Elimination of latent variables in behavioral systems: a geometric approach

Mark Mutsaers and Siep Weiland

Abstract: This paper considers computational aspects of the problem of elimination of variables. More precisely, the problem of elimination of latent variables in models is addressed in the behavioral framework. In earlier contributions on this theme, the model classes of infinitely smooth $C^\infty$ and square integrable $L_2$ linear time-invariant systems have been considered. For both system classes, conditions for elimination of latent variables have been derived. However, efficient computational scheme to eliminate distinguished variables from equations have received less attention in the literature. This paper addresses sufficient conditions for the elimination problem where algebraic state space operations are used. We provide a constructive and computable algorithm for the elimination of latent variables.

1. INTRODUCTION

The problem of elimination of variables in dynamical systems is an important one within the general field of control and systems theory. Usually, first principle models are derived by introducing auxiliary (or latent) variables that describe physical phenomena and physical conservation laws at a smaller (micro-) scale than the (macro-) scale for which the model is actually meant for. Such a modeling strategy typically results in a formal distinction between latent variables, i.e. auxiliary variables that are introduced to express physical relationships and invariants on a local scale, and manifest variables, the variables that are actually of interest to the modeler. The quest to find explicit relations between manifest variables only is of evident importance for various reasons: it simplifies analysis, reduces redundancy, allows simulations, predictions and monitoring in a more straightforward manner and allows more compact representations of the model. In addition, elimination of variables is of crucial importance in algorithms for controller synthesis, as has been shown in e.g. Trentelman et al. [2007] and Mutsaers and Weiland [2011].

The problem of elimination of latent variables is most naturally studied in the behavioral framework. See, e.g., Polderman [1997] and Polderman and Willems [1998] for infinitely smooth behaviors, and Mutsaers and Weiland [2010, 2011] for behaviors consisting of square integrable trajectories. In the latter work, necessary and sufficient conditions for eliminability of latent variables have been derived, together with an algorithm that synthesizes the induced system, using rational operators.

In this paper, we continue this line of work and present a geometric approach for the problem of eliminating latent variables in systems represented in state space form. A sufficient condition for eliminability in terms of controlled and conditioned invariant subspaces will be presented. This result is inspired by recent work in Mutsaers and Weiland [2010], where elimination problems have been considered in the setting of geometric control theory. An advantage of this approach is that it allows to view the elimination problem from a different system theoretic perspective, and that it results in explicit algorithms using state space matrices. This allows, at least in principle, to eliminate variables from more complex linear systems.

The paper is organized as follows. In Section 2, the problem is formulated. Algorithms for elimination obtained in Polderman and Willems [1998] and Mutsaers and Weiland [2011] are reviewed in Section 3 and necessary information of the geometric control theory is given. The main result is given in Section 4, which resulted in an algorithm in Section 5. In Section 6, the results are applied on an example of an active suspension system. Conclusions are drawn in Section 7.

2. PROBLEM FORMULATION

As in the general behavioral framework, systems are described by triples $\Sigma = (\mathbb{T}, \mathcal{W}, B)$. Here, $\mathbb{T}$ defines the time or the frequency domain, $\mathcal{W}$ is the signal space, and the system behavior is a subset

$$B \subseteq \mathcal{W}^T,$$

which consists of a collection of trajectories $\omega$ that evolve over $\mathbb{T}$. Latent variable systems are given by $\Sigma_\ell = (\mathbb{T}, \mathcal{W} \times \mathcal{L}, B_\ell)$, where the signal space is a Cartesian product $\mathcal{W} \times \mathcal{L}$ and where system trajectories consist of pairs $(\omega, \ell)$ with $\omega(t) \in \mathcal{W}$ and $\ell(t) \in \mathcal{L}$. Its behavior is given as:

$$B_\ell \subseteq (\mathcal{W} \times \mathcal{L})^T.$$

Here, $\omega$ is called the manifest variable and $\ell$ is the latent variable that needs to be eliminated.

* This work is supported by the Dutch Technologiestichting STW under project number EMR.7851.
Every latent variable system $\Sigma = (T, W \times L, B)$ induces a dynamical system $\Sigma_{\text{ind}} = (T, W, B_{\text{ind}})$, whose behavior

$$B_{\text{ind}} = \{ w \mid \exists \ell \text{ such that } (w, \ell) \in B \ell \}. \quad (1)$$

That is, the induced behavior of a latent variable system is simply the projection of its behavior onto the manifest variables. System classes are collections of dynamical systems and denoted by $M^w$, where $w$ is the dimension of the signal space $W$.

