Global Exact Tracking for Uncertain Multivariable Systems by Switching Adaptation *

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Abstract: In this paper, we propose an output feedback sliding mode controller to solve the problem of global exact output tracking for uncertain multivariable linear plants with non-uniform arbitrary relative degree. The controller is based on a hybrid estimation scheme, which adapts the relative degree compensation through switching between a high-gain observer and a set of locally exact differentiators. As a result, uniform global exponential practical stability and exact tracking are achieved with peaking free control signal.

Keywords: Output feedback; sliding mode control; tracking; uncertain linear systems

1. INTRODUCTION

Sliding Mode Controllers (SMC) based on High-Gain Observers (HGO) (Emelyanov et al. [1992], Esfandiari & Khalil [1992]) may be designed so that ideal sliding modes can be produced, thus avoiding chattering. However, global/semi-global stability results were obtained only with residual output errors (Levant [2006]). More recently, nonglobal exact output tracking controllers based on higher order sliding modes (HOSM) were proposed (Orlov [2005], Floquet et al. [2007], Baev et al. [2008]). To the best of our knowledge, global exact output tracking sliding mode control has remained unsolved, even in the linear case considered here.

In (Nunes et al. [2010]), we proposed a solution to the afore mentioned open problem of global exact output tracking for uncertain multivariable linear plants with non-uniform arbitrary relative degree using output feedback SMC. The result was achieved by extending a hybrid estimation scheme, originally proposed for SISO plants in (Nunes et al. [2009], Oliveira et al. [2007]), to a multivariable framework. Analogously to the original SISO version, such multivariable estimator is adaptive, in the sense of (Hespanha et al. [2003]), by selecting between a standard MIMO lead filter and a nonlinear one, which utilizes robust exact differentiation (RED), through logic-based switching. As a result, the error system becomes uniformly globally exponentially practically stable with respect to a small residual set and ultimately converges to zero.

In this paper, motivated by the improved robustness of HGOs w.r.t. unmodeled dynamics, when compared to the ordinary linear lead filters used in Nunes et al. [2010], it is shown that the MIMO lead filter can be replaced by an HGO in the above hybrid approach. Thus, the proposed output-feedback controller is based on a hybrid estimation scheme, which adapts the relative degree compensation, through switching between an HGO and a set of locally exact differentiators, in order to achieve uniform global exponential practical stability and exact tracking. In spite of the high-gain observer, global stability is guaranteed with a peaking free control signal. Moreover, in order to guarantee the existence of the sliding motion, we consider a less restrictive condition of the plant high frequency gain (HFG) matrix than in (Nunes et al. [2010]). It is worthwhile mentioning that the HGO structure allows a more natural extension to nonlinear plants (e.g., Oliveira et al. [2008]) which is not trivial for the lead filter based scheme. The proposed scheme is validated by simulations.

2. PRELIMINARIES

In what follows, all $\kappa$'s denote positive constants. $\pi(t)$ denotes an exponentially decaying function in the sense that $|\pi(t)| \leq K e^{-\lambda t}$, where $K$ possibly depends on the system initial conditions and $\lambda$ is a (generic) positive constant. $|$ stands for the Euclidean norm for vectors, or the induced matrix norm for matrices. For any measurable function $f : \mathbb{R}_+ \rightarrow \mathbb{R}^m$, $\|f\|$ denotes $\text{ess sup}_{t \geq 0} |f(t)|$, and, for any pair of times $0 \leq t_1 \leq t_2$, $\|f(t_1, \ldots, t_2)\| = \text{sup}_{t \in [t_1, t_2]} |f(t)|$. Classes $\mathcal{K}$, $\mathcal{K}_\infty$, $\mathcal{KL}$ functions are defined as usual (Jiang & Mareels [1997]).

* This work was supported in part by Brazilian founding agencies CNPq, FAPERJ and CAPES.
differential equations is assumed (Filippov [1964]). For the sake of simplicity, the symbol “s” will represent either the Laplace variable or the differential operator (d/dt), according to the context.

3. PROBLEM STATEMENT

Consider an uncertain MIMO LTI plant described by:

\[ \dot{x}_p = A_p x_p + B_p [u + d], \quad y_p = H_p x_p, \]

where \( x_p \in \mathbb{R}^m \) is the state, \( u \in \mathbb{R}^m \) is the input, \( d \in \mathbb{R}^m \) is an input disturbance, \( y \in \mathbb{R}^n \) is the output and \( A_p, B_p, H_p \) are constant uncertain matrices. All the uncertain parameters belong to some compact set \( \Omega \), such that the necessary uncertainty bounds to be defined later are available for design. As in (Hsu et al. [2002], Nunes et al. [2010]), the following assumptions are made:

(A1) \( G(s) \) is minimum phase and has full rank;

(A2) The plant is controllable and observable;

(A3) The observability index \( \nu \) of \( G(s) \), or an upper bound of \( \nu \), is known.

