Abstract: We address the problem of tracking for a general class of uncertain nonlinear MIMO systems with input quantization, without requiring any matching conditions. We consider the $L_1$ adaptive controller and analyze its performance bounds in the presence of input quantization. We study two common types of quantization, the uniform quantization and the logarithmic quantization. In both cases we provide the transient performance bounds, which are decoupled into two positive terms. One of these terms can be made arbitrarily small by increasing the rate of adaptation, while the other term can be made small by increasing the quantization density. The performance bounds imply that with sufficiently dense quantization and fast adaptation, the output of an uncertain MIMO nonlinear system can follow the desired reference input sufficiently closely. We notice that with $L_1$ adaptive control architecture fast adaptation does not lead to high-gain control and retains guaranteed time-delay margin, which is bounded away from zero. Simulations included in the paper illustrate the results.

Keywords: adaptive control, quantized systems, quantization errors

1. INTRODUCTION

The field of adaptive control emerged to provide solutions for control of systems in the presence of uncertainty and component failure. Despite the significant advances over the years, the transient performance analysis of adaptive controllers was reduced to persistence of excitation type assumption, or alternatively it was achieved via gain-scheduling, and with that defeating the point of adaptation. The recently developed $L_1$ adaptive controller has guaranteed transient and steady-state performance bounds without introducing persistency of excitation or gain-scheduling in the controller parameters. $L_1$ adaptive controller introduces a separation between adaptation and robustness, and for the first time allows for arbitrary increase of the rate of adaptation with guaranteed time-delay margin, bounded away from zero (Cao and Hovakimyan (2008, 2010); Hovakimyan and Cao (2010)). Similar to linear systems, with $L_1$ adaptive control architecture the performance of uncertain systems scales uniformly with all the variations in initial conditions, unknown parameters and reference inputs. In this paper we obtain performance bounds for the $L_1$ adaptive controller in the presence of input quantization.

As a brief review of the quantization literature (without adaptation), we note that the effect of quantization on stability was first investigated in Delchamps (1990), relaxation of the definition of stability was undertaken in Wong and Brockett (1999); Ishii et al. (2004), the design of coarse quantization was investigated in Elia and Mitter (2001), and stabilization of quantized systems was analyzed in Ishii and Başar (2005); Brockett and Liberzon (2000). In Sun et al. (2010), we obtained performance bounds for the $L_1$ adaptive controller for a SISO LTI system in the presence of input quantization. Here we extend that result to general nonlinear MIMO systems, without requiring any matching conditions. The issues addressed are not only the extension from SISO to MIMO systems, but also from linear to nonlinear, and from systems with matched uncertainties to those with both matched and unmatched uncertainties. For that, we refer to Xargay et al. (2010) for the general architecture of $L_1$ adaptive controller and analyze the performance bounds of this architecture in the presence of logarithmic as well as uniform quantization.

The paper is organized as follows. We formulate the problem in Section 2 and introduce the $L_1$ adaptive controller in Sections 3 and 4. Performances in the two cases of uniform and logarithmic quantization are studied in Sections 6 and 5, followed by simulations in Section 7. Section 8 concludes the paper.

Throughout the paper, we use $\| \cdot \|_\infty$ to denote the infinity norm of a vector, $\| \cdot \|_{L_\infty}$ for the $L_\infty$ norm of a signal, and $\| \cdot \|_{L_1}$ for the $L_1$ norm of a transfer function.

2. PROBLEM FORMULATION

We consider a general adaptive system with input quantization. As depicted in Figure 1, the state-feedback control designed (that is the output of the controller), $u_{qin}(t)$, is quantized before it is inputed to the plant, with that input denoted by $u_q(t)$. The controller itself consists of two tandem blocks: $L_1$ adaptive controller followed by a preprocessor which generates the input, $u_{qin}(t)$, to the quantizer.

The quantization block can be characterized by a map $Q : U_{qin} \rightarrow U_q, U_{qin} \subset \mathbb{R}, U_q \subset \mathbb{R}$, which maps a continuous range of values into a set of countably many elements. In
Now, given a piecewise continuous, bounded reference signal \( r(t) \), the control objective is to design an adaptive state feedback controller to ensure that \( y(t) \) tracks the output \( y_m(t) \) of the system \( y_m(s) = M(s)r(s) = C(sI - A_m)^{-1}B_mK_g(s)r(s) \), where \( K_g(s) \) is a feedforward prefilter.