The elimination problem consists of the question when latent variables in a latent variable system $\Sigma \in M^{w+\ell}$ can be completely eliminated from the system in the sense that the induced system $\Sigma_{\text{ind}}$ belongs to the same model class $M^w$. That is, we address the question under what condition on the model class $M$ the implication

$$\Sigma_{\ell} \in M^{w+\ell} \implies \Sigma_{\text{ind}} \in M^w \quad (2)$$

hold.

**Definition 1.** A latent variable system $\Sigma_{\ell}$ is called $\ell$-eliminable in the model class $M$ if (2) holds.

In this paper, we consider three model classes namely, $M_1$ as the class of $C^{\infty}$ systems with polynomial kernel representations, $M_2$ as the class of $L_2$ systems with rational kernel representations and, thirdly, $M_3$ the class of LTI systems in state space form. Specifically,

1. $M_1$ is defined by the class of $C^{\infty}$ smooth systems $\Sigma_{\ell} = (\mathbb{R}, \mathbb{R}^{w+\ell}, B_{\ell})$ that allow a representation as kernels of polynomial differential operator in the sense that:

$$B_{\ell} = \{ (w, \ell) \in C^{\infty} \mid R_1 \left( \frac{d}{dt} \right) R_2 \left( \frac{d}{dt} \right) = 0 \}, \quad (3)$$

where $R_1(\xi) \in \mathbb{R}^{p \times w}[\xi]$ and $R_2(\xi) \in \mathbb{R}^{q \times \ell}[\xi]$ are polynomial matrices. Note that trajectories are infinity often differentiable on the time domain $T = \mathbb{R}$. This class of systems has been extensively studied in e.g. Polderman and Willems [1998].

2. $M_2$ is the class of $L_2$ systems $\Sigma_{\ell} = (C, C^{w+\ell}, B_{\ell})$ whose behavior

$$B_{\ell} = \{ (w, \ell) \in L_2 \mid P_1(s) P_2(s)) = 0 \}. \quad (4)$$

Here $P_1$ and $P_2$ are stable rational operators in $\mathcal{RH}_{\infty}^+$, that define multiplicative mappings $P_1 : L_2 \to \mathcal{L}_2$ according to $(P_1 w)(s) = P_1(s)w(s)$. This means that trajectories $w$ and $\ell$ are square integrable on the imaginary axis. Equivalently, this model class can be viewed as the class of square integrable functions on time that belong to a kernel of a suitably defined convolution operator of a LTI system. See Mutsaers and Weiland [2010].

3. $M_3$ is the class of (output nulling) state space systems $\Sigma_{\ell} = (\mathbb{R}, \mathbb{R}^{w+\ell}, B_{\ell})$ of the form

$$B_{\ell} = \{ (w, \ell) \in L^{\text{loc}}_1 \mid \exists x \in L^{\text{loc}}_1 \text{ such that } v = 0 \}
\begin{cases}
\dot{x} = Ax + B_1w + B_2\ell, \\
v = Cx + D_1w + D_2\ell.
\end{cases} \quad (5)$$

Here $A$, $B_1$, $B_2$, $C$, $D_1$ and $D_2$ are real valued matrices of appropriate dimensions. $x$ denotes the state variable, which has dimension $n$.

For the first two system classes, conditions for eliminability of latent variables have been established:

**Theorem 2.** (Elimination in the class $M_1$). Any $\Sigma_{\ell} = (\mathbb{R}, \mathbb{R}^{w+\ell}, B_{\ell}) \in M_1^{w+\ell}$, with $B_{\ell}$ as in (3), is $\ell$-eliminable in the model class $M_1$.

**Theorem 3.** (Elimination in the class $M_2$).

Given is $\Sigma_{\ell} = (C_+, C^{w+\ell}, B_{\ell}) \in M_2^{w+\ell}$, with $B_{\ell}$ as in (4). Consider the equation:

$$Q = P_1 + P_2X. \quad (6)$$

The latent variable system is $\ell$-eliminable in $M_2$ if and only if $\exists X \in L_\infty$ such that $Q \in \mathcal{RH}_{\infty}^+$ and $\text{rowrank}(Q) = \text{rowrank}(P_1) + \text{rowrank}(P_2)$.

Moreover, the corresponding behavior $B_{\text{ind}}$ of the induced system $\Sigma_{\text{ind}}$ is represented by:

$$B_{\text{ind}} = \{ w \in L_2 \mid Q(s)w(s) = 0 \}, \quad (7)$$

where $Q \in \mathcal{RH}_{\infty}^+$.

This result has been shown in Mutsaers and Weiland [2010, 2011].