(A4) The left interactor matrix \( \Xi(s) \) is diagonal and \( G(s) \) has a known global vector relative degree \( \{p_1, \ldots, p_m\} \) (i.e., \( \Xi(s) = \text{diag}(s^{p_1}, \ldots, s^{p_m}) \)). The matrix \( K_p \in \mathbb{R}^{m \times m} \), finite and nonsingular, is referred to as the high frequency gain (HFG) matrix and satisfies \( K_p = \lim_{s \to \infty} \Xi(s) G(s) \).

The above assumptions are discussed and motivated in (Nunes et al. [2010]). We further assume that:

(A5) A nonsingular matrix \( S_p \) is known such that \( -K_p S_p \) is diagonally stable, i.e., there exists a diagonal matrix \( D > 0 \) such that \( DK + K^T D = -Q \), with \( Q = Q^T > 0 \) and \( K = -K_p S_p \).

Assumption (A5) is less restrictive than the previous assumption made in (Nunes et al. [2010]). As in (Hsu et al. [2002]), assumption (A5) could be relaxed by assuming the existence of a matrix \( S_p \) such that \( -K_p S_p \) is Hurwitz at the expense of the undesirable effect of peaking (Oh & Khalil [1997]) in the control law (see Lemma 6 in the appendix). In order to obtain norm bounds, consider that:

(A6) The input disturbance is assumed to be uncertain, locally integrable, uniformly bounded and with a known upper bound\(^1\) \( d(t) \geq |d(t)|, \forall t \).

Consider that a reference signal \( y_m \) is generated by the following reference model: \(^2\)

\[ y_m = W_M(s) r, \quad r, y_m \in \mathbb{R}^m, \]

\[ W_M(s) = \text{diag} \left\{ (s + \gamma_1)^{-1}, \ldots, (s + \gamma_m)^{-1} \right\} L^{-1}(s), \]

where \( \gamma_j > 0, j = 1, \ldots, m \), and \( L(s) \) is given by

\[ L(s) = \text{diag} \{ L_1(s), L_2(s), \ldots, L_m(s) \}, \]

where \( L_i(s) = s^{(p_i - 1)} + \left[ l^{(i)} \right]_0 s^{(p_i - 2)} + \cdots + \left[ l^{(i)} \right]_0 s^{(0)} + [l_{i}^{(i)}], \quad i = 1, \ldots, m \) are Hurwitz polynomials. The transfer function matrix \( W_M(s) \) has the same vector relative degree as \( G(s) \) and its HFG is the identity matrix.

\(^1\) Note that \( d \) could possibly also depend, even nonlinearly, on the state provided a bound \( d(t) \) is known.

\(^2\) The tracking of more general reference models could be obtained by simply preshaping the reference signal \( r \) through a precompensator at the input of the above model.

The main objective is to find a control law \( u \) such that the output error \( e := y_p - y_m \) tends asymptotically to zero, for arbitrary initial conditions and uniformly bounded arbitrary piecewise continuous reference signal \( r(t) \).

When the plant is known and \( d(t) \equiv 0 \), a control law which achieves the matching between the closed-loop transfer function matrix and \( W_M(s) \) is given by \( u^* = \theta^* \omega \), where the parameter vector is written as \( \theta^* = [\theta_1^T \theta_2^T \theta_3^T \theta_4^T]^T \), with \( \theta_1^T, \theta_2^T \in \mathbb{R}^{m \times (n-1) \times m}, \theta_3^T, \theta_4^T \in \mathbb{R}^{m \times m} \) and the regressor vector \( \omega = [\omega_1^T \omega_2^T \omega_3^T r^T]^T \), \( \omega_1, \omega_2 \in \mathbb{R}^{m \times (n-1)} \) is obtained from I/O state variable filters given by:

\[ \omega_1 = A(s) \Lambda^{-1}(s) u, \quad \omega_2 = \Lambda(s) \Lambda^{-1}(s) y_p, \]

where \( A(s) = [I s^{n-2} I s^{n-3} \ldots I s I]^T \), \( \Lambda(s) = \Lambda(s) \Lambda^{-1}(s) \) with \( \Lambda(s) \) being a monic Hurwitz polynomial of degree \( n - 1 \). The matching conditions require that \( \theta_4^T K = K_p^{-1} \).

The error equation can be developed following the usual approach for SISO MRAC (Nunes et al. [2009], Ioannou & Sun [1996]). Consider the following realization of (6)

\[ \tilde{\omega}_1 = \Phi \omega_1 + \Gamma u, \quad \tilde{\omega}_2 = \Phi \omega_2 + \Gamma y_p, \]

\[ \Gamma \in \mathbb{R}^{m \times (n-1) \times m}, \Phi \in \mathbb{R}^{m \times (n-1) \times m} \]

where \( \Phi = \Phi(s) \) is the output error equation can be written in input-output form as

\[ \dot{X} = A_c X + B_c u + B_c d, \quad y_p = H_c X. \]

Then, adding and subtracting \( B_0 \theta^* \omega \) and noting that there exist matrices \( \Omega_1 \) and \( \Omega_2 \) such that \( \Omega = \Omega_1 X + \Omega_2 r \), one has \( X = A_c X + B_c K p [\theta^T r + u - u^*] + B_0 d, \quad y_p = H_c X, \)

where \( A_c = A_c + B_c \theta^T \Omega_1, \quad B_c = B_c \theta^T \).

