Contribution to the stabilization of stochastic nonlinear systems with time delays

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Abstract:
In this paper, the problem of feedback stabilization of stochastic differential delay systems is considered. The systems under study are also nonlinear and nonaffine. By using a LaSalle-type theorem for stochastic systems, general conditions for stabilizing the closed-loop system with delays are obtained. In addition, stabilizing state feedback control laws are proposed. An example illustrates the proposed approach.

1. INTRODUCTION

During the last decades, the problems of stabilization and controller design for linear systems with delays has been extensively studied (see [7, 10, 16, 20]). In practice, many control processes involve delays (often due to transmission or transportation phenomena). Delays may significantly affect the closed-loop performances or even be a source of instability.

In the case of nonlinear systems with delays, the problem of stabilization is more complex. This is mainly due to the infinite dimensionality of the system state combined with the nonlinear structure of the differential equations.

In ([1]-[4]) we investigated the problem of stabilization of nonlinear, nonaffine systems involving delays in both continuous and discrete-time cases. These works have been treated in a deterministic context.

However, the presence of delays and nonlinearities are not the only sources of complexity. Indeed, various disturbances that are not measurable may arise which, in turn, limit the application of classical control systems design. This motivates the study of the stabilizability and the control design in a stochastic framework, where the state equation is described by an Itô differential delay equation driven by Wiener noise.

The stability analysis of the equilibrium positions of stochastic differential equations with delays has been extensively studied (see for instance [13],[17]). In the case of linear stochastic systems with delays, some results on stabilization have been proposed (in [6, 19], for instance). However, the stabilization of nonlinear stochastic systems with delays still remains an open problem.

In this paper, the state feedback stabilizability problem of equilibrium positions of stochastic nonlinear systems with delays is considered. The class of systems under study is nonaffine in control. Moreover, the system without drift - in other words, the related autonomous system - also involves delays. By combining a suitable mathematical formalism and a LaSalle-type theorem dedicated to stochastic systems ([14, 15]), sufficient conditions guaranteeing the stability of the closed-loop system are developed and feedback controllers for these systems are proposed. The approach adopted in this paper allows considering a rather large class of nonlinear stochastic systems. Moreover, the autonomous system (u = 0) as well as the controlled part are affected by a noise.

The organization of the paper is as follows. In Section 2 the class of systems under consideration is presented and some basic notions are recalled. Some notations are introduced and usual notions about the stability and LaSalle-type Theorem for stochastic differential systems with time delay are recalled. In Section 3, the main results are given and proved. Sufficient conditions of stabilizability and related feedback laws are proposed. Moreover, an illustrative example is presented before conclusions.

2. PROBLEM FORMULATION AND PRELIMINARIES

The following class of systems is considered:

\[ \begin{align*}
    dx(t) &= f(x(t), x(t−τ), u)dt + g(x(t), x(t−τ), u)dξ(t) \\
    x(t) &= φ(t), \quad t ∈ [−τ, 0]
\end{align*} \]

where f and g are smooth vector fields such that \( f(0,0,0) = g(0,0,0) = 0 \). In the following, \( x(t) ∈ \mathbb{R}^n \) is the state vector and \( u ∈ \mathbb{R} \) is the input vector. \( τ \) is a positive scalar that represents the delay.

The function \( φ(t) ∈ C = C(−τ,0], \mathbb{R}^n) \) represents the initial condition. \( C(−τ,0], \mathbb{R}^n) \) is the banach space of continuous function mapping \( [−τ,0] \) into \( \mathbb{R}^n \), with the norm \( \| φ \| = \sup_{t∈[−τ,0]} |φ(t)| \) where \( |φ(t)| \) stands for the Euclidean norm of \( φ(t) ∈ \mathbb{R}^n \).

\( \{ξ(t), t ≥ 0\} \) is a standard Wiener process defined on the usual complete probability space \( (Ω, F, (F_t)_{t≥0}, P) \)

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with \((\mathcal{F}_t)_{t \geq 0}\) being the complete right-continuous filtration generated by \(\xi\) and \(\mathcal{F}_0\) contains all \(P\)-null sets.