3. L1 ADAPTIVE CONTROLLER

3.1 Notations

In the following analysis, we use the notations listed below. Let \( \mathcal{R}^{k \times l}(s) \) and \( \mathcal{R} \mathcal{H}^{k \times l}(s) \) denote the set of real rational proper transfer matrices of dimensions \( k \times l \) and the set of real rational proper and Hurwitz-stable transfer matrices of dimensions \( k \times l \), respectively.

For any \( \nu > 0 \), let

\[
L_{iv} = \frac{\varrho}{\nu}I_{f_1}(\varrho),
\]

where \( \varrho \equiv \nu + \bar{\gamma}_s \), and \( \bar{\gamma}_s > 0 \) is an arbitrary, small constant. Let \( H_{zm}(s) \equiv (sI_m - A_m)^{-1}B_m, \ H_{xm}(s) \equiv (sI_m - A_m)^{-1}B_m, \ H_{zm}(s) \equiv CH_{zm}(s) = C(sI_m - A_m)^{-1}B_m, \ H_{xm}(s) \equiv CH_{xm}(s) = C(sI_m - A_m)^{-1}B_m, \) and \( \tau_0(t) \) be the signal with Laplace transform \((sI_m - A_m)^{-1}x_0\). Since \( A_m \) is Hurwitz and \( x_0 \) is finite, \( \| \tau_0 \|_{\mathcal{L}_\infty} \) is bounded.

The design of \( \mathcal{L}_1 \) adaptive controller involves a strictly proper transfer matrix \( D(s) \in \mathcal{R}^{n \times m}(s) \) and a matrix gain \( K \in \mathcal{R}^{m \times n} \), which lead to a strictly proper stable transfer matrix \( C_u(s) = \omega K (\| l_m + D(s)\omega K \|^{-1} D(s) \) with DC gain \( C(0) = I_m \). The choice of \( D(s) \) needs to ensure also that \( C_u(s)H^{-1}(s) \) is a proper stable transfer matrix, i.e., \( C_u(s)H^{-1}(s) \in \mathcal{R}^{n \times m}(s) \). For a particular class of systems, a possible choice for \( D(s) \) might be \( D(s) = \Delta_{im} \), which yields a strictly proper \( C_u(s) \) of the form \( C_u(s) = \omega K (sI_m + \omega K)^{-1} \), with the condition that the choice of \( K \) must ensure that \(-\omega K \) is Hurwitz.

For proofs of stability and performance bounds, the choices of \( D(s) \) and \( K \) need to ensure that there exists \( \rho_{x_s} > \rho_0 \), such that

\[
\| G_m(s) \|_{\mathcal{L}_1} + \| G_m(s) \|_{\mathcal{L}_\infty} \tau_0 \leq \rho_{x_s}^{-1} \| H_{zm}(s)C_u(s)K_g(s) \|_1 \| r \|_{\mathcal{L}_1} \leq \| sI_m - A_m \|^{-1} \|_{\mathcal{L}_\infty} \]

\[
\frac{L_{1p_{x_s}}}{L_{1p_{x_s}} + B_0}, \quad G_m(s) = \| H_{zm}(s)C_u(s)H^{-1}(s)C \| H_{zm}(s), \]

\[
\| \tau_0 \|_{\mathcal{L}_\infty} \leq L_{2p_{x_s}}(s) \leq \max (B_0, \| B_m \|_{\mathcal{L}_\infty}), \] and \( K_g(s) \in \mathcal{R}^{n \times m}(s) \) is the feedforward prefilter.

For the analysis results in the following sections to hold, the inequality in (5) is only a sufficient condition and is required only for the real \( \omega \). However, since \( \omega \) is unknown, (5) cannot be used to guide the filter design. More conservatively, we can make the choices of \( D(s) \) and \( K \) to ensure that for all \( \omega \in \Omega \) there exists a constant \( \rho_{x_s} > \rho_0 \), such as (5) holds.