Noting that \( A_c, B_c, H_c \) is a nonminimal realization of \( W_M(s) \) and defining the filtered disturbance signal \( \dot{d}(t) = W_d(s) d(t) \), where \( W_d(s) = K_p^{-1} W_M(s) \), the reference model can be described by

\[ \ddot{X}_M = A_c X + B_c K p [\theta^T r - d], \quad B_0 d, \quad y_p = H_c X. \]

Thus, the error state \( x_e := X - X_M \) dynamics is given by:

\[ \dot{x}_e = A_e x_e + B_e K p [u - \theta^* \omega + d], \quad \dot{e} = H_o x_e, \]

Moreover, since \( \{A_e, B_e, H_o\} \) is a realization of \( W_M(s) \), the error equation can be written in input-output form as

\[ e = W_M(s) K p [u - \theta^* \omega + d]. \]

4. UNIT VECTOR CONTROL DESIGN

For systems with uniform relative degree one, i.e. \( \rho_1 = \rho_2 = \cdots = \rho_m = 1 \), the main idea is to close the control loop with a nominal control together with a unit vector control (UVC) term to cope with uncertainties and disturbances:

\[ u = (\theta^{\text{nom}})^T \omega + U_S, \]

\[ U_S = -\theta(t) S_p \theta^T [e], \quad e \in \mathbb{R}^m, \quad S_p \in \mathbb{R}^{m \times m}, \quad \theta \in \mathbb{R}, \]

where \( \theta^{\text{nom}} \) is the nominal value for \( \theta \), \( S_p \) satisfies (A5) and the modulation function \( \theta(t) \geq 0 \) is designed to induce a sliding mode on the manifold \( e = 0 \) and is such that:

\[ \theta(t) \geq (1 + c_d) \left| S_p^{-1} [\theta^{\text{nom}} - \theta^*]^T \omega - d \right| + \delta, \]

(10)

where \( c_d \) is an appropriate positive constant and \( \delta \) is a positive constant, which can be arbitrarily small. Note that
the nominal control signal allows the reduction of the modulation function amplitude if the parameter uncertainty $|\theta^* - \theta^*|$ is small.

Since $W_M(s) = \text{diag}\{s + \gamma_1, \ldots, s + \gamma_m\}$, and $-K_S S_0$ is diagonally stable, applying Lemma 6 in the appendix one has that the above scheme is uniformly globally exponentially stable (GES) and the output error $e$ becomes identically zero after some finite time.

For systems of higher relative degree, one could use the operator $L(s)$ defined in (5), to overcome the relative degree obstacle. The operator $L(s)$ is such that $L(s)G(s)$ and $L(s)W_M(s)$ have uniform vector relative degree one. The ideal sliding variable $\sigma$ is in $\mathbb{R}^n$ is given by

$$\sigma = L(s)e = \begin{bmatrix} e^{(\tilde{\rho}_1-1)} + \cdots + e^{(\tilde{\rho}_1-1)} \tilde{u}_1 \tilde{y}_1 + \tilde{u}_0 \tilde{y}_1 \\ \vdots \\ e^{(\tilde{\rho}_m-1)} + \cdots + e^{(\tilde{\rho}_m-1)} \tilde{u}_m \tilde{y}_m + \tilde{u}_0 \tilde{y}_m \end{bmatrix}^T$$

(11)

From assumption (A4) and (8), $\sigma$ can be rewritten as:

$$\sigma = \sum_{j=1}^{m} \tilde{u}_j \tilde{y}_j T_{M} s_{j} = H_{M} e,$$

where $h_i \in \mathbb{R}^{n \times 2m-\nu}$ is the $i$-th row of matrix $H_M$. Notice that $\{A, B, \tilde{H}, \tilde{S}\}$ is a nonminimal realization of $L(s)W_M(s)$. If the control signal is given by $u = (\theta^* - \theta^*)^T \omega + d_f + \pi_{\eta}$, and then the closed-loop error system is GES and the ideal sliding variable $\sigma$ becomes identically zero after some finite time, according to Lemma 6 in the appendix. However, $\sigma$ is not directly available to implement the control law.

5. UNIT VECTOR CONTROL USING A HIGH GAIN OBSERVER

Consider the minimal order observer-form realization $\{A_M, B_M, C_M\}$ of the model $W_M(s)$ (see Khalil [1980]). Then, the the error equation (8) can be rewritten as

$$\hat{\xi} = A_M \xi + B_M K_P \left[ S + (\theta^* - \theta^*)^T \omega + d_f + \pi_{\eta} \right],$$

(12)

with $e = C_M x_e$, where the initial condition $\xi(0)$ and the exponentially decreasing signal $\pi_{\eta}(t)$ are adequate for representing the initial condition of the observable but uncontrollable modes in (8). The unstable state $\xi$ can be estimate by means of an HGO given by:

$$\hat{\xi} = A_M \xi + B_M K_P U_B = \left[ \Gamma (\varepsilon^{-1}) - H_A \right] \hat{e},$$

