Stability and Performance of Networked Control Systems with Time-multiplexed Sensors and Oversampled Observer

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Abstract: In this paper we analyze the scenario where a set of wireless sensors are time-multiplexed in order to reduce the traffic load and energy consumption in a Networked Control System. The possible choices for scheduling the transmission of measures are explored under the concept of periodic feedback patterns, i.e. repeated sequences of different measures. The length and structure of such patterns are chosen to guarantee stability of the closed loop and to minimize a predefined performance index. The control scheme is enhanced with a controller-side observer running at a rate higher than the sampling rate, for which a novel stability problem is stated and solved.

Keywords: Networked Control Systems, Wireless sensors, Periodic systems.

1. PROBLEM STATEMENT

In Networked Control Systems (NCS) reducing the energy consumption by limiting the transmission data-rates is crucial. However, less information may imply worse performance or, at worst, cause stability problems. Another issue related to feedback control system with wireless distributed sensors lies in the fact that energy consumption may be strongly unbalanced, requiring frequent battery changes for some sensors, while other are under mostly idle. Energy-aware control should implement energy balancing algorithms across the network and be able to reschedule the complete feedback strategy according to energy demands and availability. However this rescheduling capability requires proper knowledge of the system in order to recompute the stability conditions.

reconstruct the whole state of the system, which will be used for control purposes. Figure 1 describes the global scheme. Due to the characteristics of networked systems, the system is described in discrete time:

\[ x_{k+1} = Ax_k + Bu_k, \quad y_k = Cx_k, \]  

where \( x_k \in \mathbb{R}^n \), \( y_k \in \mathbb{R}^m \) and \( u_k \in \mathbb{R}^r \) are the state, the output and the control signal, respectively. \( A, B, C \) are known matrices of appropriate dimensions. As Figure 1 suggests, there exists \( m \) sensors, possibly spatially distributed, which sample each one of the components of the output vector. In each sampling period \( k \), only one sensor can use the network to send its packet. Assuming an ideal network (without noise, delays or disturbances), at instant \( k \) the sensor \( j, j = 1, ..., m \) sends the output \( y_k^j = E_j C x_k \), where \( E_j, j = 1, ..., m \), are matrices of appropriate dimensions (with elements ones and zeros) used to define any partial output.

In the controller side of the communication there is an observer trying to estimate the state of the system using the information of the received partial output (see Figure 1). The dynamics of the observer is,

\[ \tilde{x}_{k+1} = A \tilde{x}_k + B u_k + L_j \left[ y_k^j - \hat{y}_k^j \right], \]

\[ \hat{y}_k = C \tilde{x}_k, \]  

where \( \tilde{x}_k \in \mathbb{R}^n \) and \( \hat{y}_k \in \mathbb{R}^m \) are the state and the output of the observer, respectively. Matrices \( L_j, j = 1, ..., m \), are the observation gains which, in general, are different depending on the received output.

Every sampling period, the controller builds a control signal defined by \( u_k = K \tilde{x}_k \), where \( K \) is the controller matrix of appropriate dimensions.

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Finally, the error between the state of the system and the state of the observer is defined as
\[ e_k = x_k - \hat{x}_k. \quad (4) \]

The purpose of this paper is the design of the observer (2)-(3) in such a way that some stability and optimal properties for the observation error (4) were ensured. We propose a periodic observer, in which a pattern, defined off-line, is repeated at all time. This pattern indicates what output is sent in what instant.

The following definitions will be useful in future propositions and their proofs.

**Definition 1.** A measurement pattern \( \varphi_N \in \mathbb{R}^N \) is a vector whose components define which output is sent through the network. That is,
\[ \varphi_N(i) \equiv \{j_i : j_i = 1, \ldots, m\}, \quad i = 1, \ldots, N. \quad (5) \]

Hence, \( \varphi_N(i) = j_i \) implies that sensor \( j_i \) uses the network in the \( i \)-th position of the pattern. With this definition it is possible to grant priority to some outputs over the rest.