The elimination results for the classes $M_1$ and $M_2$ have lead to algorithms for synthesizing the induced system using polynomial or rational operators. The algorithms for the synthesis of the induced system consist of two steps, namely:

1. decomposing the polynomial or rational kernel operator,
2. elimination of redundant constraints on the manifest variable $w$.

The two steps in the algorithms for both system classes will be discussed in the next section.

Computations with rational operators are less complex than calculations using polynomial matrices. For more complex systems, however, state space representations are favorable. Therefore, we are interested in solving the elimination problem for the system class $M_3$.

**Problem 4.** (Elimination in the class $M_3$).

Given a latent variable system $\Sigma_{\ell} = (\mathbb{R}, \mathbb{R}^{w+\ell}, B_{\ell}) \in M_3^{w+\ell}$ with behavior $B_{\ell}$ as in (5). Find conditions such that $\Sigma_{\ell}$ is $\ell$-eliminable in the model class $M_3$, and, whenever possible, find a state space realization of its induced behavior, i.e.

$$B_{\text{ind}} = \{ w \mid \exists x \in L^{\text{loc}}_1 \text{ s.t. } \dot{x} = Ax + Bw; 0 = Cx + Dw \}.$$

The main result of this paper is a sufficient condition for this problem, that will be presented in Section 4. This also resulted in an algebraic algorithm for the construction of the induced system, that is presented in Section 5.

3. **GEOMETRIC APPROACH FOR ELIMINATION**

This section deals with Problem 4. First, we briefly review the existing algorithms for the elimination of latent variables in the system classes $M_1$ and $M_2$.

3.1 Two steps in elimination algorithms

Elimination of latent variables in the model classes $M_1$ and $M_2$ is performed in two similar steps. For the first case ($M_1$), they are:
1. **Decompose polynomial matrix** $R(\xi) \in \mathbb{R}^{P \times (w+\ell)}$: Given the latent variable system (3) with $R$ full row rank. Let $U(\xi) \in \mathbb{R}^{P \times P}[\xi]$ be a unimodular matrix such that:

$$B_{\ell} = \{(w, \ell) \in C^\infty \mid U(\frac{d}{dt})[R_{11}(\frac{d}{dt}) R_{12}(\frac{d}{dt})] (\nu) = 0 \}$$

$$= \{(w, \ell) \in C^\infty \mid [R_{11}(\frac{d}{dt}) - R_{12}(\frac{d}{dt})] 0 \} \ (\nu) = 0 \}$$

$$= \{(w, \ell) \in C^\infty \mid R_{11}(\frac{d}{dt})w + R_{12}(\frac{d}{dt})\ell = 0, \ (7) \}$$

where $R_{12}$ has full row rank. Here, we used that pre-multiplication by unimodular polynomial matrices leaves the behavior invariant (Theorem 3.6.2 of Polderman and Willems [1998]).

2. **Verify redundancy of condition**: $R_{11}w + R_{12}\ell = 0$:

Obviously, by (7) every $w \in B_{\text{ind}}$ satisfies:

$$B_{\text{ind}} = \{w \in C^\infty \mid R_{21}(\frac{d}{dt})w = 0 \}.$$ 

Conversely, the full row rank condition on $R_{12}$ implies that for any $w \in C^\infty$ there exists $\ell \in C^\infty$ such that $R_{11}(\frac{d}{dt})w + R_{12}(\frac{d}{dt})\ell = 0$, i.e. $w \in C^\infty$ is not constrained by the equation $R_{11}(\frac{d}{dt})w + R_{12}(\frac{d}{dt})\ell = 0$.

1. **Decompose the rational operator** $P \in \mathbb{R}^{H^\infty}$:

The rational operator $P$ in (4) is decomposed using a unit $U \in \mathbb{U}H^\infty_+$ as a similar way as in (7) to infer that:

$$B_{\ell} = \{(w, \ell) \in \ell_2 \mid U[P_{11} P_{12}] (\nu) = 0 \}$$

$$= \{(w, \ell) \in \ell_2 \mid P_{11} w + P_{12} \ell = 0, \ P_{21} w = 0 \}.$$ 

Here, the multiplication by units $U \in \mathbb{U}H^\infty_+$ leaves the behavior invariant, as shown in Mutsaers and Weiland [2010].