Define the constants

\[
\rho_{x_s} \equiv \| \frac{\varrho}{\nu}I_{f_1}(\varrho) \|_{\mathcal{L}_1} \leq L_{1p_{x_s}}, \quad \rho_{x_s} \equiv \min (B_0, \| B_m \|_{\mathcal{L}_\infty}), \] and \( K_g(s) \in \mathcal{R}^{n \times m}(s) \) is the feedforward prefilter.

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where $\omega_1 \max = \max_{\omega \in \Omega} \max_{i=1,\ldots,m} \|\omega_i\|_1$, and $\omega_i$ is the $i^{th}$ row of $\omega$.

Define the functions
\[ c_{de}(a, b) = \|G_{\omega}(s)\|_{L_1} + \|G_{um}(s)\|_{L_1}, \]
\[ c_{eu}(a, b) = \|\omega^{-1} C_{\omega}(s)\|_{L_1} + \|\omega^{-1} C_{u}(s)\|_{L_1} \]

3.2 $L_1$ Adaptive Controller

The $L_1$ adaptive controller consists of three elements: the state predictor, the adaptive law, and the control law. We now introduce these three elements, separately.

**State predictor:** The state predictor is generated by
\[ \dot{x}(t) = A_m \dot{x}(t) + B_m (\omega(t) u(t) + \theta(t)) x(t) + B_{um} (\theta_2(t) x_{\infty} + \sigma_1(t)), \]
\[ x(t) = x_0, \]
where $x(t) \in \mathbb{R}^{n+m}$, $\dot{x}(t) \in \mathbb{R}^m$, $\theta(t) \in \mathbb{R}^m$, $\dot{x}(t) \in \mathbb{R}^{n-m}$, and $\sigma_1(t)$ are the adaptive estimates.

**Adaptive laws:** The adaptation laws for $\omega(t)$, $\theta_1(t)$, $\sigma_1(t)$, $\dot{\theta}_2(t)$, and $\ddot{\theta}_2(t)$ are defined as
\[ \dot{\omega}(t) = \Gamma \text{Proj} \omega(t) - (\ddot{x}(t) P B_m^\top) u(t), \]
\[ \dot{\theta}_1(t) = \Gamma \text{Proj} \theta_1(t) - (\ddot{x}(t) P B_m^\top) x(t), \]
\[ \dot{\theta}_2(t) = \Gamma \text{Proj} \theta_2(t) - (\ddot{x}(t) P B_m^\top) x_{\infty}, \]
\[ \ddot{\sigma}_1(t) = \Gamma \text{Proj} \sigma_1(t) - \ddot{x}(t) P B_m^\top, \]
\[ \ddot{\theta}_2(t) = \Gamma \text{Proj} \theta_2(t) - \ddot{x}(t) P B_m^\top, \]
where $\omega(0) = \hat{\omega}_0$, $\theta_1(0) = \hat{\theta}_{10}$, $\sigma_1(0) = \hat{\sigma}_0$, $i = 1, 2$, $\ddot{x}(t) = \ddot{x}(t) - x(t)$, $\Gamma \in \mathbb{R}^+$ is the adaptation gain, $P = P^T > 0$ is the solution of the algebraic Lyapunov equation $A_m^T P + P A_m = -Q$, $Q = Q^T > 0$, and $\text{Proj} \cdot$ denotes the projection operator (Pomet and Praly (1992)).

**Control law:** We first design the control signal $u(t)$, and then the input for the quantizer $u_{qin}(t)$ is designed according to different types of quantizers. The control law is generated by
\[ u(t) = -K \chi(t), \]
which is the L1 adaptive control law.

Using the notations above, the following error dynamics can be derived from (21) and (11):
\[ \dot{x}(t) = A_m \dot{x}(t) + B_m (\omega(t) u(t) + \theta(t)) x(t) + B_{um} \eta_2(t), \]
where $x(t) = 0$. Next we show that if the adaptation gain $\Gamma$ is lower bounded by
\[ \Gamma > \frac{\theta_{m}(\omega_2)}{\lambda_{\min}(P) \gamma_0}, \]
then the truncated $L_\infty$ norm bound for $x(t)$ is proved later in Theorem 4. The following lemma summarizes this result.