Let \(C^b_{\mathcal{F}_0}([-\tau,0], \mathbb{R}^n)\) be the set of all \(\mathcal{F}_0\)-measurable bounded \(C([-\tau,0], \mathbb{R}^n)\)-valued random variables \(\phi\).

### 2.1 Preliminaries

In order to deal with the previously defined class of systems, some useful notations are introduced and usual notions about the stability of stochastic systems with time delay are first recalled as preliminaries for the further results. To that purpose, the differential stochastic system of the general form (2) is considered:

\[
\begin{align*}
  dx(t) &= F(t,x_t)dt + G(t,x_t)d\xi(t) \\
  x(t) &= \phi(t), \quad t \in [-\tau,0]
\end{align*}
\]

where \(F, G : [0, \infty) \times C([-\tau,0], \mathbb{R}^n) \mapsto \mathbb{R}^n\) is continuous with respect to the first argument, locally Lipschitz with respect to the second and satisfy \(F(0,0) = G(0,0) = 0\) for all \(t \geq 0\).

For \(t \geq \sigma - \tau\), we denote by \(x(\sigma, \phi)(t)\) its solution at time \(t\) with initial data \(\phi\) specified at time \(\sigma\), i.e., \(x(\sigma, \phi)(\sigma + \theta) = \phi(\theta), \forall \theta \in [-\tau,0]\). For \(\theta \in [-\tau,0]\),

\[ x_1(\theta) = x(t + \theta) \]

and represents the state of the delay system.

Under the previous assumptions on \(F\) and \(G\), it is known (see e.g., [13, 17, 18]) that equation (2) has a unique solution \(x(\sigma, \phi)(t)\) for \(t \geq \sigma - \tau\).

Let us introduce \(\delta\), the delay operator defined for any function \(a(.)\) by:

\[ \delta a(t) = a(t - \tau). \]

For any function \(h : \mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}^n\), the following type of notation indexed by \(\delta\), is used:

\[ h_\delta(x(t)) = h(x(t), \delta x(t)) = h(x(t), x(t - \tau)). \]

### 2.2 Notions of stability and LaSalle-type Theorem for stochastic differential delay systems

In the following, some notions of stability of equilibrium solution of stochastic differential delay equations are given. They generalize the notions initially dedicated to stochastic differential systems (cf. [5, 9, 11]).

**Definition 1.** The equilibrium solution, \(x \equiv 0\) of the stochastic differential delay equation (2) is said to be

1. **stable in probability**, if for any \(\sigma \in \mathbb{R}, \varepsilon > 0\), there is a \(b = b(\varepsilon, \sigma) > 0\) such that \(P_{t \geq \sigma}\|\phi\| < b\) implies \(P\left(\sup_{t \geq \sigma} \|x(t, \phi)\| > \varepsilon\right) = 0\).
2. **uniformly stable in probability**, if the number \(b\) is independent of \(\sigma\).

For all \(b > 0\), let us denote by \(B(0, b)\), the ball

\[ B(0, b) = \{\phi \in C([-\tau,0], \mathbb{R}^n) : \|\phi\| < b\}. \]

**Definition 2.** The equilibrium solution, \(x \equiv 0\) of the stochastic differential delay equation (2) is said to be asymptotically stable in probability, if it is stable and there exists \(b_0 = b_0(\sigma) > 0\) such that \(\phi \in B(0, b_0)\) implies \(P\left(\lim_{t \to +\infty} \|x(t, \phi)\| = 0\right) = 1\).

In practice, we can use the following LaSalle-type Theorem for stochastic differential delay system (cf. [14, 15]). This is a stochastic version of the well-known LaSalle Theorem (cf. [8, 12]).