2. **Verify redundancy of condition**: $P_{11}w + P_{12}\ell = 0$:

The condition (6), together with the rank condition on $Q$ stated in Theorem 3, implies that

$$\{w \in \ell_2 \mid P_{21}w = 0 \} \subset \{w \in \ell_2 \mid \exists \ell \in \ell_2 \text{ such that } P_{11}w + P_{12}\ell = 0 \}.$$ 

Consequently,

$$B_{\text{ind}} = \{w \in \ell_2 \mid P_{21}w = 0 \}.$$ 

Both steps of the algorithms are depicted in Fig. 1. In Mutsaers and Weiland [2010] it is shown that the second step of the synthesis algorithm for $\mathcal{L}_2$ systems can be reformulated in geometric terms as a disturbance decoupling problem. However, for the first step we still have to do computations using polynomial or rational operators. To obtain a complete state space solution, we consider the elimination problem in the class $M_3$ in the remainder of the paper.

3.2 A geometric condition for elimination

In this subsection, we focus on latent variables systems with behaviors represented as in (5). For simplicity of the arguments, we first consider the case where $D_1 = 0$ and $D_2 = 0$. The state space system is then described by:

$$\Sigma : \begin{cases} \dot{x} = Ax + B_1w + B_2\ell, \\ v = Cx, \end{cases} \tag{9}$$

where the state $x(t) \in X = \mathbb{R}^n$ and the manifest and latent variable are $w(t) \in \mathbb{W} = \mathbb{R}^w$ and $\ell(t) \in \mathcal{L} = \mathbb{R}^\ell$, resp. We view $v(t) \in \mathcal{V} = \mathbb{R}^p$ as an artificial output variable. This means for the class $M_3$ that $B_{\ell} = \{w, \ell) \in \ell^{p\infty} \mid \exists x \in \ell_1^{w\infty} \text{ such that (9) holds with } v = 0 \}.$

*Step 1: Decompose the state space representation:*

Reviewing the first step of the existing algorithms, we see that for the classes $M_1$ and $M_2$ the interconnection with the unimodular (or unit) $U$ results in a decomposition of the polynomial (or rational) operator $R$ (or $P$). As illustrated in Fig. 1(a), this interconnection results in a transform of the output variable $v$ in (9) according to

$$\tilde{v} = U v, \text{ where the zero-block in (7) or (8) implies that the latent variable } \ell \text{ is not influencing a specific component of the output } \tilde{v}.$$ 

For the class $M_3$, this transformation can be interpreted as a disturbance decoupling estimation problem (DDEP), which is depicted in Fig. 2 and is well known in geometric control theory. Consider the problem where we want to design an estimator $\Sigma_{\tilde{v}}$ that fulfills a similar requirement, namely that the latent variables $\ell$ are not visible on a specific component (of maximal dimension) of the artificial output $\tilde{v}$.

Fig. 2. Disturbance Decoupling Estimation Problem.
This dynamical estimator is given by:

\[ \Sigma_e : \begin{cases} \dot{\hat{x}} = A\hat{x} - L(v - C\hat{x}), \\ \tilde{v} = C\hat{x}, \end{cases} \]

where \( \hat{x} \) is the estimate of the state in \( \Sigma \), \( \tilde{v} \) is the estimation of the artificial output, \( L \) is the Luenberger observer gain. Let \( \tilde{v} \) denote the error made in estimating \( v \), as \( \tilde{v} = v - \tilde{v} \). The observer gain \( L \) is designed in such a way that the estimation error \( \tilde{v} \) is maximally independent of \( \ell \) in the sense that there exists a partitioning \( \tilde{v} = (\tilde{v}_1, \tilde{v}_2) \), with \( \tilde{v}_2 \) of maximal dimension, such that \( \tilde{v}_2 \) is independent of \( \ell \).

As in (7) and (8), we factorize the output space \( \mathcal{V} = V_1 \times V_2 \) (and the corresponding estimate \( \hat{\mathcal{V}} \) and estimation error \( \mathcal{V} \)) in the dynamics of (9) as:

\[ \begin{bmatrix} \tilde{v}_1 \\ \tilde{v}_2 \end{bmatrix} = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} x. \] (10)

The exact partitioning of \( C \) into \( C_1 \) and \( C_2 \) (resulting in the partitioning of \( v \)) will be made clear later on in this subsection.

To make \( \tilde{v}_2 \) independent of \( \ell \), we need to introduce the notion of conditioned invariant subspaces. We recall that a subspace \( S \subset \mathcal{X} \) is called \((C, A)\) invariant (or conditioned invariant) if there exists \( B : \mathcal{V} \to \mathcal{X} \) such that \( (A + LC)S \subset S \). One can show that \( S \) is \((C, A)\) invariant if and only if \( A(S \cap \ker C) \subset S \). It is easy to see that \((C, A)\) invariant subspaces are closed under intersections, which means that there exists a smallest \((C, A)\) invariant subspace that contains a subspace \( Z \subset \mathcal{X} \). We denote this subspace by \( S^*_C(A)(Z) \). Hence, \( S^*_C(A)(Z) \) satisfies \( Z \subset S^*_C(A)(Z) \), \((A + LC)S^*_C(A)(Z) \subset S^*_C(A)(Z) \) for some \( L \), and there does not exist a subspace \( S \) of smaller dimension than \( S^*_C(A)(Z) \) with these two properties.