(13)

with $\hat{e} = C_M \hat{x} - e$ being the observer output error, where $K^n$ is the nominal value of the gain $K_P$. $\Gamma (\varepsilon^{-1})$ is the block diagonal $\{\Gamma_1, \ldots, \Gamma_m\}$, with $\Gamma_i = \begin{bmatrix} \tilde{a}_i^{\tilde{\rho}_i-1} & \cdots & \tilde{a}_i 0 \\ \vdots & \ddots & \vdots \\ \tilde{a}_i 0 & \cdots & \tilde{a}_i^{\tilde{\rho}_i-1} \end{bmatrix}$,

$$L_i(s) = \begin{bmatrix} s^{\tilde{\rho}_i} & a_i^{\tilde{\rho}_i-1} s^{\tilde{\rho}_i-1} & \cdots & a_i 0 \\ \vdots & \hdots & \ddots & \vdots \\ \tilde{a}_i 0 & \cdots & \tilde{a}_i^{\tilde{\rho}_i-1} \end{bmatrix},$$

with $H_M = \{H_1, \ldots, H_m\}$, and $H_i = \begin{bmatrix} a_i^{\tilde{\rho}_i-1} & \cdots & a_i \end{bmatrix}^T$.

The coefficients $\tilde{a}_i$ in the observer feedback vector, must be chosen such that $N_0(s) = s^{\rho_i} + \tilde{a}_1 s^{\rho_i-1} + \cdots + \tilde{a}_0$ is Hurwitz and $\varepsilon > 0$. It is possible to design a matrix $H_M$ such that $\{A_M, B_M, H_M\}$ is a realization of the transfer function $L(s)W_M(s)$. Thus, for plants of higher relative degree, $\sigma$ can be estimated by

$$\hat{\sigma}_h = H_M \hat{\xi}, \quad H_M = \text{block diag} \{H_1, \ldots, H_m\},$$

(14)

with $H_M$ given by

$$H_M = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & a_{\tilde{\rho}_i-1} & a_i \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 1 & a_i \tilde{a}_1 \cdots a_i \tilde{a}_0 \end{bmatrix},$$

Defining the estimation error state as $\hat{\xi} = \xi - \xi$, one has:

$$\hat{\xi} = A_{\xi} (e^{-}\varepsilon) + B_M K_P \gamma \hat{\xi}, \quad \hat{\xi}_n = H_M \hat{\xi},$$

(15)

$$\tilde{U} = (K_P')^{-1} K_P \left( (\theta^* - \theta^*)^T \omega - d_f - \pi_{\eta} \right) + [I - (K_P')^{-1} K_P] \Upsilon_S,$$

where $\hat{\xi}_n = \hat{\sigma}_h - \sigma$, $A_{\xi} (e^{-}\varepsilon) = \text{block diag} \{A_1, \ldots, A_m\}$, $B_M = \text{block diag} \{B_1, \ldots, B_m\}$, with

$$A_{\xi} = \begin{bmatrix} -\tilde{a}_i^{\tilde{\rho}_i-1} / \varepsilon & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots \\ -\tilde{a}_i^{\tilde{\rho}_i-1} / \varepsilon & \cdots & \cdots & \cdots & 0 \end{bmatrix}, \quad B_M = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ -\tilde{a}_i^{\tilde{\rho}_i-1} / \varepsilon \end{bmatrix},$$

Replacing $\sigma$ by its estimate $\hat{\sigma}_h$, the control law would be given by $u = (\theta^* - \theta^*)^T \omega - \theta(t) \frac{\partial \theta^*}{\partial \theta}$. However, in the stability analysis of the closed-loop error system, with state $z^T = \left[ x_e^T \hat{\xi}^T \right]$, we will assume the presence of an absolutely continuous uniformly bounded output disturbance $\beta(t)$ of order $e$. Taking this into account, the plant input can be represented as

$$u = (\theta^* - \theta^*)^T \omega + U_S, \quad U_S = \theta(t) S_P \frac{\partial \theta^*}{\partial \theta} \beta(t),$$

(16)

The following result is instrumental in the stability analysis of the proposed controller (Section 7).

**Theorem 1.** Consider the plant (1) with input signal (16) and the reference model (3)-(5). Suppose that assumptions (A1) to (A6) hold and that (10) is satisfied. If the disturbance $\beta(t)$ is absolutely continuous and bounded by $|\beta_{\alpha}(t)| \leq K_{\beta} R_{\beta}$, then for sufficiently small $\varepsilon > 0$, the error system (8), (11), (15) and (16), with state $z^T = \left[ x_e^T \hat{\xi}^T \right]$, is uniformly globally exponentially practically stable with respect to a residual set of order $e$, i.e., there exist constants $c_\varepsilon, a > 0$ such that $|z(t)| \leq c_\varepsilon e^{a(t-\varepsilon)} |z(0)| + O(\varepsilon)$ holds $\forall z(t), \forall t \geq t_0$. (Proof: see Appendix.)