**Definition 2.** Interval \([\varphi_N(i), \varphi_N(j)]\) denotes all the consecutive samples between \( \varphi_N(i) \) and \( \varphi_N(j) \).

### 1.1 Literature review and open problems

In the literature there are related works as Rehbinder and Sanfridson (2004) for optimal periodic linear quadratic regulator; Zhang and Hristu-Varsakelis (2006); Jiang et al. (2008) for periodic observers; Hristu-Varsakelis and Zhang (2008) for periodic Kalman filters or Sinopoli et al. (2004); Gupta et al. (2007) for Kalman filter subject to intermittent observations.

However, none of these works solve the problem of finding an optimal pattern which manages the sent of the outputs. In Zhang et al. (2006); Hristu-Varsakelis and Zhang (2008) the communication sequences are chosen in such a way that the reachability and observability of the system are preserved. However, for a concrete system may exist several patterns that preserve the observability and reachability. How to choose the optimal one among them still remains an open problem. Finally, in Lu et al. (2003) a periodic pattern is chosen minimizing an \( H_{\infty} \) norm.

Moreover, for the best of our knowledge, there does not exist any work in which the observer is running at a frequency higher than the sampling rate, suitable when the control law can change faster than the data arrival rate. Those problems are studied in Section 2.

### 2. OVERSAMPLED OBSERVER

By letting the controller and observer run at a higher frequency than the transmission of partial outputs, there will be sampling instants without external feedback. Between two consecutive transmissions, the sensors remain asleep for \( P_o \) observation periods. So the dynamics of the observer differs for the two types of periods, observation periods (OP) and measurement periods (MP).

\[ (MP) : \quad \dot{x}_{k+1} = A \hat{x}_k + Bu_k + L_i \left[ y_k - \hat{y}_k \right]; \]
\[ (OP) : \quad \dot{x}_{k+1} = A \hat{x}_k + Bu_k. \]

Figure 2 shows the complete pattern for a case in which three observation periods have been introduced.

\[ \begin{array}{cccc}
\varphi_N(1) & \varphi_N(2) & \varphi_N(3) & \varphi_N(1) \\
(\text{OP}) & (\text{OP}) & (\text{OP}) & (\text{OP}) + (\text{OP}) \\
\downarrow & \downarrow & \downarrow & \downarrow \\
\hat{x}_k & \hat{x}_k & \hat{x}_k & \hat{x}_k \\
\uparrow & \uparrow & \uparrow & \uparrow \\
k & k + P_o + 1 & k + (P_o + 1)N & k + (P_o + 1)N \\
\end{array} \]

**Fig. 2.** Time scheduling for the oversampled observer

Analyzing both sets of equations, and defining the augmented state \( z_k^T = [x_k^T \; e_k^T] \), the following proposition gives the complete evolution of the system along a full period. The proof is given in Appendix A.

**Proposition 1.** Given a measurement pattern \( \varphi_N \in \mathbb{R}^N \), the evolution of the system from \( k \) to \( k + (P_o + 1)N \) is defined as
\[
\begin{align*}
z_{k+(P_o+1)N} & = \left( A + BK \right)^{(P_o+1)N} z_k \\
& = \Xi_{(P_o+1)N} z_k, \quad \forall k
\end{align*}
\]

where \( \Xi_{11} = A + BK \), \( \Xi_{22} = \prod_{j_i \in [\varphi_N(N), \varphi_N(1)]} [\theta_{j_i}], \Xi_{12} \) is a matrix with complex structure.

#### 2.1 Stability

The following lemma establishes the stability of the system.

**Lemma 2.** The discrete-time system with evolution given by Proposition 1 is asymptotically stable if and only if the eigenvalues of the following matrices are inside the unit circle:

- \( \Xi_{11} = A + BK \)
- \( \Xi_{22} = \prod_{j_i \in [\varphi_N(N), \varphi_N(1)]} [\theta_{j_i}] \)

The proof is immediate using fairly extended eigenvalue properties of triangular matrices.