Using this, we can apply the result for DDEP on this specific problem (see Schumacher [1980]):

**Lemma 5.** Given the system in (9) and the partitioning in (10), then, \( \exists L : \mathcal{V} \to \mathcal{X} \) that makes the transfer from \( \ell \) to \( \tilde{v}_2 \) zero if and only if

\[ S^*_C(A)(\text{im} B_2) \subset \ker C_2. \] (11)

For the computation of \( S^*_C(A)(\text{im} B_2) \), we do not need the partitioning of \( v \) in (10). However, to check the condition for DDEP, we need to define this partitioning. To create the largest zero block, as in (7) or (8), and also to make the second step for eliminating the latent variable \( \ell \) easier, consider the following. Let \( \{r_1, \ldots, r_q, \ldots, r_p\} \) be a basis of \( \mathcal{V} \) in such a way that \( \text{span}\{r_1, \ldots, r_q\} = C_1 S^*_C(A)(\text{im} B_2) \), where \( q = \dim C S^*_C(A)(\text{im} B_2) \). Define \( R_1 = [r_1, \ldots, r_q] \) and \( R_2 = [r_{q+1}, \ldots, r_p] \) and factorize the variable \( v \in \mathcal{V} \) with respect to this basis according to

\[ v = \begin{bmatrix} R_1 \\ R_2 \end{bmatrix} \begin{bmatrix} \tilde{v}_1 \\ \tilde{v}_2 \end{bmatrix}. \]

Note that \( R = [R_1 \ \ R_2] \) is a basis matrix, its inverse \( T = R^{-1} \) exists and admits a partitioning according to:

\[ \begin{bmatrix} \tilde{v}_1 \\ \tilde{v}_2 \end{bmatrix} = Tv = \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} v. \]

Now define

\[ C_1 := T_1 C \quad \text{and} \quad C_2 := T_2 C, \] (12)

and observe that

\[ C_2 S^*_C(A)(\text{im} B_2) = T_2 CS^*_C(A)(\text{im} B_2) = T_2 R_1 \ker C_2 = 0, \]

i.e.

\[ S^*_C(A)(\text{im} B_2) \subset \ker C_2, \]

which, by Lemma 5, shows that \( \tilde{v}_2 \) is decoupled from the variable \( \ell \). We make two observations:

1. the dimension of \( \mathcal{V}_1 \) equals the number of non-zero constraints (on \( \ell \)) that have to be eliminated after the partitioning in step 1.
2. when a decomposition of \( v \) is made, the maximal dimension of \( \tilde{v}_2 \) for which \( \tilde{v}_2 \) is independent of \( \ell \), is \( p - \dim (CS^*_C(A)(\text{im} B_2)) \).

With this construction, the interconnected system of Fig. 2 is given by:

\[ \Sigma_{\text{int}} : \begin{cases} \dot{x} = (A + LC)\hat{x} + B_1 w + B_2 \ell, \\ \tilde{v}_1 = C_1 \hat{x}, \end{cases} \] (13)

We claim that \( B_\ell \) is represented by (13) in the following sense.

**Lemma 6.** Let \( B_\ell \) be defined by (9). Then \( B_\ell = \{w, \ell\} \subset L^1 = \|x\| L^1 \) such that (13) holds with \( \tilde{v}_1 = 0 \) and \( \tilde{v}_2 = 0 \).

**Proof:** For all nonsingular matrices \( S, T \) and for all matrices \( L \), we have that \( \begin{bmatrix} A & B_1 & B_2 \\ C & 0 & 0 \end{bmatrix} \) represents \( B_\ell \) if and only if

\[ \begin{bmatrix} S^{-1}(A + LC)S^{-1}B_1 \\ TCS \\ 0 \\ 0 \end{bmatrix} \]

represents \( B_\ell \). Here, \( S = I \), \( T = [T_1 \ T_2] \) and \( L \) the matrix that makes \( S^*_C(A)(\text{im} B_2) \) \((A + LC)\)-invariant.

