Using a PSO algorithm for tuning a PID$^\beta$ controller applied to a heat system

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Abstract: In this paper we study the control of a heat diffusion system in the perspective of fractional control and using a particle swarm optimization (PSO) algorithm. PSO is one of the latest evolutionary optimization techniques, inspired by social behavior of bird flocking or fish schooling. It has the better ability of global searching and has been successfully applied in many areas of engineering and science. Simulations are presented assessing the performance of the proposed fractional algorithms.

Keywords: PID control; Fractional; Modelling; Optimization problems; Error criteria.

1. INTRODUCTION

Particle swarm optimization (PSO) is an optimization technique developed by Dr. Eberhart and Dr. Kennedy in 1995, inspired by social behavior of bird flocking or fish schooling. The PSO scheme optimizes searching by virtue of the swarm intelligence produced by the cooperation and competition among the particles of a species. The social system is discussed through the collective behaviors of simple individuals interacting with their environment and each other. Examples of this are the bird flock or fish school. Some applications of PSOs are found in the field of nonlinear dynamical systems, data analysis, electrical engineering, function optimization, artificial neural network training, fuzzy control and many others in real world applications [Eberhart and Kennedy (1995), Kennedy and Eberhart (1995), Eberhart and Shi (2001), Trelea (2003), Reis and Machado (2007), Pires et al. (2007), Shayeghi et al. (2008) and Zhan et al. (2009)].

The concept of differentiation and integration to no integer order dates to 1695, when Leibniz mentioned it in a letter to L’Hopital. Since then, many scientists developed the area and notable contributions have been made, both in theory and in applications [Oldham and Spanier (1974) and Podlubny (1999a)]. In fact, fractional calculus (FC) is a generalization of integration and differentiation to a non-integer order $\alpha \in C$, being the fundamental operator $^{a}D^\alpha_t$, where $a$ and $t$ are the limits of the operation [Oldham and Spanier (1974) and Podlubny (1999a)]. These fractional concepts constitute a useful tool for describing several physical phenomena in almost all areas of science and engineering, such as heat, magnetism, flow, mechanics or fluid dynamics. Presently, the ability of the FC is being recognized for better modelling and control of many dynamical systems. In fact, during the last years FC has been used increasingly to model the constitutive behavior of materials and physical systems exhibiting hereditary and memory properties. This is the main advantage of fractional derivatives in comparison with the classical integer-order counterpart, where these effects are neglected.

It is well known that the fractional operator $^{0.5}$ appears in several types of problems [Battaglia et al. (2001)]. The transmission lines, the heat flow or the diffusion of neutrons in a nuclear reactor are examples where the half-order operator is the fundamental element. Moreover, diffusion is one of the three fundamental partial differential equations of mathematical physics [Courant and Hilbert (1962)]. Therefore, the control of such systems having in mind FC concepts is an important subject.

In this paper we investigate the control of a heat diffusion system based on fractional algorithms. The adoption of fractional controllers has been justified by its superior performance, particularly when used with fractional dynamical systems, such as the case of the heat system under study. The fractional PID controller, or $PI^D\beta$ controller, involves an integrator of order $\alpha \in R^+$ and a differentiator of order $\beta \in R^+$. It has been demonstrated the good performance of this type of controller, in comparison with the conventional PID algorithm.

Bearing these ideas in mind, the paper is organized as follows. Section 2 gives the fundamentals of fractional control systems. Section 3 introduces the heat diffusion system control strategies and discusses the results. Finally, section 4 draws the main conclusions and addresses perspectives towards future developments.

2. FRACTIONAL CONTROL SYSTEMS

Fractional control systems are characterized by differential equations that have, in the dynamical system and/or in the control algorithm, an integral and/or a derivative of fractional order. Due to the fact that these operators are defined by irrational continuous transfer functions, in the Laplace domain, or infinite dimensional discrete transfer functions, in the $Z$ domain, we often encounter evaluation problems in the simulations. Therefore, when analyzing fractional systems, we usually adopt continuous or discrete integer-order approximations of fractional operators [Podlubny (1999b) and Barbosa et al. (2006)]. The following two subsections provide a background for the remaining
of the article by giving the fundamental aspects of the FC, and the discrete integer approximations of fractional operators.