**Theorem 3.** Assume that there are functions \(V \in C^{1,2}(0, \infty) \times \mathbb{R}^n, (0, \infty)\), \(\gamma \in L^1((0, \infty), (0, \infty))\) and functions \(w_1, w_2 \in C(\mathbb{R}^n, [0, \infty))\) such that

\[ LV(t, x, y) \leq \gamma(t) - w_1(x) + w_2(y), \quad x, y \in \mathbb{R}^n, t \geq 0 \]

\[ w_1(x) \geq w_2(x) \quad x \in \mathbb{R}^n \]

and \(\lim_{|x| \to \infty} \inf_{0 \leq t \leq \infty} V(x, t) = \infty\)

Then, \(Ker(w_1 - w_2) \neq \emptyset\) and

\[ \lim_{t \to \infty} d(x(t, \phi), Ker(w_1 - w_2)) = 0 \quad \text{a.s. (almost surely)} \]

for every \(\phi \in C^b_{\mathcal{F}_0}([-\tau,0], \mathbb{R}^n)\).

**Remark 4.** The infinitesimal generator associated to system (5) is defined, in this theorem, by

\[ \mathcal{L}_\delta V(t, x) = \frac{\partial V(t, x)}{\partial t} + \gamma(t) - w_1(x) + w_2(y) \]

\[ + \frac{1}{2} Tr \left( [G(t, x, \delta x) G^T(t, x, \delta x)] \right) \]

where \(\nabla\) denotes the gradient and \(\langle ., . \rangle\) designates the usual scalar product. The matrix \(\nabla V(t, x) = \frac{\partial^2 V(t, x)}{\partial x^2}\) is the Hessian matrix of the second order partial derivatives of \(V(t, x)\) with respect to \(x\).
\[ \mathcal{L}V(t, x, y) = \frac{\partial V(t, x)}{\partial t} + \langle F(t, x, y), \nabla V(t, x) \rangle + \frac{1}{2} \text{Tr} \left( G(t, x, y) G^T(t, x, y) \nabla^2 V(t, x) \right) \]

and \( L_1([0, \infty), [0, \infty)) \) is defined by:

\[ \sup_{|u| \leq 1} |H_{\delta}(x(t), u)|^2 + |\nabla L_{1,\delta} V(x(t))|^2 + 2 \int_0^{\infty} \gamma(t) dt < \infty. \]

Note that the notation \( \mathcal{L}V(t, x) \) which will be used throughout the paper and defined by \( (6) \), corresponds to \( \mathcal{L}V(t, x, \delta x) \) in Theorem 3.

We shall now state and prove our main results.

### 3. MAIN RESULTS

In the following, we propose to develop sufficient conditions to guarantee the stabilizability of system \((1)\) and derive feedback control laws. For this purpose, it is assumed that \( f \) can be developed in the form:

\[ f(x(t), x(t-\tau), u) = f_0(x(t), x(t-\tau)) + a^\alpha f_\alpha(x(t), x(t-\tau)) + a^\beta g_\delta(x(t), x(t-\tau), u) \]

and

\[ g(x(t), x(t-\tau), u) = g_0(x(t), x(t-\tau)) + a^\beta g_\beta(x(t), x(t-\tau)) + a^{\beta+1} g_{\beta+1}(x(t), x(t-\tau), u). \]

where \( \alpha, \beta \in N \) and \( 1 \leq \alpha, \beta \). \( f_0 \) and \( g_0 \) are functions defined by:

\[ f_0(x(t), x(t-\tau)) = f(x(t), x(t-\tau), 0) \]

and

\[ g_0(x(t), x(t-\tau)) = g(x(t), x(t-\tau), 0). \]

\( f_\alpha, f_{\alpha+1}, g_\beta \) and \( g_{\beta+1} \) are smooth functions.

Note that since \( f \) and \( g \) are smooth, such expansion are possible. Indeed, with the integer \( \alpha \) defined by:

\[ \alpha = \inf \{ k \in N : \frac{\partial^k}{\partial u^k} f(x, \delta x, 0) \neq 0 \} \]

\( f_\alpha \) can be written as:

\[ f_\alpha(x(t), x(t-\tau)) = \frac{1}{\alpha!} \frac{\partial^\alpha f}{\partial u^\alpha}(x(t), x(t-\tau), 0) \]

and \( f_{\alpha+1}(x(t), x(t-\tau), u) \) correspond to the rest of the Taylor expansion. A similar remark stands for \( g \) (and the integer \( \beta \)) as well. Thus

\[ \beta = \inf \{ k \in N : \frac{\partial^k}{\partial u^k} g(x, \delta x, 0) \neq 0 \} \]

In order to simplify the presentation, it is supposed that \( \alpha < \beta \) and that \( \alpha \) is odd. Notice that the assumption of an odd \( \alpha \) generalizes the case of vector fields affine in the input \( u \). The other cases are considered in the following of this paper.