**Corollary 2.** For all $R > 0$, there exists $\varepsilon > 0$ sufficiently small such that for some finite time $T$, the error state $z(t)$ is steered to an invariant compact set $D_R := \{ z : |z| \leq R \}$.

**Corollary 3.** The signals $e^{(\rho_i)}(t)$, $i = 1, \ldots, m$ are uniformly bounded. Moreover, if $|x_e(t)| \leq R, \forall t \geq T$, then, there exist constants $C_{\rho_i} > 0$ such that $|e^{(\rho_i)}(t)| \leq C_{\rho_i}$. (Proof: see the proof of [Nunes et al., 2009, Corollary 2].)

6. MIMO ROBUST EXACT DIFFERENTIATOR

In the previous section, a UVC using an HGO to estimate $\sigma$ was analyzed. From Theorem 1 the convergence of the error state is only guaranteed to a residual set of order $e$. In order to achieve exact tracking, one can use a MIMO extension of the recently proposed exact differentiator.
based on HOSM (Levant [2003]), to be referred simply by MIMO RED. The idea is to use a RED of order \( p_j = \rho_j - 1 \) for each output \( e_j \in \mathbb{R}, j = 1, \ldots, m \) as follows:

\[
\begin{align*}
\dot{z}_i^j &= \psi_i^j, \quad \psi_i^j = -\lambda_i^j |z_i^j - v_i^j|^{\rho_i^j - 1} \operatorname{sgn}(z_i^j - v_i^j) + e_i^j, \\
\dot{v}_i^j &= -\lambda_i^j |v_i^j|^{\rho_i^j - 1} \operatorname{sgn}(v_i^j),
\end{align*}
\]

where \( i = 0, \ldots, p_j \) and \( v_i^j = e_j(t) \). A similar approach was considered in (Fridman et al. [2008]) to build a MIMO RED based observer. The following theorem, presented in (Levant [2003]), characterizes the convergence properties of each individual RED.

**Theorem 4.** (Levant [2003]). Consider system (17). Let the input signal \( e_j(t) \in \mathbb{R} \) be a function defined on \([0, \infty)\) with the \( p_i \) derivative having a known Lipschitz constant \( C_{\rho_i^j} > 0 \). If the parameters \( \lambda_i^j \), \( i = 0, \ldots, n \) are properly chosen, then in the absence of input noise the following equalities are true after a finite time transient process:

\[
\zeta_i^j = \psi_i^j, \quad i = 0, \ldots, p_j, \quad j = 1, \ldots, m.
\]

Under the foregoing conditions, the above differentiator can provide the exact \( e_j \) (derivatives). According to Levant [2003], the RED’s performance is asymptotically optimal in the presence of small Lebesgue-measurable input noise and parameters \( \lambda_i^j \) can be selected adopting a suggested practical rule. Thus, using a MIMO RED, composed by \( m \) REDs of order \( p_1 - 1 \) for each output \( e_j \), the following estimate for \( \sigma \) can be obtained:

\[
\hat{\sigma}_g = \hat{\sigma}_h + [1 - \alpha(\tilde{\nu}_h)] \hat{\sigma}_h.
\]

Then, a control signal \( u = (g_{\text{nom}})^T \omega - g(t)S_p \hat{\sigma}_g / |\hat{\sigma}_g| \) could be used, however, Theorem 4 would only guarantee local convergence of the error state to zero, since the signals \( e_j(t) \), \( j = 1, \ldots, m \) should be uniformly bounded.

7. **GLOBAL EXACT TRACKING CONTROLLER BASED ON SWITCHED ADAPTIVE ESTIMATION**

In order to guarantee global exponential stability with respect to a small residual set of order \( \varepsilon \) and to achieve global convergence of the error state to zero, we show that the UVC using an HGO (Section 5) can be combined with the MIMO RED (Section 6). The proposed control scheme is based on a hybrid compensator which consists of an adaptive convex combination of the HGO estimate (14) and the MIMO RED estimate (18) according to:

\[
\hat{\sigma}_g = \alpha(\tilde{\nu}_h) \hat{\sigma}_h + [1 - \alpha(\tilde{\nu}_h)] \hat{\sigma}_r,
\]

where \( \tilde{\nu}_h = \hat{\sigma}_h - \hat{\sigma}_h \) is the difference between both estimates. The Switching Adaptation Function (SAF) \( \alpha(\tilde{\nu}_h) \) is a continuous, state dependent modulation which assumes values in the interval \([0, 1]\) and allows the controller to smoothly change from one estimator to the other.