**Example.** Consider the discrete-time system from Ishii and Francis (2002)

\[ x_{k+1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & -3 \\ 0 & 0 & -2 \end{bmatrix} x_k + \begin{bmatrix} 1 \ 0 \\ 0 \ 1 \\ 0 \ 1 \end{bmatrix} u_k, \]
\[ y_k = \begin{bmatrix} 0 \ 1 \ 0 \\ 1 \ 0 \ 0 \end{bmatrix} x_k. \quad (7) \]

observed with pattern \( \varphi_2 = [1, 2] \). Assume that the observer gains are \( L_1 = [0 - 2.2 - 0.8]^T \) and \( L_2 = [0.5 \ 0 \ 0]^T \). Using Lemma 2, the stability of the observation error without observation periods is ensured. The eigenvalues of \( \Xi_{22} are \lambda(\Xi_{22}) = \{0, 0.4, 0.5\} \). However, with \( P_o = 1 \), the eigenvalues of \( \Xi_{22} are \lambda(\Xi_{22}) = \{0, -2, 0.5\} \), so the observation error is unstable. However, choosing, for example, \( L_1 = [0 - 2 - 0.67]^T \) and \( L_2 = [1 \ 0 \ 0]^T \) with \( P_o = 1 \) yields \( \lambda(\Xi_{22}) = \{0, 0.83, 0.09\} \) so the observer is stable, but a reduction of data through the network is achieved.
2.2 Observer Design

In order to ensure the stability of the whole NCS, the eigenvalues of two matrices must lay inside the unit circle. Assuming that the system is stabilizable, we have to design the observer in such a way that the error dynamics becomes asymptotically stable. An LMI-based method is presented using ideas from periodic system theory.

A p-periodic discrete-time system is described by \( x_{k+1} = A_k x_k \), where \( x_k \) represents the state and \( A_k \) are p-periodic matrices, that is, \( A_{k+p} = A_k \), \( \forall k \).

An important result related to these systems is the so-called Periodic Lyapunov Lemma, see Bittanti et al. (1985), actually an extension of the Lyapunov Lemma for periodic matrix valued systems. It states that system (1) is observable at instant \( k \) if and only if matrix

\[
Q := \begin{bmatrix}
C' \\
C'A' \\
\vdots \\
C'A'^{n-1}
\end{bmatrix}
\]

has rank equal to \( n \), where \( A' := A^N \) and

\[
C' := \begin{bmatrix}
E_k C \\
E_{k+1} CA \\
\vdots \\
E_{k+N-1} CA^{N-1}
\end{bmatrix}.
\]

The proof is omitted due to space limitation, but it can be made by applying Cayley-Hamilton theorem as in the classical test. Finally, we introduce the definition of complete observability.

Definition. System (1) is completely observable if and only if it is observable for all \( k = 1, \ldots, N \).

Therefore, if a pattern achieves the observability of the system (1), it will be able to stabilize the observer. The observability test is clearly easier, computationally spoken, than the LMI-based stability test, so we can study off-line every possible pattern of length \( N \).

Minimization of a cost index

If we need to compare the quality of some stable patterns, we can establish a numerical index. This index could be defined taking into account the observation error.

\[
J(i) = \sum_{k=0}^{i} c_k^T Q c_k.
\]

being \( Q \) a positive definite matrix. A similar idea was firstly introduced in Havre and Skogestad (2003), but for optimal feedback stabilization of a system with only one unstable pole. The dependence of the index (11) on the chosen pattern is very complex, but it is possible to compare them, at least, in a numerical way. In the following, assume that \( P_0 = 0 \).