### 3.1 Case : \( \alpha < \beta \) and \( \alpha \) is odd

Let us denote by \( L_{0,\delta} \) and \( L_{1,\delta} \), the differential operators defined for all \( \Xi \in C^2(R^n) \) by:

\[ L_{0,\delta} \Xi(x(t)) = \langle f_{0,\delta}(x(t)), \nabla \Xi(x(t)) \rangle + \frac{1}{2} \text{Tr} \left( g_{0,\delta}(x(t), x(t-\tau), u) g_{0,\delta}^T(x(t), x(t-\tau), u) \nabla^2 \Xi(x(t)) \right) \]

and

\[ L_{1,\delta} \Xi(x(t)) = \langle f_{0,\delta}(x(t)), \nabla \Xi(x(t)) \rangle + \frac{1}{2} \text{Tr} \left( g_{1,\delta}(x(t), x(t-\tau), u) g_{1,\delta}^T(x(t), x(t-\tau), u) \nabla^2 \Xi(x(t)) \right). \]

We introduce \( H(x(t), x(t-\tau), u) \) defined by:

\[ H(x(t), x(t-\tau), u) = \langle f_{0,\delta}(x(t)), \nabla \Xi(x(t)) \rangle + \frac{1}{2} \text{Tr} \left( g_{0,\delta}(x(t), x(t-\tau), u) g_{0,\delta}^T(x(t), x(t-\tau), u) \nabla^2 \Xi(x(t)) \right) \]

where we use the notation \( (4) \) and set

\[ f_{\alpha+1,\delta}(x(t), x(t-\tau), u) = f_{\alpha+1}(x(t), x(t-\tau), u) \]

and

\[ g_{\beta+1,\delta}(x(t), x(t-\tau), u) = g_{\beta+1}(x(t), x(t-\tau), u) \]

in order to get more compact expressions.

We suppose that there exists a Lyapunov function \( V \in C^2([0, \infty), [0, \infty)) \) and functions \( a \) and \( a_d \in C([0, \infty)) \) such that:

\[ L_{0,\delta} V(x(t)) \leq -a(x(t)) + a_d(x(t-\tau)) \]

where

\[ a_d(x) \leq a(x) \quad \forall x \in R^n. \]

Let us denote by \( ^M \) the set:

\[ ^M = \text{Ker}(a - a_d) \cap \text{Ker}(L_{1,\delta} V) \]

We then have the following result:

**Theorem 5.** Let \( u(x(t), x(t-\tau)) = -\psi(x(t), x(t-\tau)) L_{1,\delta} V(x(t)) \)

where \( \psi \in C^\infty(R^n) \) is a function satisfying:

\[ \psi(x(t), x(t-\tau)) \leq \frac{1}{\sup_{|u| \leq 1} |H_\delta(x(t), u)|^2 + |L_{1,\delta} V(x(t))|^2 + 2} \]

and

\[ H_\delta(x(t), u_\delta(x(t))) = H(x(t), x(t-\tau), u_\delta(x(t))) \]
with \( u_\delta(x(t)) = u(x(t), x(t-\tau)) \).
If the set \( \mathcal{M} \) defined by (14) is reduced to the origin, then the trivial solution of the system (1) with (15) is almost surely asymptotically stable.

