MPC oriented experiment design

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Abstract: In this contribution we outline an experiment procedure tailored for Model Predictive Control (MPC). The design criterion takes the MPC criterion into account explicitly. The Scenario Approach is used to handle the fact that there is no explicit expression for the MPC criterion nor to the performance degradation due to the use of an estimated model (due to the constraints). The approach is illustrated on a railcar example.

Keywords: System identification, prediction error methods, MIMO, identification algorithms, control applications.

1. INTRODUCTION

Applications oriented system identification is a research topic that has attracted significant interest since the birth of the field of system identification in the mid 1960s. In particular there was a surge of interest in conjunction with the developments in robust control between 1985-1995, giving rise to the area of “identification for control” (Gevers (1991); Schrama (1992); Goodwin et al. (1992); Gevers (1993); Bayard et al. (1992); Jacobsen (1994); Zang et al. (1995); Van den Hof and Schrama (1995); de Ćallafon and Van den Hof (1997); Böling and Måkiliä (1998); Goodwin (1999); Rivera and Jun (2000); Malan et al. (2001); Eker and Nikolaou (2002). A key outcome of this effort was the recognition of the importance of experiment design which lead to iterative approaches trying to achieve experimental conditions such that the bias error was appropriately distributed over frequencies to suit control applications. A consequence of these developments was that results on computational methods for optimal experiment design were revisited and extended (Lindqvist and Hjalmarsson, 2000, 2001; Hildebrand and Gevers, 2003; Jansson and Hjalmarsson, 2004, 2005).

In Barenthin et al. (2005) it is illustrated that optimal input design may result in significant savings in experimental efforts in control applications. Through some simple examples, it was advocated in Hjalmarsson (2005) that it is possible to combat the curse of complexity, i.e. that the model uncertainty grows with the system complexity so that for highly complex systems the model becomes virtually useless, by careful experiment design and that this also allows simple models to be used (as long as only a limited amount of system properties are to be extracted from the measurements). Following up on Hjalmarsson (2005), the dual role of a “good” input as 1) an enhancer of system properties of interest, and 2) as an attenuator of properties of little or no interest was formalized in Hjalmarsson et al. (2006) and further developed in Mårtenson (2007). In particular it was shown that, under certain conditions, an input that is designed to be optimal for a scalar cost function and for a full order model, results in experimental data for which also reduced order models can be used to consistently identify the property of interest.

In Hjalmarsson (2009a,b) it is argued that the main resource to cope with system complexity is the experiment design and a general framework for applications oriented experiment design is outlined.

In summary, applications oriented system identification has turned out to be a very complex issue with many facets and there is to date no entirely satisfying general methodology even though many of the important characteristics of the problem are well recognized.

Since its introduction, model predictive control, MPC has grown very popular and become widely used in industry. In fact, most petrochemical plants and refineries have implemented MPC (Zhu, 2006). Because MPC relies on good process models, system identification for MPC has become an important issue. In this contribution we adapt the approach in Hjalmarsson (2009a,b) to experiment design for MPC. In Section 2 we outline the basic ideas of the applications oriented experiment design presented in Hjalmarsson (2009a,b) and the specifics for the MPC case are outlined in Section 3. The method relies on a performance degradation cost which for the MPC case lacks a closed form solution (due to finite horizon and the explicit consideration of constraints). Therefore, the Scenario Approach, presented in Section 4, is used to estimate the performance degradation. Section 5 presents the experiment design procedure for MPC which is then illustrated by an example in Section 6.

2. BASIC IDEAS

The quality of a model will directly influence the performance of an application where the model is being used. We assume that if an exact mathematical model of the true system, $S_0$, were available for the design of the application, the desired performance would be obtained. However, when the used model does not correspond to the true system, the performance of the application will degrade. We measure this in terms of a performance degradation “cost” $\text{V}_{\text{app}}(M)$ which has global minimum $\text{V}_{\text{app}}(M) = 0$ at $M = S_0$, c.f. Gevers and Ljung (1986); Forssell and Ljung.
For example, in MPC the performance degradation can be measured by
\[ V_{\text{app}}(M) := \sum_{t=1}^{N} \| y_{\text{ideal}}(t) - y_M(t) \|^2, \]
where \( y_{\text{ideal}} \) is the output response when the true system model is used in the MPC and where \( y_M \) is the response when model \( M \) is used instead.