Here, we propose a switching adaptation law \( \alpha(\cdot) \) so that the resulting system becomes equivalent to the UVC using an HGO with a uniformly bounded output disturbance of order \( \varepsilon \) and after some finite time only the MIMO RED is used to estimate the ideal sliding variable \( \sigma \). Thus, global practical stability and convergence to the compact set \( D_R \) are guaranteed, according to Theorem 1, independently of the MIMO RED behavior, provided its signals remain bounded. Specifically, \( \alpha(\cdot) \) is designed such that \( |\hat{\sigma}_g - \hat{\sigma}_h| \leq \varepsilon K_R \), as follows:

\[
\alpha(\tilde{\nu}_h) = \left\{ \begin{array}{ll}
0, & |\tilde{\nu}_h| - \varepsilon_M > \Delta \\
(\tilde{\nu}_h)/\varepsilon_M - \Delta, & \varepsilon_M - \Delta \leq |\tilde{\nu}_h| < \varepsilon_M \\
1, & |\tilde{\nu}_h| \geq \varepsilon_M
\end{array} \right.
\]

where \( 0 < \Delta < \varepsilon_M \) is a boundary layer used to smooth the switching function, and \( \varepsilon_M := \varepsilon K_R \) with \( K_R \) being an appropriate positive design parameter. Within \( D_R \), the convergence of the MIMO RED can be guaranteed and an upper bound \( \xi_g \) for the HGO estimation error \( \xi_g \) can be determined. Thus, selecting \( K_R \) such that \( \varepsilon_M - \Delta > \xi_g \), it follows that after some finite time only the MIMO RED is active (\( \alpha = 0 \)), providing exact estimation of the ideal sliding variable \( \sigma \), as desired.

The control law is given by:

\[
u = (g_{\text{nom}})^T \omega + u_S, \quad u_S = -g(t)S_p \hat{\sigma}_g / |\hat{\sigma}_g|,
\]

where the modulation function \( g(t) \) satisfies (10). The sta-

\[\text{Fig. 1. UVC based on a hybrid compensator.}\]

\[\text{Fig. 2. Control block diagram.}\]

\[\text{Fig. 3. Time response.}\]

8. **SIMULATION RESULTS**

In order to illustrate the proposed control strategy, we consider that the plant (2) is given by

\[
G(s) = \frac{\kappa(s+2)}{\kappa(s+1)(s+3)} - \frac{\kappa}{(s-1)(s+1)(s+3)^2} - \frac{1}{\mu^2+1} - \frac{1}{\mu^2+1} - \frac{1}{\mu^2+1}
\]

where the constant \( \kappa \in [4, 10] \) is uncertain and the constant \( \mu \) is associated with an unmodeled dynamics.
The input disturbance is considered uncertain for the control design and is given by \( d = [\text{sqw}(5t) \ \text{sqw}(3t)]^T \), where \( \text{sqw}(\cdot) \) denotes a unit square wave. To perform the simulations the actual parameter \( \kappa \) is set to 10 while \( \kappa^{\text{nom}} = 7 \) is chosen for control purposes.

8.1 Nominal plant without unmodeled dynamics (\( \mu = 0 \))

The nominal linear system has non-uniform relative degree (\( \rho_1 = 2, \rho_2 = 3 \)) and HFG matrix given by \( K_p = \begin{bmatrix} 1 & \kappa \\ 0 & 1 \end{bmatrix} \).

Thus, assumption (A5) is satisfied with \( S_p = I \), which would not be the case if we had considered the more restrictive assumption \( K_pS_p + S_p^2K_p^T > 0 \) made in [Nunes et al. [2010]]. The reference signal and model are chosen as 
\[
r = [\sin(t) \ sin(0.5t)] \quad \text{and} \quad W_M = \text{diag} \left\{ \frac{1}{(s+1)^2}, \frac{1}{(s+1)^2(s+2)} \right\}.
\]

In the absence of disturbances, model matching is achieved with \( u^* = \theta^T \omega \). For \( \kappa^{\text{nom}} = 7 \) and \( \kappa \in [4,10] \), \( \left| (\theta^{\text{nom}} - \theta^*)^T \right| \leq 2 \). Then, \( \varrho(t) \) (see (21)) satisfies (10), with \( c_2 = 2.25, \ |d| = |W_a(s) * d(t)| < 1 \) and \( \delta = 0.1 \).

Other design parameters are: II/0 filters (6): \( \nu = 3 \) and \( \lambda(s) = (s + 2)^2; \ L(s) = \text{diag} \{ (s + 1), (s + 1)^2 \} \) HGO (14): \( \epsilon = 0.01; \ K_p^{\text{nom}} = K_p \) with \( \kappa = 7 \), \( N_a^{[1]} = s^2 + 2s + 1 \), \( N_a^{[2]} = s^2 + 3s^2 + 3s + 1 \), \( H_a^{[1]} = [2]^T; \ H_a^{[2]} = [4 \ 5 \ 2]^T \), \( H_b^{[1]} = [-1 \ 1]^T; \ H_b^{[2]} = [4 - 2 \ 1]^T \); MIMO RED (17)-(18): \( \lambda_0^{[1]} = 1.5C_s^{1/2}; \ A_1^{[1]} = 1.1C_s^{1} \) and \( C_2^{[1]} = 30 \); \( \lambda_0^{[2]} = 3C_s^{1/2}; \lambda_1^{[2]} = 1.5C_s^{1/2}; \lambda_0^{[3]} = 1.1C_s^{1} \) and \( C_3^{[2]} = 100 \); switching function (20): \( \varepsilon_M = 60 \epsilon \) and \( \Delta = 10\epsilon \). We consider the following plant initial conditions: \( y_p(0) = 8, \dot{y}_p(0) = -13, \ \ddot{y}_p(0) = 20 \). The remaining system initial conditions are set to zero. The Euler Method with step-size \( h = 10^{-5} \) is used for numerical integration.