**Proof:**

With the control law (15), the closed-loop system is of the form:

\[
dx(t) = f_0(x(t), x(t-\tau)) \, dt \\
+ u_\alpha(x(t), x(t-\tau), u(t), x(t-\tau)) \, dt \\
+ u^{\alpha+1}_\delta(x(t), x(t-\tau)),
\]

\[
f_{\alpha+1}(x(t), x(t-\tau), u(t), x(t-\tau)) \, dt \\
+ g_0(x(t), x(t-\tau)) \, d\xi(t) \\
+ u^2(x(t), x(t-\tau)) g_\beta(x(t), x(t-\tau)) \, d\xi(t) \\
+ u^{\alpha+1}_\delta(x(t), x(t-\tau)),
\]

\[
g_{\beta+1}(x(t), x(t-\tau), u(t), x(t-\tau)) \, d\xi(t).
\]

The infinitesimal generator associated to this system (17) is given by:

\[
\mathcal{L}_\delta V(x(t)) =< f_{\alpha,\delta}(x(t)), \nabla V(x(t)) > \\
+u_\alpha^2(x(t)) < f_{\alpha,\delta}(x(t)), \nabla V(x(t)) > \\
+u^{\alpha+1}_\delta(x(t)) < f_{\alpha+1}(x(t), \delta x(t), u_\delta(x(t))), \nabla V(x(t)) > \\
+\frac{1}{2} \text{Tr}\left(g_{0,\delta}(x(t)) + u^2_\delta(x(t)) g_{\beta,\delta}(x(t))
\right)
\]

\[
+u^{\beta+1}_\delta(x(t)) \nabla_{\delta x(t)} u_\delta(x(t))
\]

\[
+u^{\alpha+1}_\delta(x(t)) \nabla_{\delta x(t)} u_\delta(x(t))
\]

\[
\cdot \nabla^2 V(x(t))
\]

Using (15)(30) and the definition of \( H \) given by (12), we get

\[
\mathcal{L}_\delta V(x(t)) = L_{0,\delta} V(x(t)) + u^2_\delta(x(t)) L_{1,\delta} V(x(t)) \\
+u^{\alpha+1}_\delta(x(t)) H(x(t), x(t-\tau), u_\delta(x(t)))
\]

It is easy to check that \( u \) satisfies the following inequalities

\[
u^2(x(t), x(t-\tau)) L_{1,\delta} V(x(t)) \leq 0,
\]

\[
|u(x(t), x(t-\tau))| \leq \frac{1}{2}
\]

and \( |\psi(x(t), x(t-\tau)) H_\delta(x(t), u_\delta(x(t)))| \leq \frac{1}{2} \),

with \( u_\delta(x(t)) = u(x(t), x(t-\tau)) \).

Then

\[
\mathcal{L}_\delta V(x(t)) = L_{0,\delta} V(x(t)) \\
-\psi(x(t), x(t-\tau)) \alpha (L_{1,\delta} V(x(t)))^{\alpha+1} \\
+\psi(x(t), x(t-\tau)) \alpha^{\alpha+1} (L_{1,\delta} V(x(t)))^{\alpha+1}.
\]

By definition of \( \psi \) with (30) and (20),

\[
[\psi(x(t), x(t-\tau)) H_\delta(x(t), u_\delta(x(t)))] \geq \frac{1}{2} > 0
\]

and

\[
\mathcal{L}_\delta V(x(t)) = L_{0,\delta} V(x(t)) \\
-\psi(x(t), x(t-\tau)) \alpha (L_{1,\delta} V(x(t)))^{\alpha+1} \\
\cdot \left[1 - \psi(x(t), x(t-\tau)) H_\delta(x(t), u_\delta(x(t)))\right] \leq L_{0,\delta} V(x(t)) - \frac{\psi(x(t), x(t-\tau)) \alpha}{2} (L_{1,\delta} V(x(t)))^{\alpha+1} \leq 0.
\]

Therefore, we can deduce that the system (1)(15) is stable in probability.

By applying the stochastic version of LaSalle Theorem (cf. [14, 15]), we can deduce that the solution \( x(t) \) tend to the set asymptotically \( \mathcal{N} = \{ x \in \ker(L_\delta V) \} \) with probability one.