A model is then deemed to be acceptable if its performance degradation is sufficiently small. This leads to a set of acceptable models
\[ \mathcal{E}_{\text{app}} := \left\{ M : V_{\text{app}}(M) \leq \frac{1}{\gamma} \right\}. \]
Here \( \gamma \) is a user specified constant that determines the required accuracy. We will call this constant, the desired accuracy since increasing \( \gamma \) leads to tighter specifications (and hence a smaller set \( \mathcal{E}_{\text{app}} \)).

This leads to the simple idea that the objective of applications oriented system identification is to produce a model which belongs to \( \mathcal{E}_{\text{app}} \).

In this paper a stochastic framework is adopted where disturbances are assumed to be random variables. Resulting parameter estimates (which correspond to models) will then also be random variables, typically with unbounded support, and as a consequence it is not possible to provide 100% guarantee that the resulting model belongs to \( \mathcal{E}_{\text{app}} \). Instead, this has to be relaxed to a certain level of probability, e.g. 99%.

In parameter identification, the model parameters, say \( \theta \), correspond to the model \( M(\theta) \) and the objective is then that the parameter estimate, \( \hat{\theta} \), say, belongs to the set of model parameters that corresponds to \( \mathcal{E}_{\text{app}} \) with a certain (high) level of probability \( p \), i.e.
\[ \text{Probability} \left( \left\{ \hat{\theta} : M(\hat{\theta}) \in \mathcal{E}_{\text{app}} \right\} \right) \geq p. \tag{1} \]
Condition 1 is in general very difficult to impose, therefore it is usually replaced by (Hjalmarsson, 2009a,b)
\[ M(\theta) \in \mathcal{E}_{\text{app}}, \text{ for all } \theta \in U \]
where \( U \) is a given confidence ellipsoid of \( \hat{\theta} \).

### 2.1 Applications oriented experiment design

One thus has to ensure that the experiment design is such that (1) holds. A natural objective is to try to minimize the experimental resources required to ensure this. The experiment design problem in applications oriented parametric system identification can be formulated as follows:
\[
\begin{align*}
\min & \text{Experimental Resources} \\
\text{s.t.} & M(\theta) \in \mathcal{E}_{\text{app}}, \text{ for all } \theta \in U \\
\end{align*}
\tag{2}
\]
Examples of resources are mean experimental time or used energy. There are two key issues associated with solving this problem in practice:

i) The set of acceptable models \( \mathcal{E}_{\text{app}} \) has to be determined.

ii) It must be, at least, numerically possible to solve problem (2).

### 3. MPC ORIENTED EXPERIMENT DESIGN

In regards to the issues in the preceding section, methods to compute \( \mathcal{E}_{\text{app}} \) for MPC are detailed in Wahlberg et al. (2010). In the next section we describe an alternative method based on the scenario approach. A hampering factor is that the system parameters need to be known. An adaptive method to cope with this is presented in Gerencsér et al. (2009) for ARX systems but this method still has to be adapted for MPC. For ii) in Section 2.1, we are relying on results in Jansson and Hjalmarsson (2005) where it is shown that optimal experiment design problems can be solved by convex optimization when the system is linear and time-invariant if the input spectrum is linearly parameterized.

### 4. THE SCENARIO APPROACH

In this section we briefly describe the Scenario Approach for solving a robust convex problem, or, more generally, a semi-infinite convex optimization problem, i.e., one where the number of decision variables is finite, but the number of constraints is infinite.

Consider the following general Robust Convex Program:
\[
\begin{align*}
\min_{\gamma \in \mathbb{R}^d} & \quad c^T \gamma \\
\text{s.t.} & \quad f_\delta(\gamma) \leq 0, \quad \delta \in \Delta.
\end{align*}
\]
where \( f : \mathbb{R}^d \rightarrow \mathbb{R} \) is convex for every \( \delta \in \Delta \).