Fig. 2(a) shows that the proposed control scheme achieves precise tracking despite the disturbance \( d \). From the plot of \( \alpha(\cdot) \) in Fig. 2(b), one notes that after a short transient only the MIMO RED remains active as expected. It is important to stress that with the same parameters and initial conditions, if only the RED is used \( (\alpha(\cdot) \equiv 0, \forall t) \) the system becomes unstable. Stability can be recovered by increasing the MIMO RED parameters. However, the larger the parameters, the higher sensitivity to noises (Levant [2003]). From a practical point of view, an advantage of the hybrid estimation scheme is that it does not require large RED parameters to guarantee global stability.

8.2 Nominal plant with unmodeled dynamics

Now, we consider that \( \mu = 0.8 \). The same control design developed for the nominal plant without unmodeled dynamics is considered, except for the switching function parameters, which are now set to \( \varepsilon_M = 250 \epsilon \) and \( \Delta = 10\epsilon \). The initial conditions are the same of the previous example.

As can be seen in Fig. 3, the tracking performance of the controller using the hybrid estimation scheme is clearly superior to that obtained using only the high-gain observer. Considering, the same example, if the switching scheme proposed in [Nunes et al. [2010]] were used with a time constant \( \tau = 0.01 \) for the MIMO lead filter, the system would become unstable. To ensure stability the time constant should be reduced to \( \tau = 0.002 \). Moreover, to enhance performance this time constant should be reduced even more, making the scheme more sensitive to noise.

The main motivation for using asymptotic observers in SMC is that ideal sliding modes can be realized in spite of unmodelled dynamics through the observer (Bondarev et al. [1985]). Since such ideal sliding loop is not achieved using only lead filters, it is expected that the replacement of lead filters by HGOs improves the robustness of the hybrid estimation scheme to unmodelled dynamics.

9. CONCLUSIONS

An output feedback sliding mode tracking controller for uncertain MIMO linear plants with non-uniform arbitrary relative degree has been proposed. The controller is based on a hybrid estimation scheme, which combines a high-gain observer with locally exact differentiators in such way that uniform global exponential practical stability with respect to a small residual set is guaranteed, as well as, ultimate exponential exact tracking of a reference signal generated by a reference model. Improved robustness of the new controller w.r.t. unmodeled dynamics is discussed.

REFERENCES
Lemma 6. Consider the MIMO system
\[
\sigma(t) = M(s)K[u + d(t)],
\]
where \(M(s) = \text{diag}\left\{ (s+\gamma_1)^{-1}, \ldots, (s+\gamma_m)^{-1}\right\} \), \(\gamma_i > 0\), \(K \in \mathbb{R}^{m \times m}\) is the HFG matrix and is such that \( \gamma_i > 0\) is an appropriate constant, \(\gamma_i \geq 0\) is an arbitrary constant, then, the inequality
\[
|\sigma(t)| + |x_\sigma(t)| \leq c|x_\sigma(0)|e^{-\lambda t}
\]
holds for some positive constants \(c, \lambda\), where \(x_\sigma(t)\) is the state of any stabilizable and detectable realization of (A.1) (possibly nonminimal). Moreover, if \(\delta > 0\), then \(\sigma(t)\) becomes identically zero after some finite time \(t_\delta \geq 0\).

Proof: is similar to the proof of [Nunes et al., 2010, Lemma 1].

Lemma 7. Consider the MIMO system
\[
\sigma(t) = -A_m\sigma(t) + K[u + d(t) + v], \quad A_m, K \in \mathbb{R}^{m \times m}
\]
where \(A_m = \text{diag}\{\gamma_1, \ldots, \gamma_m\}\), \(|\sigma| \leq Re^{-\lambda t} - K\) is diagonally stable, \(u = \sigma(t) + d(t), \pi(t)\) and \(d(t)\) are LI and \(\frac{d}{dt} \sigma(t) = \beta(t)\), where \(\beta(t)\) is absolutely continuous \((\forall t)\). If \(\sigma(t) \geq (1 + \epsilon_d)\|d(t)\| \forall t\), for some appropriate \(\epsilon_d \geq 0\), then \(\sigma\) and \(\pi\) are bounded by
\[
|\sigma(t)| \leq c_1|\sigma(0)|e^{-\gamma t} + c_2\left(Re^{-\min\{\gamma_i, \lambda\}} + \beta(t)\right)
\]
(A.4) for some positive constants \(c_1, c_2\) and \(\gamma \geq \gamma_i\).

Proof: Consider the function \(V(\sigma) = \sigma^T D\sigma\). Since \(-K\) is diagonally stable, then it follows that
\[
V = 2\sigma^T D A_m\sigma - \sigma^T DK\sigma + \sigma^T K^T D A_m\sigma + 2\sigma^T DK(d + \pi).
\]
Noting that \(2\sigma^T D A_m \leq -2\gamma\sigma^T D\sigma = -2\gamma V\) the proof follows the same steps presented in the proof of [Hsu et al., 2002, Lemma 2].