Proposition 6. Given a pattern with length \( N \) and a positive definite matrix \( Q \), the cost index \( J = \sum_{k=0}^{\infty} c_k^T Q c_k \) can be calculated as

\[
J = c_0^T \left[ \sum_{n=0}^{\infty} (\alpha_k^n)^T \Phi_N(\alpha N)^n \right] c_0,
\]

where

\[
\alpha_k := A_k^T P_k^{-1} A_k P_k^{-1} W_k.
\]

2.3 Pattern design

In Section 2.1, we had shown the conditions that must be verified so that the dynamics of the observation error were asymptotically stable given the observation matrices \( L_i \). In Section 2.2, a LMI-based design method was proposed to obtain the observation matrices given a measurement pattern. The next step in this process is, among all possible patterns, which one stabilizes the system?

It is obvious that those patterns which do not observe the dynamics of the observation error could be discard. There exist some researches in the literature on controllability and observability of generic periodic system, see Guo and Qiao (2004). However, for the case under study, only the output matrix \( E_k C \) is periodic. We can enunciate here an easier observability test.

Lemma 5. System (1) is observable at instant \( k \) if and only if matrix

\[
P := \begin{bmatrix}
P_k - P_k^T A_k P_k^{-1} W_k P_k^{-1} A_k P_k^{-1} W_k
\end{bmatrix} > 0,
\]

\( \forall k \in \{1, \ldots, p\} \).

The dynamics of the error between state of the plant and of the observer can be described as a periodic system. The following theorem proposes an LMI-based design procedure to obtain the observation matrices.

Theorem 4. The dynamics of the observation error, given by Proposition 1, is asymptotically stable if and only if there exists an \( N \)-periodic positive definite matrix \( P_k \) and an \( N \)-periodic matrix \( W_k \) of appropriate dimensions such that the following LMIs are satisfied,

\[
\begin{bmatrix}
P_k - (A_k P_k^T)^T A_k P_k^{-1} W_k P_k^{-1} A_k P_k^{-1} W_k
\end{bmatrix} > 0,
\]

\( \forall k \in \{1, \ldots, N\} \), where \( P_0 = P_N \). Then, the observation matrices are defined as \( L_k = P_k^{-1} W_k \).

Proof. To prove Theorem 4 we need to apply Schur complement to the Periodic Lyapunov Lemma (8). Then substitute \( A_k \) by \( \tilde{A}_k \) and define \( W_k = P_k L_k \) to obtain conditions (9).

As no conservatism has been added, Theorem 4 can also be used to find the maximum number of observation periods \( (P_o) \) that can be introduced between two consecutive samplings.
\[ \alpha_N = \vartheta_N \vartheta_{N-1} \ldots \vartheta_2 \vartheta_1, \]
\[ \Phi_N = \sum_{i=1}^{N} \alpha_i^T Q \alpha_i. \]

The proof is immediate by substituting the evolution of the system (6) with \( P_0 = 0 \) in the cost index (11).

From Proposition 6, it turns out the dependence between the index and the initial condition of the observation error. It is more interesting the situation in which the initial condition of the observation error is bounded. That is, we know the upper bound of \( \|e_0\|^2 \). For such case, we need to bound the cost index, employing the relation:

\[ x^T Ax \leq \lambda_{\text{max}} \{A\}\|x\|^2. \]

So, the cost index (12) is bounded by \( J \leq \lambda_{\text{max}} \{\beta_N\}\|e_0\|^2 \), with

\[ \beta_N = \sum_{n=0}^{\infty} (\alpha_N^T)^n \Phi_N (\alpha_N)^n. \]

For a periodic pattern, there are two characteristics we have to choose: 1) the length of the pattern and 2) its structure. The length of the pattern is critical, in such a way that, if it is not fixed a priori the number of combinations grows until infinity. Between all possible patterns of length \( N \) we can choose the one who minimizes the maximum eigenvalue of matrix \( \beta_N \).

Remark. Due to the pattern stability, matrix \( \alpha_N \) has all its eigenvalues inside the unit circle. So, the infinite series can be replaced by a finite one without incurring in practical errors, as we will see in the examples. Finally, this finite series can be calculated numerically.