Therefore, we can conclude that, as is the case in the classes \( M_1 \) and \( M_2 \), interconnection with the estimator does not change the behavior. Moreover, if we compare this result with the one obtained for \( M_2 \), we have that:

\[ \begin{bmatrix} P_1(s) = C_1(sI - A - LC)^{-1}B_1, \\ P_2(s) = C_1(sI - A - LC)^{-1}B_2, \\ P_{21}(s) = C_2(sI - A - LC)^{-1}B_1 \\ P_{22}(s) = 0, \end{bmatrix} \]

under the assumption that \( A + LC \) is stable.\(^1\) From Fig. 2, one can see that the operator \( U \) is represented, by the mapping from \( v \) to \( \hat{v} \), as:

\[ U(s) = -C(sI - A - LC)^{-1}L - I. \]

**Step 2:** Verify redundancy of condition on \( \ell \):

Using the result in Lemma 6 from Step 1, observe that any \( w \in B_{\text{end}} \) necessarily satisfies

\[ \begin{cases} \dot{\hat{x}} = (A + LC)\hat{x} + B_1 w, \\ \dot{x} = C_2 \hat{x}, \end{cases} \] (14)

while any \( \{w, \ell\} \subset B_\ell \) satisfies (14) and

\[ \Sigma' : \begin{cases} \dot{\hat{x}} = A\hat{x} + B_1 w + B_2 \ell, \\ \dot{x} = C_2 \hat{x}, \end{cases} \] (15)

where \( A := A + LC \).

\(^1\) If \( A + LC \) is not stable, we can replace the DDEP with DDEPS, where closed-loop stability will be guaranteed.
We now consider the problem under what conditions in (15) does not impose constrains on the variable \( w \). That is, consider the question when
\[
\dot{\bar{x}} = (A + LC)x + B_1w, \quad 0 = C_2\bar{x},
\]
vanishes for all time. We solve this problem by searching for a feedback law \( \ell = F\bar{x} \) such that \( \bar{v}_1 \in \Sigma'' \) is independent of \( w \). In control relevant terms, this amounts to solving the disturbance decoupling problem (DDP) for \( \Sigma'' \) where \( w \) is viewed disturbance, \( \ell \) as control variable and \( \bar{v}_1 \) as output.

A subspace \( V \subset X \) is said to be \((A, B_2)\) controlled invariant if there exists \( F : X \to \mathcal{L} \) such that \( (A + B_2F)V \subset V \). Equivalently, if \( AV \subset V + \im B_2 \). It can be shown that \((A, B_2)\) invariant subspaces are closed under addition. This means that a largest \((A, B_2)\) controlled invariant subspace exists that is contained in a subspace \( K \). This largest controlled invariant subspace is denoted \( \mathcal{V}_{AB_2}(K) \).

The result of DDP applied to the required condition to solve Step 2 of the elimination problem is given as follows:

**Lemma 7.** Given the system in (15). Then DDP is solvable if and only if
\[
\im B_1 \subset \mathcal{V}_{AB_2}^*(\ker C_1).
\]

When computing \( \mathcal{V}_{AB_2}^*(\ker C_1) \) that satisfies (16), we find the state feedback matrix \( F : X \to \mathcal{L} \) such that \( \ell = F\bar{x} \). Then, the interconnection with \( \Sigma'' \) results in
\[
\Sigma : \{ \dot{\bar{x}} = (A + LC)x + B_1w, \quad \bar{v}_1 = C_1\bar{x} \}.
\]

If the condition (16) is satisfied, we can conclude that the constraint (15) on the variables \( w, \ell \) does not impose a constraint on \( w \). That is, the induced behavior associated with (15) is simply \( L_{\mathcal{V}_{AB_2}} \) (i.e. \( \{w \mid \exists \ell \in L_{\mathcal{V}_{AB_2}}, \exists x \in L_{\mathcal{V}_{AB_2}} \} \)).

4. MAIN RESULT

When combining Lemma 5 with Lemma 7, we obtain the following result as sufficient condition to eliminate latent variables in the model class \( M_3 \) (cf. Problem 4):

**Theorem 8.** (Elimination in the class \( M_3 \)).

Given a latent variable system \( \Sigma_\ell = (\mathbb{R}, \mathbb{R}^{w+\ell}, B_2) \in M_3 \), with behavior \( B_2 \) represented by \( \im B_2 \) by (9). Let \( L : V \to X \) be a mapping that makes \( \mathcal{S}_{CA}^*(\im B_1) \) \((A + LC)\)-invariant and let \( C_1, C_2 \) be defined by (12). If
\[
\im B_1 \subset \mathcal{V}_{AB_2}^*(\ker C_1),
\]
with \( \lambda = A + LC \), then \( \Sigma_\ell \) is \( \ell \)-eliminable in \( M_3 \) and its induced behavior is
\[
B_{\text{ind}} = \{ w \in L_{\mathcal{V}_{AB_2}} \mid \exists x \in L_{\mathcal{V}_{AB_2}} \text{ s.t. } \{ \dot{\bar{x}} = A\bar{x} + B_1w, \quad 0 = C_2\bar{x} \} \}.
\]