Appendix B. PROOF OF THEOREM 1

The \(\xi\) subsystem (15) is ISpS with respect to the input \(x_\sigma\). Applying the transformation \(x_\xi = T(\xi, \tilde{\xi})\) with \(T(\xi, \tilde{\xi}) = \text{diag}\{T_1[i], \ldots, T_m[i]\}\) with
\[
T_1[i] = \frac{\pi_{i-1}}{\sum_{j=0}^{i-1} |\pi_j|}, \quad T_m[i] = \frac{\pi_{m-1} \cdots \pi_0}{\sum_{j=0}^{m-1} |\pi_j|}.
\]

The system (15) can be rewritten as
\[
\dot{x}_\xi = \sum_{i=1}^{m} x_{\xi[i]} T_{\xi[i]} K_{p_{\xi[i]}},
\]
where \(x_{\xi[i]} = T_{\xi[i]}(\xi_{\xi[i]}^{[m]} + x_{\xi[i]}^{[m]}), A_{\xi[i]} = \frac{\pi_{i-1}}{\sum_{j=0}^{i-1} |\pi_j|} - \frac{\pi_{i-1}}{\sum_{j=0}^{i-1} |\pi_j|} = 0.\)

\section{Appendix A. AUXILIARY LEMMAS}

Lemma 6. Consider the MIMO system
\[
\sigma(t) = M(s)K[u + d(t)],
\]
where \(M(s) = \text{diag}\left\{ (s+\gamma_1)^{-1}, \ldots, (s+\gamma_m)^{-1}\right\} \), \(\gamma_i > 0\), \(K \in \mathbb{R}^{m \times m}\) is the HFG matrix and is such that \(\gamma_i > 0\) is an appropriate constant, \(\gamma_i \geq 0\) is an arbitrary constant, then, the inequality
\[
|\sigma(t)| + |x_\sigma(t)| \leq c|x_\sigma(0)|e^{-\lambda t}
\]
holds for some positive constants \(c, \lambda\), where \(x_\sigma(t)\) is the state of any stabilizable and detectable realization of (A.1) (possibly nonminimal). Moreover, if \(\delta > 0\), then \(\sigma(t)\) becomes identically zero after some finite time \(t_\delta \geq 0\).

Proof: is similar to the proof of [Nunes et al., 2010, Lemma 1].

Lemma 7. Consider the MIMO system
\[
\sigma(t) = -A_m\sigma(t) + K[u + d(t) + v], \quad A_m, K \in \mathbb{R}^{m \times m}
\]
where \(A_m = \text{diag}\{\gamma_1, \ldots, \gamma_m\}\), \(|\sigma| \leq Re^{-\lambda t} - K\) is diagonally stable, \(u = \sigma(t) + d(t), \pi(t)\) and \(d(t)\) are LI and \(\frac{d}{dt} \sigma(t) = \beta(t)\), where \(\beta(t)\) is absolutely continuous \((\forall t)\). If \(\sigma(t) \geq (1 + \epsilon_d)\|d(t)\| \forall t\), for some appropriate \(\epsilon_d \geq 0\), then \(\sigma\) and \(\pi\) are bounded by
\[
|\sigma(t)| \leq c_1|\sigma(0)|e^{-\gamma t} + c_2\left(Re^{-\min\{\gamma_i, \lambda\}} + \beta(t)\right)
\]
(A.4) for some positive constants \(c_1, c_2\) and \(\gamma \geq \gamma_i\).

Proof: Consider the function \(V(\sigma) = \sigma^T D\sigma\). Since \(-K\) is diagonally stable, then it follows that
\[
V = 2\sigma^T D A_m\sigma - \sigma^T DK\sigma + \sigma^T K^T D A_m\sigma + 2\sigma^T DK(d + \pi).
\]
Noting that \(2\sigma^T D A_m \leq -2\gamma\sigma^T D\sigma = -2\gamma V\) the proof follows the same steps presented in the proof of [Hsu et al., 2002, Lemma 2].

Appendix B. PROOF OF THEOREM 1

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\[
T_1[i] = \frac{\pi_{i-1}}{\sum_{j=0}^{i-1} |\pi_j|}, \quad T_m[i] = \frac{\pi_{m-1} \cdots \pi_0}{\sum_{j=0}^{m-1} |\pi_j|}.
\]

The system (15) can be rewritten as
\[
\dot{x}_\xi = \sum_{i=1}^{m} x_{\xi[i]} T_{\xi[i]} K_{p_{\xi[i]}},
\]
where \(x_{\xi[i]} = T_{\xi[i]}(\xi_{\xi[i]}^{[m]} + x_{\xi[i]}^{[m]}), A_{\xi[i]} = \frac{\pi_{i-1}}{\sum_{j=0}^{i-1} |\pi_j|} - \frac{\pi_{i-1}}{\sum_{j=0}^{i-1} |\pi_j|} = 0.\)