2.1 Fundamentals of Fractional Calculus

The mathematical definition of a fractional derivative and integral has been the subject of several different approaches [Oldham and Spanier (1974), Podlubny (1999a) and Petras and Vinagre (2002)]. One commonly used definition for the fractional derivative is given by the Riemann-Liouville definition ($\alpha > 0$):

$$D_{\alpha}f(t) = \frac{1}{\Gamma(n - \alpha)} \frac{d^n}{dt^n} \int_{0}^{t} \frac{f(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, \quad n - 1 < \alpha < n$$

where $f(t)$ is the applied function and $\Gamma(x)$ is the Gamma function of $x$. Another widely used definition is given by the Grünwald-Letnikov approach ($\alpha \in \mathbb{R}$):

$$D_{\alpha}^{t}f(t) = \lim_{h \to 0} \frac{1}{h^\alpha} \sum_{k=0}^{\lfloor t/h \rfloor} (-1)^k \binom{\alpha}{k} f(t-kh)$$

where $h$ is the time increment and $[x]$ means the integer part of $x$.

The "memory" effect of these operators is demonstrated by (1) and (2), where the convolution integral in (1) and the infinite series in (2), reveal the unlimited memory of these operators, ideal for modeling hereditary and memory properties in physical systems and materials.

An alternative definition to (1) and (2), which reveals useful for the analysis of fractional control systems, is given by the Laplace transform method. Considering vanishing initial conditions, the fractional differintegration is defined in the Laplace domain, $F(s) = L\{f(t)\}$, as:

$$L\{D_{\alpha}^{t}f(t)\} = s^\alpha F(s), \quad \alpha \in \mathbb{R}$$

An important aspect of fractional-order algorithms can be illustrated through the elemental control system represented in Fig. 1, with open-loop transfer function

$$G(s) = Ks^{-\alpha} (1 < \alpha < 2)$$

in the forward path [Machado (1997)]. The open-loop Bode diagrams of amplitude and phase have correspondingly a slope of $-20\alpha$ dB/dec and a constant phase of $-\alpha\pi/2$ rad over the entire frequency domain. Therefore, the closed-loop system has a constant phase margin of $PM = \pi(1 - \alpha/2)$ rad that is independent of the system gain $K$. Likewise, this important property is also revealed through the root-locus ($K \geq 0$) depicted in Fig. 2. In fact, when $1 < \alpha < 2$, the root-locus follows the relation $\pi - \pi/\alpha = \cos^{-1} \zeta$, where $\zeta$ is the damping ratio, independently of the gain $K$. Therefore, the closed-loop system will be robust against gain variations exhibiting step responses with an iso-damping property [Battaglia et al. (2001), Barbosa et al. (2004) and Chen and Moore (2005)].

![Fig. 1. Elemental feedback control system of fractional order $\alpha$.](image)

![Fig. 2. Root-locus of $G(j\omega)$ for $1 < \alpha < 2$, $K \geq 0$.](image)

### 2.2 Approximations of Fractional Operators

In this paper we adopt discrete integer-order approximations to the fundamental element $s^\alpha$ ($\alpha \in \mathbb{R}$) of a fractional-order control (FOC) strategy. The usual approach for obtaining discrete equivalents of continuous operators of type $s^\alpha$ adopts the Euler, Tustin and Al-Alsouli generating functions [Vinagre et al. (2004), Chen et al. (2004) and Barbosa et al. (2006)].

It is well known that rational-type approximations frequently converge faster than polynomial-type approximations and have a wider domain of convergence in the complex domain. Thus, by using the Euler operator

$$w(z^{-1}) = (1 - z^{-1})/T_c \quad \text{and} \quad \text{performing a power series expansion of} \quad [w(z^{-1})]^n = [(1 - z^{-1})/T_c]^n \quad \text{gives the discretization formula corresponding to the Grünwald-Letnikov definition (2)}$$

$$D^\alpha (z^{-1}) = \left(1 - z^{-1}\right)^\alpha / T_c = h^\alpha (0) + h^\alpha (1) z^{-1} + \ldots = \sum_{k=0}^{\infty} h^\alpha (k) z^{-k}$$

where $T_c$ is the sampling period and $h^\alpha (k)$ is the impulse response sequence given by the expression ($k \geq 0$):

$$h^\alpha (k) = \left(1/T_c\right)^\alpha \binom{k - \alpha - 1}{k}$$

A rational-type approximation can be obtained through a Padé approximation to the impulse response sequence (5) $h^\alpha (k)$, yielding the discrete transfer function:

$$H(z^{-1}) = \frac{b_0 + b_1 z^{-1} + \ldots + b_m z^{-m}}{1 + a_1 z^{-1} + \ldots + a_n z^{-n}} = \sum_{k=0}^{\infty} h(k) z^{-k}$$

where $m \leq n$ and the coefficients $a_k$ and $b_k$ are determined by fitting the first $m+n+1$ values of $h^\alpha (k)$ into the impulse response $h(k)$ of the desired approximation $H(z^{-1})$. Thus,
we obtain an approximation that has a perfect match to the desired impulse response $h^n(k)$ for the first $m + n + 1$ values of $k$ [Barbosa et al. (2006)]. Note that the above Padé approximation is obtained by considering the Euler operator but the determination process will be exactly the same for other types of discretization schemes.

3. HEAT DIFFUSION

The heat diffusion is governed by a linear partial differential equation (PDE) of the form:

$$\frac{\partial c}{\partial t} = k_c \left( \frac{\partial^2 c}{\partial x^2} + \frac{\partial^2 c}{\partial y^2} + \frac{\partial^2 c}{\partial z^2} \right)$$

(7)

where $k_c$ is the diffusivity, $t$ is the time, $c$ is the temperature and $(x,y,z)$ are the space Cartesian coordinates. The system (7) involves the solution of a PDE of parabolic type for which the standard theory guarantees the existence of a unique solution [Machado et al. (2006)].

For the case of a planar perfectly isolated surface we can apply a constant temperature $C_0$ at $x = 0$ and we can analyze the heat diffusion along the horizontal coordinate $x$. Under these conditions, the heat diffusion phenomenon is described by a non-integer order model, yielding a transfer function of type:

$$C(x,s) = \frac{C_0}{s} G(s) , \quad G(s) = e^{-\sqrt{\pi} s}$$

(8)

where $x$ is the space coordinate, $C_0$ is the boundary condition and $G(s)$ is the system transfer function.

The solution of system (8) in the time domain yields:

$$c(x,t) = C_0 \operatorname{erfc} \left( \frac{x}{2 \sqrt{kt}} \right) = C_0 \left( 1 - \frac{2}{\sqrt{\pi}} \int_0^{\sqrt{kt}} e^{-z^2} dz \right)$$

(9)

In our study, the simulation of the heat diffusion is accomplished by adopting the explicit numerical integration based on the discrete approximation to differentiation as [Gerald and Wheatley (1999)]:

$$c(j + 1, i) = c[j, i + 1] + c[j, i - 1] + (1 - 2r) \cdot c[j, i]$$

(10)

where $r = k_c \Delta t (\Delta x^2)^{-1}$, $\{\Delta x, \Delta t\}$ and $\{i, j\}$ are the increments and the integration indices for space and time, respectively.

Figure 3 depicts the polar diagrams for both $G_A(j\omega)$ (i.e., analytical solution according to (8) and $G_N(j\omega)$ (i.e., numerical solution from (10)) implementations, when $x = 3.0$ m and $k_c = 0.042$ m²s⁻¹. It is verified that the results obtained through the numerical approach differ from the analytical results at low frequencies [Farlow (1993) and Gerald and Wheatley (1999)]. In the simulations we adopt (10) and, therefore, the smaller gain of $G_N(j\omega)$ may be interpreted as the introduction of extra losses at low frequencies.

3.1 Control and optimization strategies

The generalized PID controller $G_c(s)$ has a transfer function of the form [Podlubny (1999b)]:

$$G_c(s) = K_p + \frac{K_i}{s^\alpha} + K_D s^\beta, \quad \alpha, \beta > 0$$

(11)

where $\alpha$ and $\beta$ are the orders of the fractional integrator and differentiator, respectively. The parameters $K_p$, $K_i$, and $K_D$ are correspondingly the proportional, integral, and derivative gains of the controller. Clearly, taking $(\alpha, \beta) = \{(1, 1), (1, 0), (0, 1), (0, 0)\}$ we get the classical {PID, PI, PD, P} controllers, respectively. Other PID controllers are possible, namely: PD² controller, PI² controller, PID³ controller, and so on. The PI²D² controller is more flexible and gives the possibility of adjusting more carefully the closed-loop system characteristics [Chen (2006) and Podlubny (1999a)].