As mentioned in Section 3.2, we considered the case where \( D_1 = 0 \) and \( D_2 = 0 \). As shown in e.g. Stoorvogel and van der Woude [1991], the computations for controlled and conditioned invariant subspaces can also be performed with non-zero feed-through terms. Then, \( \mathcal{S}_{CA}^*(\im B_1) \) needs to be replaced by the smallest subspace \( \mathcal{S}_{CA}^* \subset X \) for which there exists \( L \) such that
\[
(A + LC)\mathcal{S}_{CA}^* \subset \mathcal{S}_{CA}^* \text{ and } \im (B_1 + LD_1) \subset \mathcal{S}_{CA}^*.
\]
Similarly, \( \mathcal{V}_{AB_2}^*(\ker C_1) \) is replaced by the largest subspace \( \mathcal{V}_{AB_2}^* \) for which there exists \( F \) such that
\[
(A + B_2F)\mathcal{V}_{AB_2}^* \subset \mathcal{V}_{AB_2}^* \text{ and } \mathcal{V}_{AB_2}^* \subset \ker(C_1 + D_1F).
\]
(Notation is shortened to improve readability) Then, Theorem 9. (Elimination in the class \( M_3 \) with \( D_1 \neq 0 \)).

Given a latent variable system \( \Sigma_\ell = (\mathbb{R}, \mathbb{R}^{w+\ell}, B_2) \in M_3 \), with behavior \( B_2 \) as represented by (5). Let \( L : V \to X \) be a mapping that makes \( \mathcal{S}_{CA}^*(\im B_1) \) \((A + LC)\)-invariant and let \( C_1, C_2 \) and \( D_2 := T_D \) be defined using (12). If
\[
\im B_1 \subset \mathcal{V}_{AB_2}^*,
\]
with \( A = A + LC \), then \( \Sigma_\ell \) is \( \ell \)-eliminable in \( M_3 \) and its induced behavior is
\[
B_{\text{ind}} = \{ w \in L_{\mathcal{V}_{AB_2}} \mid \exists x \in L_{\mathcal{V}_{AB_2}} \text{ s.t. } \{ \dot{\bar{x}} = A\bar{x} + B_1w, \quad 0 = C_2\bar{x} + D_2w \} \}.
\]

We denote that the conditions in Theorems 8 and 9 are only sufficient for elimination of latent variables in the class \( M_3 \). Observe that the redundancy condition in step 2 of Section 3.2 results in a causal mapping from \( w \) to \( \ell \), which does not need to be the case for the problem of elimination. From this observation, we state that:

**Conjecture 10.**

The condition (18) is sufficient for \( \ell \)-eliminability. We conjecture that the condition \( \im B_1 \subset \mathcal{V}_{AB_2}^* \), where \( \mathcal{V}_{AB_2}^* \) is the largest almost controlled invariant subspace (see e.g. Willems [1982]) is necessary and sufficient.

5. ALGORITHM FOR ELIMINATION

The algorithm that constructs, when possible, the induced system using state space representations, is inspired on the result obtained in Theorem 8. This is in contrast to the existing algorithms that directly make use of the polynomial differential or rational operators. Since we now start making use of state space representations for behaviors, such that it is possible to handle more complex dynamical systems with the algorithm discussed in this section.

**Algorithm 1.** (Elimination using geometric approach).

**Given:** Latent variable system \( \Sigma_\ell \), consisting of manifest and latent variables \( w, \ell \), resp., in the system class \( M_3 \) as introduced in Section 2.

**Find, when possible:** Induced manifest system \( \Sigma_{\text{ind}} \), consisting of manifest variables \( w \) only, in \( M_3 \).

**Aim:** Use advantage of state space representations.

**Step 1:** Compute \( \mathcal{S}_{CA}^*(\im B_2) \) and the corresponding Luenberger matrix \( L \). Set \( A = A + LC \) of \( \Sigma_\ell \) as in (13).

**Step 2:** Apply the partitioning of \( C \) into \( C_1 \) and \( C_2 \) as proposed in (12).

**Step 3:** Compute \( \mathcal{V}_{AB_2}^*(\ker C_1) \) and verify whether \( \im B_1 \subset \mathcal{V}_{AB_2}^*(\ker C_1) \). If this is not the case, we can not use the proposed approach for elimination of \( \ell \) in \( \Sigma_\ell \). The algorithm stops here, but this does not imply that there does not exist a \( \Sigma_{\text{ind}} \in M_3 \).