In a robust optimization framework, the set \( \Delta \) represents the uncertainty in the parameter \( \delta \), since it corresponds to the collection of all admissible values that such parameter may take. The role of the optimal solution of \( \text{RCP} \) is, therefore, to minimize a certain quantity, \( c^T \gamma \), while satisfying a given set of constraints under all possible situations (i.e., values of \( \delta \)).

The description of \( \text{RCP} \) involves the satisfaction of an infinite number of constraints, i.e., one per each value of \( \delta \in \Delta \). This corresponds to a convex optimization problem which, except for some particular cases, is in general computationally intractable (Ben-Tal and Nemirovski, 1998).

The scenario approach (Calafiore and Campi, 2006) provides a tractable relaxation of \( \text{RCP} \), which consists in selecting a small number of these constraints to include in the optimization problem. To do this, the scenario approach presumes a probabilistic description of the uncertainty, in other words, a probability distribution \( P_r \) over \( \Delta \). The method then extracts, at random, \( N \) instances or ‘scenarios’ of the uncertainty parameter \( \delta \) according to the probability \( P_r \), and it considers only the corresponding constraints in the scenario optimization problem.

The scenario-based approximation is formally described next.

### Scenario-Based Optimisation (Calafiore and Campi, 2006)

Extract \( N \) independent identically distributed samples \( \delta^{(1)}, \ldots, \delta^{(N)} \in \Delta \), according to the probability \( P_r \) and solve the scenario convex program:
\[
\begin{align*}
\min_{\gamma \in \mathbb{R}^d} & \quad c^T \gamma \\
\text{s.t.} & \quad f_{\delta^{(i)}}(\gamma) \leq 0, \quad i = 1, \ldots, N.
\end{align*}
\]
The resulting program, \( \text{SCP}_N \), is a standard finite dimensional convex optimization problem with a finite number of constraints. The computational cost of \( \text{SCP}_N \) can be quite reasonable, provided \( N \) is not large.

Since the scenario approach involves a random selection of constraints, a remaining question is how the solution of
SCP\textsubscript{N} relates to that of RCP. The answer can be posed in the following probabilistic sense (Calafiore and Campi, 2006):

The scenario-based optimization program SCP\textsubscript{N} provides a solution \( \gamma^{\text{opt}} \) which, with high probability, say 1 − \( \beta \), satisfies all the constraints in \( \Delta \), except for a fraction with a small probability, say \( \epsilon \) (with respect to the probability measure \( P_\gamma \)).

Here \( \beta \) is denoted as the ‘confidence parameter’ and \( \epsilon \) is the ‘violation parameter’. These variables are user choices which determine the minimum number of scenarios \( N \) to be randomly selected.

The minimum number of scenarios \( N \) required for SCP\textsubscript{N} to give a reasonable approximation to the solution of RCP in the sense stated above (in terms of the probabilities \( \beta \) and \( \epsilon \)) has been studied in several publications (see e.g. Alano et al. (2007, 2008); Calafiore and Campi (2006); Campi and Garatti (2007)). To date, the tightest bound on the minimum \( N \) has been established in (Campi and Garatti, 2007), according to which \( N \) has to satisfy

\[
\sum_{i=0}^{d} \frac{N}{i} \epsilon^i (1 - \epsilon)^{N-i} \leq \beta.
\]

5. OPTIMAL INPUT DESIGN FOR MPC

We will consider input design for prediction error identification of models of the type

\[
x(t + 1) = A(\theta)x(t) + Bu(t) \\
y(t) = x(t) + e(t),
\]

where the \( A \)-matrix is parameterized by \( \theta \), \( u(t) \) is the input used for identification and \( e(t) \) is white noise with covariance matrix \( \Sigma_0 \). We also assume that there is a \( \theta_0 \) such that the model corresponds to the true system. The parameter estimates are then asymptotically, in the number of samples \( N \), Gaussian with covariance matrix given by (Barentin et al., 2008)

\[
P = \left[ \frac{1}{2\pi} \int_{-\pi}^{\pi} \left( \frac{\partial \text{vec} G(\theta, e^{j\omega})}{\partial \theta} \right)^T \Phi_u(\omega) \otimes \Sigma_0^{-1} \left( \frac{\partial \text{vec} G(\theta, e^{j\omega})}{\partial \theta} \right) d\omega \right]^{-1}
\]