In previous studies developed by the authors [Jesus and Machado (2007), Jesus et al. (2008) and Jesus and Machado (2008)], the control of system of Fig. 4 was analyzed adopting the classical integer-order PID tuned by the Ziegler-Nichols open loop (ZNOL) heuristics. It was verified that the system under the action of a PID controller does not produce satisfactory results, giving rise to a significant overshoot, and a large settling time. Moreover, the step response reveals a considerable time delay. The poor results obtained indicate that the method of tuning the system may not be the most adequate for the control of the heat system under consideration. In fact, the system fractional dynamics suggests the adoption of other configurations, namely the use of a fractional PID³ controller. In this line of thought, in this work we propose a fractional PID³ controller tuned through the minimization of the indices ISE (integral square error) and ITSE (integral time square error), by applying a PSO algorithm in order to achieve a superior control performance of the heat diffusion control system of Fig. 4.

It is important to mention that a reliable execution and analysis of a PSO usually requires a large number of simulations to provide that stochastic effects have been properly considered. Therefore, the experiments consist on executing the PSO several times $N_{PSO}$, in order to
generate a combination of controller gains parameters, that leads to a better transient and steady state responses.

In the PSO algorithm, all particles represent a potential solution to a problem, which is performed by adjusting their position taking into account both personal and group experiences [Eberhart and Kennedy (1995), Kennedy and Eberhart (1995) and Reis and Machado (2007)]. In each iteration, the velocity is actualized by expression (12a). The new particle position is found by adding their actual position with the new velocity, as shown in (12b).

\[ v_{k+1} = w \cdot v_k + c_1 \cdot r_1 \cdot (lbp - cp_k) + c_2 \cdot r_2 \cdot (gbp - cp_k) \]  
\[ cp_{k+1} = cp_k + v_{k+1} \]  

(12) where \( v \) is the particle velocity, \( w \) is the inertia weights, \( c_1 \) and \( c_2 \) are the individual and sociality weights coefficients for modelling attractive forces from the local and global best, respectively; \( r_1 \) and \( r_2 \) are aleatory numbers between [0, 1], \( lbp \) is the local best position, \( cp \) is the current position, and \( gbp \) is the global best position [Shi and Eberhart (1998) and Eberhart and Shi (2000)].

The inertia term, forces the particle to move in the same direction as before by adjusting the old velocity. The cognitive term (personal best), forces the particle to go back to the previous best position. On the other hand, the social learning term (global best), forces the particle to move to the best previous position of its neighbors.

The PSO optimizes an objective function by iteratively improving a swarm of solutions, called particles, based on special management of memory. Each particle is modified by referring to the memory of individual swarm’s best information. Due to the collective intelligence of these particles, the swarm is able to repeatedly improve its best observed solution and converging to an optimum.

The optimization fitness function corresponds to the minimization of one of these two indices that measure the response error, namely the ISE and the ITSE criteria, defined as:

\[ J_{ISE} = \int_0^\infty [r(t) - c(t)]^2 \, dt \]  
\[ J_{ITSE} = \int_0^\infty t |r(t) - c(t)|^2 \, dt \]  

(13) We can use other integral performance criteria such as the integral absolute error (IAE) or the integral time absolute error (ITAE). In the present case the ISE and the ITSE criteria have produced the best results and are adopted in the sequel [Jesus and Machado (2007)]. Furthermore, the ITSE criterion enable us to study the influence of time in the error generated by the system.

We establish the following values for the PSO parameters: population number \( PN = 40, c_1 = c_2 = 2, w = 1 \) and a maximum number of iterations \( T_{Max} = 200 \).

### 3.2 PID\(^3\) Control

In this subsection we analyze the closed-loop system with a fractional PI\(^D\) controller given by the transfer function (11) with \( \alpha = 1 \). The fractional-order derivative term \( K_D s^\alpha \) in (11) is implemented by using a 4\(^{th}\)-order Padé discrete rational transfer function of type (6). It is used a sampling period of \( T_c = 0.1 \) s. The PID\(^3\) controller is tuned by minimization of the ISE (13a) or, alternatively, by the ITSE (13b) criteria.

A step reference input \( R(s) = 1/s \) is applied at \( x = 0.0 \) m and the output \( c(t) \) is analyzed for \( x = 3.0 \) m. The heat system is simulated for 250 seconds. Fig. 5 illustrates the variation of the fractional PID parameters \( (K_p, K_i, K_d) \) as function of the derivative order \( \beta \), when minimizing the ISE or the ITSE criteria and for the best case within a sample of 10 experiments.