**Lemma 3.** (Xargay et al. (2010)). Let the adaptation gain be lower bounded as in (26), and the projection be confined to the bounds in (27). Given the system in (2) and the $L_1$ adaptive controller defined by (11)-(13) subject to (5), if there exist $\rho_5$ and $\rho_6$, such that $\|x(t)\|_{L_\infty} \leq \rho_5$, $\|u(t)\|_{L_\infty} \leq \rho_6$, then we have $\|x(t)\|_{L_\infty} \leq \gamma_0$, where $\gamma_0$ was introduced in (26).

The proof is similar to Lemma 3 in Xargay et al. (2010) and is omitted.
5. LOGARITHMIC QUANTIZATION

Let $Q_{\log}$ be the quantization function of the logarithmic quantizer. If the input signal is $u_{qin}(t) \geq 0$, the quantization function $u_q(t) = Q_{\log}(u_{qin}(t))$ is defined by

$$u_q(t) = u_q, \quad \text{if } u_q \leq u_{qin}(t) < u_{q} + \epsilon,$$

where the constant $u_q > 0$ and the density constant $0 < \rho < 1$ are the parameters of the quantizer (Hayakawa et al. (2009)). For $u_{qin}(t) < 0$, $u_q(t) = -Q_{\log}(-u_{qin}(t))$. The logarithmic quantization function $Q_{\log}$ is shown in Figure 2.

![Logarithmic Quantization Function, $\Delta = 5$, $\rho = 0.7$](image)

Let $M_1 = 1$ and $M_2 = \rho$. We know from (28) that $M_2|u_{qin}| \leq |u_q| \leq M_1|u_{qin}|$. Thus, $|u_q - M_1|u_{qin}| |u_{qin}| \leq M_2 - M_2|u_{qin}|$. Since $u_q$ and $u_{qin}$ always have the same sign, we have

$$|u_q - M_1|u_{qin}| \leq M_2 - M_2|u_{qin}|. \tag{29}$$

Following Hayakawa et al. (2009), we choose

$$u_q = \frac{M_1 - M_2}{M_1 + M_2} u_{qin}. \tag{30}$$

Let

$$u_{qin} = \frac{M_1}{M_1 + M_2} u. \tag{31}$$

Since $u_q = u + u_{qe}$ and $M_2|u_{qin}| \leq |u_q| \leq M_1|u_{qin}|$, we have

$$|u_{qin}| \leq \Delta |u_q| \leq \Delta |u| \tag{32}$$

where $\Delta = \frac{M_1 - M_2}{M_1 + M_2}$. Since the inequality above holds for all $t \in (0, \infty)$, we have $|u_{qin}| \leq \Delta |u_q| \leq \Delta |u|$.

Note that $\Delta = \frac{M_1 - M_2}{M_1 + M_2} = \frac{1}{1 + \rho}$ is also a constant representing the coarseness of the quantizer as $\rho$ in (28). When the quantizer is finer, $\rho$ increases and $\Delta$ decreases, and when the quantizer is coarser, $\rho$ decreases and $\Delta$ increases, with $0 < \rho < 1$, $0 < \Delta < 1$.

Given the quantization density $\Delta$, we introduce some more notation. Let $\rho_{\log}$ be defined as

$$\rho_{\log} \triangleq \rho_x + \gamma_{\log}, \tag{33}$$

where $\gamma_{\log} > 0$ is a small constant. Also define

$$\gamma_{\log} \triangleq \gamma_{x_{\log}} + \gamma_{q_{\log}} + \epsilon, \tag{34}$$

and $\Delta$ are sufficiently small so that $\gamma_{x_{\log}} = \gamma_{x_{\log}} + \gamma_{q_{\log}} + \epsilon \leq \gamma_{x_{\log}}$. Similarly, let

$$\rho_{u_{\log}} \triangleq \rho_{u} + \gamma_{u_{\log}}, \tag{35}$$

$$\gamma_{u_{\log}} \triangleq \frac{1}{1 + \Delta|u_{x_{\log}}|} \left[ |\epsilon| \left| C_{u_{\log}}(L_4 p_{x_{\log}}, L_2 p_{u_{\log}}) + 1 + 2\epsilon_1 |u_{x_{\log}}| \right| C_{u_{\log}}(L_4 p_{x_{\log}}, L_2 p_{u_{\log}}) + 1 + 2\epsilon_1 |u_{x_{\log}}| \right] \Delta \rho_{u_{\log}}. \tag{36}$$

