed as originating from the measurements of one valid output trajectory of a \( \text{PWL} \). Gaussian noise with zero mean and variance \( \sigma^2 \) was added to the obtained data.

Finally, Algorithm 1 was applied to the resulting data. In the numerical experiments we first tested the performance of the algorithm on artificial \( \text{PWL} \) systems. Furthermore, we studied the algorithm on the Lorenz system and on the Tyson-Novak model for the cell cycle of budding yeast.

Simulations on artificial \( \text{PWL} \) systems We generate an artificial \( \text{PWL} \) system containing \( K \) affine subsystems, each of the same dimension \( n_q = N \). The state switching is obtained by partitioning the state space in \( K \) subsets \( \{V_1, \ldots, V_K\} \). The switching is generated by using the following switching law: the discrete mode \( i \) is active, if the continuous state belongs to \( V_i \). In the absence of noise, and if the subsystems are sufficiently sampled, the estimation algorithm is usually able to find the system parameters, though occasionally it gets stuck in a local minimum. When applying zero-mean Gaussian noise \( \mathcal{N}(0, \sigma^2) \), the accuracy of the reconstruction decreases as \( \sigma \) increases.

Figure 1 below shows the result on a dataset of dimension \( N = 2 \) with \( K = 5 \) subsystems and \( M = 1800 \) data points with a SNR (signal-to-noise ratio) of 5%. Two subsystems could not be reconstructed, but the correlation between the three reconstructed systems with the best fitting original subsystems is 98.4.

Chaotic systems We applied Algorithm 1 to the Lorenz systems Lorenz (1963). This example provides insight how the identification algorithm performs on systems that are not piecewise affine by nature. The Lorenz attractor can be visualized approximately as consisting of two linear subsystems Left and Right, with two different central attractors. A third region can be defined that involves the transition between the Left and Right subsystem. In Figure 2 the application of Algorithm 1 to the Lorenz systems with \( M = 5000 \) data points is presented for \( K = 2, 3 \). For \( K = 2 \) the two main parts of the Lorenz attractor are found. For \( K = 3 \) also the transition region from the left to the right attractor is found. Applying more than 3 subsystems does not improve the performance.

Biological cell cycle models We applied Algorithm 1 to data simulated using the Tyson-Novak model Chen et al.

5. CONCLUSIONS

We presented some basic results on realization theory of PWLs and a practical identification algorithm. Analysis of the correctness and convergence of the presented algorithm remains a topic of future research.

REFERENCES


Fig. 1. PWL with noise: the identified system

Fig. 2. Lorenz systems

Fig. 3. Tyson-Novak system

(2004) for the yeast cell cycle. The simulated data was used to construct a PWL approximation of the system using Algorithm 1. This resulted in a clear partition of the cell cycle dynamics as a function of the selected number of classes K. The best qualitative results were obtained for K = 2, see Figure 3. Moreover, for K > 3 the results did not improve. It was found that the obtained two discrete modes correspond to the phases S (DNA synthesis) and M (mitosis) of the cell cycle. The reconstruction remains valid under noise with SNR at most 10%.