![Figure 5](image)

Fig. 5. The PID\(^3\) parameters \( (K_p, K_i, K_d) \) versus \( \beta \) for the ISE and ITSE criteria, \( x = 3.0 \) m, \( k = 0.042 \) m\(^2\)s\(^{-1}\).

Figure 6 shows the ISE and ITSE errors as function of \( \beta \), for the best case within \( N_{PSO} = 10 \). The graph shows that the ISE and ITSE criteria lead to similar curves, with only a slight scale factor between the two indices. This scale factor is due to the influence of time in the ITSE computation. We have a lower error for \( \beta_{ISE} = 0.4 \) and \( \beta_{ITSE} = 0.6 \). Moreover, the curves indicate a large influence of a weak derivative order on the system dynamics.

Figure 7 illustrates the step responses of the closed-loop system, for the PID\(^3\) tuned in the ISE and ITSE perspectives for the best cases of \( \{\beta_{ISE},\beta_{ITSE}\} = \{0.4; 0.6\} \). The
controller parameters, corresponding to the minimization of those indices, lead to the values of ISE: \( \{K_p, K_i, K_d, \beta\} = \{-105.22, 0.17, 266.11, 0.4\} \) and ITSE: \( \{K_p, K_i, K_d, \beta\} = \{-26.97, 0.11, 151.76, 0.6\} \).

![Fig. 6. Error \( J \) for the ISE and ITSE indices versus \( \beta \), \( x = 3.0 \text{ m}, k = 0.042 \text{ m}^2\text{s}^{-1} \).](image1)

The step responses with the PID\(^3\) (Fig. 7) present a reasonable transient behavior and a zero steady-state error showing the effectiveness of the proposed fractional algorithm when used for the control of the heat system.

Figure 8 shows the variation of the transient response specifications, namely the settling time \( t_s \), the rise time \( t_r \), the peak time \( t_p \), and the percent overshoot \( ov(\%) \) versus \( \beta \), for the closed-loop response tuned through the minimization of the ISE and ITSE indices, respectively.

The charts reveal several different regions. It is clear that it is impossible to minimize simultaneously all specifications. However, for \( \beta_{\text{ISE}} \approx 0.4 \) and \( \beta_{\text{ITSE}} \approx 0.8 \) we get a good compromise between all specifications.

In conclusion, for \( 0.4 \leq \beta \leq 0.8 \) we get the best controller tuning, superior to the performance revealed by the classical integer-order scheme studied in previous works [Jesus et al. (2006) and Jesus and Machado (2007)]. These results demonstrate the effectiveness of the fractional-order algorithms when used for the control of fractional-order systems. However, the time delay observed in the step response, tell us that this kind of structure may not be the most adequate for the control of the heat diffusion system. In fact, the system suggests the adoption of a Smith predictor scheme, that is very effective in improving the control of processes having time delays.

![Fig. 7. Step responses of the closed-loop system for the ISE and ITSE indices, with a PID\(^3\) controller, \( x = 3.0 \text{ m}, k = 0.042 \text{ m}^2\text{s}^{-1} \).](image2)

![Fig. 8. Parameters \( t_s, t_r, t_p, ov(\%) \) versus \( \beta \) for the step responses of the closed-loop system for the ISE and ITSE criteria with a PID\(^3\) controller, \( x = 3.0 \text{ m}, k = 0.042 \text{ m}^2\text{s}^{-1} \).](image3)

4. CONCLUSIONS

This paper presented the fundamental aspects of application of the FC theory in the control of diffusion systems. In this line of thought, it was studied a heat diffusion system described through the fractional-order operator \( s^{0.5} \). The dynamics of the system was analyzed in the perspective of FC and using a PSO algorithm.

We presented a PID\(^3\) control strategy, that gives better results than those obtained with the conventional PID controller. This result points out the use of a fractional
PID algorithm inserted in a Smith Predictor structure (which is used to compensate systems with time delay).

Moreover, at the PSO algorithm the particles update themselves with the internal velocity. They also have memory, thus retaining part of their previous state, which is important to the algorithm [Reis and Machado (2007) and Eberhart and Kennedy (1995)]. Furthermore, this memory behavior of the particles have similarities with that occurs in the fractional order calculus. In fact, the PSO memory enables the algorithm to adjust its searching strategy dynamically in accordance with the current situation.

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