Theorem 4. Let the adaptive gain be lower bounded as in (26) and the projection be confined to the bounds in (27). Given the closed-loop system with the L1 adaptive controller defined via (11)-(13), subject to the $L_1$-norm condition in (5), and the closed-loop reference system in (16), if $||x_0|| \leq \rho_x$, then we have

$$\|x|| \leq \rho_x, \quad \|u|| \leq \rho_u, \quad \|\bar{x}|| \leq \gamma_0, \tag{37}$$

where $\gamma_{x_{\log}}$ and $\gamma_{u_{\log}}$ are defined in (34) and (36).

**Proof.** See Appendix.

Remark 5. The performance bounds for the state and the control signal in (37) contain a term linear in the quantization parameter $\Delta$. When $\Delta$ goes to zero and there is no quantization, this term vanishes and the bounds reduce to the ones in Theorem 1 of Xargay et al. (2010).

Remark 6. To obtain the result in Theorem 4, the selection of $C_{u}(s)$ should satisfy the condition in (5). Without the unmatched uncertainty, $G_{unif}(s) = 0$, this condition can always be satisfied by increasing the bandwidth of $C_u(s)$. However, in the presence of unmatched uncertainty, the existence of $\rho_x$ is not obvious. When the unmatched uncertainty is small, the left-hand-side is small and $\rho_x$ can be determined. If the unmatched uncertainty increases, the positive invariant set shrinks, which agrees with intuition.

6. UNIFORM QUANTIZER

For a uniform quantizer, every quantization interval has a fixed length. Let $Q_{\text{unif}}$ be the quantization function for the uniform quantizer. If the control signal is $u_{qin}(t) \geq 0$, the quantization function $u_q(t) = Q_{\text{unif}}(u_{qin}(t))$ is defined by

$$u_q(t) = u, \quad \text{if } u - \frac{l}{2} \leq u_{qin}(t) \leq u + \frac{l}{2}, \tag{38}$$

where $u_{q+1} = u + l$, $u_0 > 0$ is a constant, and $l$ is the length of the quantization interval. For $u_{qin}(t) < 0$, $u_q(t) = -Q_{\text{unif}}(-u_{qin}(t))$. This uniform quantization function $Q_{\text{unif}}$ is shown in Figure 3.

![Uniform Quantization Function, $l = 2$](image)
Recall that \( u_{q}(t) = u(t) - u(t) \) as in (31). If we pick \( u_{q}(t) = u(t), \) then from (38) we have the quantization error bound
\[
\|u_{q}\|_{\infty} \leq \frac{1}{l},
\]
which shows that the error introduced by a uniform quantizer is bounded by a constant.

Given the quantization interval length \( l, \) we introduce some more notations. Let \( \rho_{\text{unif}} \) be defined as
\[
\rho_{\text{unif}} \triangleq \rho_{x} + \gamma_{\text{unif}},
\]
\[
\gamma_{\text{unif}} \triangleq \gamma_{x} + \gamma_{q_{\text{unif}}} + \epsilon,
\]
\[
\gamma_{z_{\text{unif}}} \triangleq \frac{1-\epsilon_{d}(\delta, L_{2}x_{\text{f}}, L_{2}x_{\text{f}})}{1-\epsilon_{d}(\delta, L_{2}x_{\text{f}}, L_{2}x_{\text{f}})} l,
\]
where \( \epsilon \leq \gamma_{\text{unif}} \) is a small positive constant, \( \gamma_{x}, \gamma_{c}, \) and \( \epsilon_{d}(a, b) \) are defined in (9) and (10), and \( \gamma_{0} \) and \( \Delta \) are sufficiently small, so that \( \gamma_{z_{\text{unif}}} = \gamma_{x_{\text{unif}}} + \gamma_{q_{\text{unif}}} + \epsilon \leq \gamma_{x_{\text{unif}}} \).