\[
G(\theta, e^{j\omega}) = (e^{j\omega} I - A(\theta))^{-1} B
\]
evaluated at the true parameter \( \theta_0 \). \( P^{-1} \) is a linear function of the input spectral density \( \Phi_u(\omega) \). Therefore, a linear parameterization of \( \Phi_u(\omega) \) gives a convex optimization problem. Two suggested choices are Partial correlation parameterization and Finite dimensional spectrum parameterization (Jansson and Hjalmarsson, 2005).

The particular input design problem will be to minimize the total input power while guaranteeing, with a (high) probability \( \alpha \), that the parameter estimates result in a degradation of the application cost that is considered acceptable. We formulate the problem as

\[
\min_{\Phi_u(\omega)} \text{tr} \int_{-\pi}^{\pi} \Phi_u(\omega) d\omega
\]

s.t. \( (\theta_0 - \theta)^T N \frac{1}{\kappa} P^{-1} (\theta_0 - \theta) \geq \gamma V_{\text{app}}(\theta) \) \( \forall \theta \)

\[
\Phi_u \geq 0, \quad \forall \omega,
\]

where \( \kappa \) depends on \( \alpha \) and is determined from the \( \chi^2 \) distribution. The last constraint ensures that \( \Phi_u(\omega) \) corresponds to an actual spectrum. This can be reformulated as an LMI, e.g. by the KYP lemma for the finite dimensional parametrization of the spectrum and by a Toeplitz condition for the partial correlation parameterization (Jansson and Hjalmarsson, 2005).

The remaining issue is to choose \( V_{\text{app}}(\theta) \) and to characterize the set of acceptable models given by

\[
V_{\text{app}}(\theta) = \frac{1}{N} \sum_{t=1}^{N} \|y(t, \theta_0) - y(t, \hat{\theta})\|_2
\]

with minimum at \( \hat{\theta} = \theta_0 \).

We will employ the scenario approach to approximate the set of parameters that give models that belong to \( \mathcal{E}_{\text{app}} \). Let

\[
\Theta = \left\{ \theta : V_{\text{app}}(\theta) \leq \frac{1}{\gamma} \right\}
\]

This set is centered around the true parameter value \( \theta_0 \) and typically contains infinitely many parameter values for which the application degradation is sufficiently small. To get a computationally tractable optimization problem, random samples of \( \theta \) are taken from \( \Theta \). If the degradation is sampled at sufficiently many points an acceptable approximation of \( \mathcal{E}_{\text{app}} \) is obtained. The number of samples needed depends on the number of parameters and the required level of approximation and it should satisfy (4).

This means that we can now formulate (5) as

\[
\min_{\Phi_u(\omega)} \text{tr} \int_{-\pi}^{\pi} \Phi_u(\omega) d\omega
\]

s.t. \( (\theta_0 - \theta_k)^T N \frac{1}{\kappa} P^{-1} (\theta_0 - \theta_k) \geq \gamma V_{\text{app}}(\theta_k), \) \( k = 1, \ldots, N_k \)

\[
\Phi_u \geq 0, \quad \forall \omega,
\]

where \( \theta_k \in \Theta \) are samples taken from, say, a uniform distribution on \( \Theta \).

One advantage of MPC is the ability to easily handle constrained input and output signals in the controller. It will be expected that different constraints on the input and output will affect the application degradation. Because the set of acceptable models will be inside the application degradation set, the constraints can also be expected to influence the optimal input design. Even though the proposed method does not directly include these constraints in the actual input design, the constraints only influence indirectly through \( V_{\text{app}}(\theta) \), some techniques are available to include them at a later stage, when generating the input signals in the time domain, see, e.g. Rojas et al. (2007).

6. A RAILCAR EXAMPLE

In this section we illustrate the proposed method with a numerical example of input design for identification of a model of a railcar that is to be used for temperature control. The example is introduced in Barcelli and Bemporad (2009).
Fig. 1. Layout of the railcar as presented in Barcelli and Bemporad (2009).