**Result:** Set \( \Sigma_{\text{ind}} \) with manifest behavior \( B_{\text{ind}} \) represented by:
\[
\Sigma_{\text{ind}} : \{ \dot{\bar{x}} = (A + LC)x + B_1w, \quad 0 = C_2\bar{x} \}.
\]
Since the elimination problem of latent variables is given by a problem in geometric control theory, we can make use of the MATLAB toolbox that comes with Basile and Marro [1992].

6. EXAMPLE

The motivating example for applying the algorithm discussed in the previous section is the model for active suspension of a transport vehicle, as depicted in Fig. 3, taken from Weiland et al. [1997]. The model is given by:

\[
m_2 \ddot{q}_2 + b_2 (\dot{q}_2 - \dot{q}_1) + k_2 (q_2 - q_1) - F = 0,
\]

\[
m_1 \ddot{q}_1 + b_2 (\dot{q}_1 - \dot{q}_2) + k_2 (q_1 - q_2) + k_1 (q_1 - q_0) + F = 0,
\]

where \( F \) is a force acting on the chassis mass \( m_2 \) and where \( m_1 \) is the axle mass. The chosen values for those masses are \( m_1 = 2, m_2 = 10, b_2 = 3, k_1 = 1, \) and \( k_2 = 5 \) are the values of the damper and springs in the system. Here, \( q_1, q_2 \) and \( q_0 \) are the positions of the axle, chassis and the ground, respectively.

In this example, we specify the partitioning of manifest and latent variables as:

\[ w = [q_2 - q_1, q_0]^T \quad \text{and} \quad \ell = [\dot{q}_2, q_1 - q_0, F]^T. \]

The output nulling state space representation of the latent variable system, according to (5), is given by the system matrices:

\[
A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -3 & -2.5 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0.5 & 0.6 & -0.5 & -0.6 \end{bmatrix},
B_1 = \begin{bmatrix} 0 \\ 0.5 \\ 0.5 \\ 0.1 \end{bmatrix},
B_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},
C = \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix},
D_1 = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix},
D_2 = \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix}.
\]

**Step 1** of the algorithm results in \( S_{CA}^* (\text{im} B_1) \) and the Luenberger matrix \( L \):

\[
S_{CA}^* (\text{im} B_1) = \begin{bmatrix} -0.9806 & 0 & 0.9806 \\ 0 & -0.016 & 0 \\ 0.1961 & 0 & 0 \\ 0 & 0 & -0.1961 \end{bmatrix};
L = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0.016 & 0 \\ 0 & 0 & 0 \\ 0 & -0.0801 & 0 \end{bmatrix}.\]

We can apply **Step 2**, where the partitioning of \( C \) into \( C_1 \) and \( C_2 \) is made using the transformation:

\[
T = \begin{bmatrix} 0.5015 & 0 & -0.4180 \\ 0 & 1.0198 & 0 \\ 0.6402 & 0 & 0.7682 \end{bmatrix}.
\]

**Step 3** asks for the computation of the largest controlled invariant and smallest conditioned invariant sub-space \( \mathcal{V}_{AB}(\text{ker} C_1) \), which here is equal to \( \mathbb{R}^4 \), so obviously the condition for \( \text{im} B_1 \) holds. Hence, \( \Sigma_{\text{ind}} \in \mathcal{M}_3 \) can be obtained by substituting the results from both steps in \( B_{\text{ind}} \) of Theorem 9.

7. CONCLUSIONS

In this paper, we have discussed the problem of eliminating variables in a system that consists of desired manifest and to-be-eliminated latent variables for the classes of infinitely smooth \( \mathcal{C}^\infty (M_1) \) and square integrable \( L_2 (M_2) \) systems. This problem is well studied in the behavioral framework. For both classes, algorithms are developed consisting of two disjoint steps that need computations with polynomial or rational matrices.

For systems with higher complexity, it is desirable to have a constructive algorithm using state space representations, which makes computations more efficient and easier. Therefore, we have continued the research on looking for relations between the problem of elimination and problems in geometric control theory, as started in Mutsaers and Weiland [2010]. In that work, one of steps in the algorithm for \( L_2 \) systems has been shown to be equivalent with a disturbance decoupling problem.

In this paper, we have shown that there exists a sufficient condition in terms of geometric control theory for the problem of elimination in the class of systems represented using state space matrices of the latent variable system. This has been illustrated using an example.

REFERENCES