If \( x \) is an element of \( \mathcal{N} \) then with (22), and the fact that

\[
[1 - \psi(x(t), x(t-\tau)) H_\delta(x(t), u_\delta(x(t)))] > 0
\]

we can deduce that

\[
L_{0,\delta} V(x) = 0 \quad \text{and} \quad L_{1,\delta} V(x) = 0.
\]

From the condition (13) on \( L_{0,\delta} V(x) \), it follows that:

\[
a(x) - a_\delta(x) = 0 \quad \text{and} \quad L_{1,\delta} V(x) = 0.
\]

Therefore, \( x \) is an element of \( \mathcal{M} \). Since \( \mathcal{M} = \{0\} \), the almost sure attractivity of the origin is proved. Consequently, the closed-loop system is almost surely asymptotically stable.

This completes the proof of the Theorem 5. \( \square \)

3.2 Case : \( \beta < \alpha \) and \( \beta \) is odd

When \( \beta < \alpha \) and \( \beta \) is odd, then by proceeding as above, a result on stabilizability of system (1) can also be established.

Indeed, let us denote by \( L_{2,\delta} \) the second order differential operator defined for all \( \Xi \in C^2(R^n) \) by:

\[
L_{2,\delta} \Xi(x(t)) = \frac{1}{2} \text{Tr}\left(g_{0,\delta}(x(t)) g_{\beta,\delta}(x(t))
\right)
\]

\[
+ g_{\beta,\delta}(x(t)) g_{0,\delta}(x(t)) |\nabla^2 \Xi(x(t))|.
\]
and set
\[ K(x(t), x(t-\tau), u) = u^{\alpha-\beta-1} <f^\alpha,\delta(x(t), u), \nabla \Xi(x(t)) > + \begin{array}{c} 1  \end{array} Tr \left\{ \left[ g_0,\delta(x(t)) \right] \left[ g_{\alpha+1,\delta}(x(t), u) \right] \right\} (23) \]

It is assumed that there exists a Lyapunov function \( V \in C^2(R^n, [0, \infty)) \) and functions \( a \) and \( a_d \in C(R^n, [0, \infty)) \) such that condition (13) is satisfied.

Let \( \tilde{\mathcal{M}} = Ker(a - a_d) \cap Ker(L_{2,\delta} V) \) (24)

Then, we have the following result.

**Theorem 6.** Let
\[ u(x(t), x(t-\tau)) = -\tilde{\psi}(x(t), x(t-\tau)) L_{2,\delta} V(x(t)) \] (25)

where \( \tilde{\psi} \in C^\infty(R^n \times R^n; [0, \infty)) \) is a function satisfying:
\[ \tilde{\psi}(x(t), x(t-\tau)) \leq \frac{1}{\sup_{|u| \leq 1} |K_\delta(x(t), u)|^2 + |L_{2,\delta} V(x(t))|^2 + 2} \] (26)

and \( K_\delta(x(t), u_d(x(t))) = K(x(t), x(t-\tau), u_d(x(t))) \).

If the set \( \tilde{\mathcal{M}} \) defined by (24) is reduced to the origin, then the trivial solution of the system (1) with (25) is almost surely asymptotically stable.

**Remark 7.** The proof of Theorem 6 is analogous to the proof of Theorem 5. Indeed, it can be established that the infinitesimal generator associated to system (17) with (25) satisfy
\[ L_\delta V(x(t)) = L_{0,\delta} V(x(t)) + u_d^2(x(t)) L_{2,\delta} V(x(t)) + u_d^{\beta+1}(x(t)) K(x(t), x(t-\tau), u_d(x(t))) \] (27)

The rest of the proof is similar with \( L_1,\delta V(x(t)) \) replaced by \( L_{2,\delta} V(x(t)) \) and \( H(x(t), x(t-\tau), u_d(x(t))) \) by \( K(x(t), x(t-\tau), u_d(x(t))) \).