6.1 The problem

Consider the railcar presented in Figure 1. The railcar is divided into passenger areas with seats and a table, a corridor and fore and aft antechambers. Each section has a temperature sensor and the passenger areas all have their own heater and air conditioner. The goal of the MPC is to adjust the temperature of each passenger area to its own temperature reference.

We model the interactions in the railcar using the heat conduction equation, and the change in temperature in a section as the sum of the heat flow from other sections plus the heat flow from heaters/air conditioning. This gives

\[
\frac{dT_i}{dt} = \sum_j Q_{ij}(t) + Q_{ui}
\]

\[
Q_{ij}(t) = \theta_{ij}(T_i(t) - T_j(t))
\]

where \(T_i(t)\) is the temperature in section \(i\), \(Q_{ij}\) is the heat flow between sections \(i\) and \(j\) and \(Q_{ui}\) is the heat flow from heater \(i\). The parameter \(\theta_{ij}\) relates to the thermal conductivity, heat capacity and the geometry of the sections. All sections are chosen such that \(\theta_{ij} = \theta_{ji}\) which in total given seven different parameters to be identified. These parameters are presented in Table 1. The sampling period is chosen to be faster than the dynamics of the system and the model is discretized using the Euler method. The final model of the railcar becomes

\[
T(t + 1) = A(\theta)T(t) + Bu(t)
\]

\[
y(t) = T(t) + e(t),
\]

where \(e(t)\) is white Gaussian noise with covariance matrix \(\Sigma_0\). In the MPC implementation, the measurements are assumed to be noise free and the MPC control problem is

\[
V(x(t)) = \min \bigg\{ x^T(t + M)P x(t + M) + \sum_{k=0}^{M-1} x^T(k)Q x(k) + u^T(k)Ru(k) \bigg\}
\]

s.t. \(x(k+1) = Ax(k) + Bu(k)\), \(k = 0, \ldots, M - 1\)

\[
y(k) = C x(k), \quad k = 0, \ldots, M - 1
\]

\[
u_{\min} \leq u(k) \leq u_{\max}.
\]

For more details on the MPC aspects of the example we refer to Barcelli and Bemporad (2009).

From the layout of the railcar it is clear that the parameters \(\theta_6\) and \(\theta_7\) will enter in the model in the same way. This means that there will be identifiability problems. However, it is only the sum of the parameters that matter for the control problem; for instance it will not matter for our model if the insulation in the roof is double that of the insulation in the floor, since the total heat flow through roof and floor will be the same. Therefore, \(V_{\text{app}}(\theta)\) will be constant along the sum of \(\theta_6\) and \(\theta_7\). This in turn means that for the input design problem the loss of identifiability is not an issue. An illustration of this is shown in Figure 2.

The goal of the identification is to find parameter estimates \(\hat{\theta}\) such that the identified model belongs to \(\mathcal{E}_{\text{app}}\) with probability \(\alpha\). We use (5) as the application degradation cost and choose

\[
\gamma = \frac{100}{\sum_{k=0}^{N} ||y(\theta_0)||^2}
\]

which gives a maximal acceptable degradation of the MPC criterion of 1% compared to when \(\theta_0\) is used.

6.2 Input design

The discussed input design problem for the railcar example becomes

\[
\min_{\Phi_\omega} \text{tr} \int_{-\pi}^{\pi} \Phi_\omega d\omega
\]

s.t. \([\theta_0 - \theta_k]^T \frac{N}{P-1} [\theta_0 - \theta_k] \geq \gamma V_{\text{app}}(\theta_k), \quad k = 1, \ldots, N_k\]

\[
\Phi_\omega(\omega) \geq 0, \quad \forall \omega
\]

where \(\Theta\) is the set of parameters such that the application degradation is sufficiently small. The remaining issue is the parameterization of the input spectral density.

We choose the input as a multivariable FIR spectrum and the spectral density is parameterized as
The total input power required to achieve the identification goal for four different types of input spectra. All other parameters of the optimization are kept the same in all four experiments.