Example. Optimal pattern design

Consider the discrete-time system from Hristu-Varsakelis and Zhang (2008). In that paper, the choice of the pattern was made based on stability properties only.

\[ x_{k+1} = \begin{bmatrix} 1 & 0.1 & 0 & 0 \\ 0.1 & 1.25 & 0 & 0 \\ 1 & 0.1 & 1/6 & 0.5 \\ 0 & 0 & 0 & 1.25 \end{bmatrix} x_k + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} u_k, \]
\[ y_k = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} x_k. \quad (13) \]

There exist two possible outputs to send through the network \( y_k^1 = [1 \ 0] y_k \) and \( y_k^2 = [0 \ 1] y_k \). The weighting matrix is chosen as \( Q = \text{diag} \{1, 1\} \). It is assumed that \( e_0 = x_0 \), that is, the initial condition for the observer is exactly zero.

The objective is to choose and design the optimal pattern among all possible ones. We call optimal pattern the one which minimizes the cost index (12). To calculate the cost index, a finite horizon is needed. Here, this horizon is chosen larger enough to neglect the errors.

Figure 3 depicts the lower maximum eigenvalue obtained for different lengths. Remember that the cost is bounded by

\[ J < \lambda_{\text{MAX}} \{e_0\}. \]

It can be seen that from some value, the larger pattern length does not imply the lower costs. For this example, a length of 7 is enough. More precisely, the optimal pattern is \([1 \ 2 \ 1 \ 2 \ 2 \ 2] \).

3. EXPERIMENTAL APPLICATION

3.1 Platform description, modeling and control

The control strategy described above is applied to a two-degree-of-freedom direct drive robot, which has been designed and developed by the Department of Systems Engineering and Automation at University of Seville, see Millán et al. (2010). The objective will be to maintain the robot at its upright equilibrium point, similar as an inverted pendulum.

The robot configuration is schematically shown in Figure 4. The first link (which is between both motors) will be termed as shoulder whereas the second link (which is between the smaller motor and the edge of the robot) is the elbow.

The selected control software is the xPC Target environment Mosterman et al. (2005) with MATLAB/Simulink. It is well-known that the dynamics of a robotic manipulator is extremely non-linear. To apply the results of this paper, the robot will be operated around the unstable upright equilibrium defined by \( \dot{q}_e = [\pi \ 0] \), \( \dot{\dot{q}}_e = [0 \ 0] \). To linearize the system and obtain matrix \( A \) and \( B \) of equation (1) a mean square iterative identification procedure has been followed.

In order to perform the tests an initial control based on feedback linearization is applied, which steers the robot from its stable downward position (both links stopped
Fig. 6. Position errors for the oversampled observer with different number of observation periods in their lower positions) to the surroundings of its unstable upright equilibrium. This controller is applied assuming the state completely accessible. Once that position is reached, periodic observer plus linear controller is switched.

3.2 Experimental results

The experiment is the same in all cases. The observer is switched (from the nonlinear control) from the upright equilibrium point. Then, an additive disturbance in the torque is applied at \( t = 10 \) seconds.

Although the linear model suggests that the system is observable from each of the output (classical observer), experiments show that this is not true for the real robot. For instance, Figures 5b and 5c show that it is not possible to observe the position of the elbow or the velocity of the shoulder. The only component of the state that can be observed is the position of the shoulder, as Figure 5a shows.

To compare the throughput of the oversampled observer we insert additional observation periods and repeat the same experiment. The pattern is chosen as \( \varphi = [3, 2, 3, 4] \), that is, the position of the shoulder is the only component of the state vector which is being observed (the others are directly measured). Figure 6 depicts the position error of the shoulder when different number of observation periods are included. As expected, as the number of observation periods grows, the error is higher. The constant steady-state error are due to the model errors.

Finally, we prove a different pattern and compare its throughput with the one proposed before. The observation error are depicted in Figure 7.