3.3 Case : \( \alpha = \beta \) and \( \alpha, \beta \) are odd

If \( \alpha = \beta \) and \( \alpha, \beta \) are odd, then by introducing \( L_{3,\delta} \) the second order differential operator defined for all \( \Xi \in C^2(R^n) \) by :
\[ L_{3,\delta} \Xi(x(t)) = L_{1,\delta} \Xi(x(t)) + L_{2,\delta} \Xi(x(t)) \]

and setting
\[ J(x(t), x(t-\tau), u) = \begin{cases} f^a_{\alpha+1,\delta}(x(t), u), \nabla \Xi(x(t)) > \\ \frac{1}{2} Tr \left\{ \left[ g_0,\delta(x(t)) \right] \left[ g_{\alpha+1,\delta}(x(t), u) \right] \right\} \end{cases} (28) \]

we can derive the following result.

**Theorem 8.** Suppose there exists a Lyapunov function \( V \in C^2(R^n, [0, \infty)) \) such that condition (13) is satisfied.

If the set \( S = Ker(a - a_d) \cap Ker(L_{2,\delta} V) \) is reduced to \( \{0\} \), then the system (17) is almost surely stabilizable by means of the feedback law:
\[ u(x(t), x(t-\tau)) = -\tilde{\psi}(x(t), x(t-\tau)) L_{2,\delta} V(x(t)) \] (29)

where \( \tilde{\psi} \in C^\infty(R^n \times R^n; [0, \infty)) \) is such that
\[ \tilde{\psi}(x(t), x(t-\tau)) \leq \frac{1}{\sup_{|\xi| \leq 1} |J_\delta(x(t), u_d(x(t)))|^2 + |L_{2,\delta} V(x(t))|^2 + 2} \] (30)

and \( J_\delta(x(t), u_d(x(t))) = J(x(t), x(t-\tau), u_d(x(t))) \).

**Remark 9.** The proof is analogous to the previous one.

In Theorem 5 (respectively Theorem 6) there is no assumption on the parity of \( \beta \) (respectively \( \alpha \)). In fact, we suppose in both cases, that the smallest order, \( min(\alpha, \beta) \), is odd. In the other case, when the smallest order is even, the previous results cannot be applied directly. This case is out of the scope of this paper and will be the subject of further investigations. When both \( \alpha \) and \( \beta \) are odd, we can apply Theorem 5 or Theorem 6. But the result proposed in Theorem 8 has the advantage to use the richness of the decomposition (8) and (9) of the nonlinearities \( f \) and \( g \) in the structure of the control law (29).

In order to illustrate the main results developed in this section, an example is proposed.

**Example**

Consider the following example
\[ \begin{cases} dx(t) = [-2x^3(t) + u(t)] dt \\ + [2x^2(t-\tau) + x(t)] du \end{cases} \] (31)

where \( \mu \in N, \mu \geq 1 \).

We consider again the Lyapunov function \( V(x) = x^2 \).

Differentiating \( V \) in the sense of Itô, leads to
\[ L_\delta V(x(t)) = \begin{cases} -4x^4(t) + 4x^4(t-\tau) \\ + u^3(2x^{\mu+1}(t) + 4x(t)x^2(t-\tau)) \end{cases} \] (32)
Here
\[ L_{0,\delta}V(x(t)) = -4x^4(t) + 4x^4(t-\tau) \]
\[ L_{3,\delta}V(x(t)) = 2x^{\mu+1}(t) + 4x(t)x^2(t-\tau). \]
and
\[ J(x(t), x(t-\tau), u) = \cdots \]

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is almost surely asymptotically stable if the set \( S = \{0\} \).

If \( \mu \) is odd then we cannot deduce directly the attractivity of the origin.

If \( \mu \geq 2 \) and \( \mu \) is even, then \( S = \{0\} \) and we can deduce that the closed-loop system (31)(33) is almost surely asymptotically stable.

4. CONCLUSIONS

In this paper, we have considered the problem of feedback stabilization of a class of nonlinear stochastic time-delayed systems. The case of systems with discrete time delay has been considered. The Stochastic version of the Invariance Principle of LaSalle for stochastic differential delayed systems has been used in order to tackle the feedback stabilization problem. In addition, sufficient conditions for guaranteeing the asymptotic stability of the closed-loop system have been obtained. Stabilizing state feedback control laws have been derived from these conditions. The case of systems with multiple delays will be the subject of further studies.

REFERENCES