<table>
<thead>
<tr>
<th>Spectrum type</th>
<th>White</th>
<th>FIR, M = 0</th>
<th>FIR, M = 1</th>
<th>FIR, M = 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>225</td>
<td>190</td>
<td>114</td>
<td>102</td>
</tr>
</tbody>
</table>

\[ \Phi_u(\omega) = \sum_{k=-M}^{M} C_k e^{i\omega k} \]

where \( C_k \) are square matrices with as many rows as inputs. This means that the last constraint can be handled using the KYP lemma as described in Jansson (2004). With this parameterization the optimization problem becomes

\[
\min_{C_k} \text{tr} C_1 \\
\text{s.t. } [\theta_0 - \theta_k]^T \frac{N}{\kappa} P^{-1} [\theta_0 - \theta_k] \geq \gamma V_{\text{app}}(\theta_k), \\
k = 1, \ldots, N_k \\
\sum_{k=-M}^{M} C_k e^{i\omega k} \geq 0, \ \forall \omega
\]

6.3 Numerical Illustration

We now study input design for a railcar with a total of 8 different seating areas, 4 corridor sections and forward and aft ante chambers, both divided into 3 sections each. This could also correspond to the case where seating areas in the original train are grouped two and two. Temperature sensors measure the temperatures in all sections, and the goal of the MPC is to control the temperature in the 8 seating areas. In total the model has 18 states, 8 inputs and 18 outputs. The total number of samples is \( N = 1000 \), and \( \alpha = 0.95 \) which corresponds to a 95% confidence ellipsoid.

We consider parameterizations of the input spectrum with \( M = 0, 1, 2, N = 0 \) corresponds to an input that is temporally white but possibly spatially colored, while \( M > 0 \) allow for temporal coloring. The largest possible \( M \) in the current implementation is, for this model size, 2 due to memory requirements. In Figure 3 the resulting designs are shown. Because of the symmetric geometry of the railcar the input spectrum is also symmetric, hence only the upper left \( 4 \times 4 \) block of the spectrum is shown. It is clear that there is spatial coloring in the optimal input. The problem is implemented in Matlab using CVX (Grant and Boyd, 2011) and solved with ScDuMi.

What can we see is that there is for all designs more power in the inputs corresponding to the seating areas that are next to ante chambers. This makes sense since there are internal walls between the seating areas and ante chambers with heat transfer that need to be identified. Since there is no actuation in the ante chambers more power is needed in the adjoining seating areas.

For comparison we have also found the both temporally and spatially white input that gives the required application degradation. The resulting input power is found to be much higher that what is required with the optimal inputs. Table 2 shows the different optimal values for the four designs. It is also clear that when we allow for temporal coloring in the input, it is possible to reduce to total power even further.

7. CONCLUSIONS

MPC has rapidly become a very popular control strategy widely used in industry. Therefore, tools for finding good process models to be used in MPC is now an important issue. In this contribution we present a strategy for experiment design for identification when the goal is to use the model in MPC applications. Ideas from applications oriented system identification have been tailored for the MPC case.

The lack of explicit solutions to the MPC criterion and the performance degradation cost, due to constraints and finite horizons, an approximation scheme based on the Scenario Approach is used. This results in a finite dimensional convex optimization problem.

The proposed experiment design strategy is used on a numerical example of a railcar where temperature control is implemented using MPC. Optimal inputs for the identification experiments are designed and show that one can expect the required input power in the optimal designs to be lower than for a white input, if the same accuracy is wanted.

The proposed method has two major shortcomings. The first is the fact that the input and output constraints are not directly included in the input design. However, as pointed out, there are methods to include them later in the signal generation phase. The second is the memory requirements for the optimization problem. The size of the problem grows quite fast with the size of the process model. Exploiting the structure of the model, in this case the matrices are quite sparse, should be a way to reduce this problem.

REFERENCES


Fig. 3. The upper left \( 4 \times 4 \) block of the optimal input spectrum for identification of the railcar model parameters with different spectrum parameterization, \( M = 0 \) (solid), \( M = 1 \) (dashed), \( M = 2 \) (dotted).

Table 2. The total input power required to achieve the identification goal for four different types of input spectra. All other parameters of the optimization are kept the same in all four experiments.