4. CONCLUSION

We have analyzed in depth the different feedback possibilities available when a set of wireless networks embedded in a control loop are time-multiplexed in order to avoid collisions and reduce the total network traffic and energy consumption. Finally, as a result of the performance analysis, new criteria for feedback pattern design in scheduled networks have been set. The results have been illustrated with simulations and in a real laboratory 2-DOF robot. Further work should consider the computation of the performance index of the feedback patterns without the need of numerical estimations.

REFERENCES


Appendix A. PROOF OF PROPOSITION 1

Suppose that there are $m$ sensors and $N$ outputs, with $N \geq m$. Further suppose that $P_o$ observation periods are introduced between two consecutive samples. The ordering of the measurements is defined by $\varphi_N \equiv \{j_i : j_i \in \{1, ..., m\} \text{ and } i = 1, ..., N\}$.

The evolution of the augmented state for a measurement period can be written as

$$z_{k+1} = \begin{bmatrix} A & BK \\ -BK & (A - L_j E_j C) \end{bmatrix} z_k = \Lambda_j z_k.$$  \hfill (A.1)

For an observation period, the following relations hold,

$$z_{k+1} = \begin{bmatrix} A & BK \\ 0 & A \end{bmatrix} z_k = \Delta z_k.$$  \hfill (A.2)

Suppose that from instant $k$ to $k + P_o - 1$, no output is received from the system. After $P_o$ observation periods, the following relations hold,

$$z_{k+P_o} = \begin{bmatrix} (A + BK)^{P_o} & \Psi P_o \\ 0 & A^{P_o} \end{bmatrix} \begin{bmatrix} z_k \\ e_k \end{bmatrix} = \Delta^{P_o} z_k,$$  \hfill (A.2)

where $\Psi = \sum_{i=1}^{P_o} (A + BK)^{P_o-i} BK A^{-1-i}$. It is easy to obtain this matrix by multiplying matrix $\Delta$ by itself $P_o$ times.

Suppose that, after $P_o$ observation periods, the controller receives a measurement. Hence the complete evolution of the observation periods plus one measurement period can be obtained using (A.1) and (A.2).

Finally, we have to prove that the following equality is true:

$$(\Lambda_j, \Delta^{P_o}) = \Xi_{(P_o+1)N},$$  \hfill (A.3)

where $\Xi_{(P_o+1)N}$ has been defined previously in the hypothesis of the proposition. To do that, we write $\Xi_{(P_o+1)N}$ (hereinafter) using its block structure, that is, $\Xi = \{\Xi_{ij} : i, j = 1, 2\}$, and we study the equivalence between the different blocks.

$$z_{k+(P_o+1)N} = \begin{bmatrix} (A + BK)^{P_o+1} & 0 \\ 0 & \prod_{j_i \in \varphi_N} [\delta_{j_i}] \end{bmatrix} \Xi_{12} \Xi_{12} \Xi_{21} \Xi_{22} \Xi_{22} \Xi_{12}.$$  \hfill (A.4)

$\Xi_{11}$: Due to the zero block in position (2, 1), when multiplying $N$ times $\Lambda_j, \Delta^{P_o}$, in position (1, 1) we obtain $(A + BK)^{P_o+1} N$.

$\Xi_{21}$: Because of the structure of the matrices $\Lambda_j, \Delta^{P_o}$, it is obvious that multiplying them, we obtain a zero block in position (2, 1).

$\Xi_{22}$: In this position we have a similar structure as in position (1, 1), due to the zero block. However, the matrices in block (2, 2) of $\Xi_{12}$ are different to each other. So, here, the ordering of the matrix products must be preserved. Multiplying from $\varphi_N$ until $\varphi_N(1)$, we obtain the same block that in matrix $\Xi_{(P_o+1)N}$.

$\Xi_{12}$: This block has complex structure. Although it could be found by multiplying the matrices, we have not done it because it does not affect to stability or further developments.

This completes the proof.