Similarly, let
\[
\rho_{\text{unif}} \triangleq \rho_{x} + \gamma_{x_{\text{unif}}},
\]
\[
\gamma_{\text{unif}} \triangleq \gamma_{x} + \gamma_{q_{\text{unif}}} + \epsilon,
\]
\[
\gamma_{z_{\text{unif}}} \triangleq \frac{1-\epsilon_{d}(\delta, L_{2}x_{\text{f}}, L_{2}x_{\text{f}})}{1-\epsilon_{d}(\delta, L_{2}x_{\text{f}}, L_{2}x_{\text{f}})} l,
\]
\[
\epsilon \leq \gamma_{\text{unif}} \] is a small positive constant, \( \gamma_{x}, \gamma_{c}, \) and \( \epsilon_{d}(a, b) \) are defined in (9) and (10), and \( \gamma_{0} \) and \( \Delta \) are sufficiently small, so that \( \gamma_{z_{\text{unif}}} = \gamma_{x_{\text{unif}}} + \gamma_{q_{\text{unif}}} + \epsilon \leq \gamma_{x_{\text{unif}}} \).

We have the following theorem.

**Theorem 7.** Let the adaptation gain be lower bounded as in (26) and the projection be confined to the bounds in (27). Given the closed-loop system with the \( L_{1} \) adaptive controller defined by (11)-(13), subject to the \( L_{1} \)-norm condition in (5), and the closed-loop reference system in (16), if \( \| x(0) \|_{\infty} \leq \rho_{0} < \rho_{x} \), then we have
\[
\| x_{\text{ref}} \|_{\infty} \leq \rho_{x},
\]
\[
\| x_{\text{ref}} \|_{\infty} \leq \rho_{x_{\text{unif}}},
\]
\[
\| \tilde{x} \|_{\infty} \leq \gamma_{x_{\text{unif}}},
\]
\[
\| x \|_{\infty} \leq \| x_{\text{ref}} \|_{\infty} \leq \rho_{x_{\text{unif}}},
\]
\[
\| x - x_{\text{ref}} \|_{\infty} \leq \gamma_{x_{\text{unif}}},
\]
\[
\| u_{q} - u_{\text{ref}} \|_{\infty} \leq \rho_{u_{\text{unif}}},
\]
\[
\| x - y_{\text{ref}} \|_{\infty} \leq \gamma_{x_{\text{unif}}},
\]
\[
\| u_{\text{ref}} \|_{\infty} \leq \| x_{\text{ref}} \|_{\infty} \leq \rho_{x_{\text{unif}}},
\]
\[
\| u_{\text{ref}} \|_{\infty} \leq \| x_{\text{ref}} \|_{\infty} \leq \rho_{x_{\text{unif}}},
\]
\[
\| x - x_{\text{ref}} \|_{\infty} \leq \gamma_{x_{\text{unif}}},
\]
\[
\| y - y_{\text{ref}} \|_{\infty} \leq \rho_{\text{unif}},
\]
The proof is similar to that of Theorem 4 and is thus omitted.

**Remark 8.** The performance bounds in (45) are decoupled into two terms, one of which depends linearly on \( l, \) representing the length of the quantization interval for the uniform quantizer, and the other one is independent of the quantizer’s parameters. When \( l \) decreases to zero and there is no quantization, the corresponding term vanishes, and the performance bounds reduce to the ones in Theorem 1 in Xurgay et al. (2010).

### 7. SIMULATIONS

Consider the system
\[
\dot{x}(t) = (A_{m} + A_{\Delta}) x(t) + B_{m} \omega_{q}(t) + f_{\Delta}(x(t), t),
\]
\[
y(t) = C x(t), \quad x(0) = x_{0},
\]
\[
u_{q}(t) = Q(u_{\text{ref}}(t)),
\]
where
\[
A_{m} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix}, \quad B_{m} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},
\]
while \( A_{\Delta} \in \mathbb{R}^{3 \times 3} \) and \( \omega \in \mathbb{R}^{2 \times 2} \) are unknown constant matrices satisfying
\[
\| A_{\Delta} \|_{\infty} \leq 1, \quad \| \omega \|_{\infty} \leq 1.
\]

For example, we pick
\[
A_{\Delta} = \begin{bmatrix} 0.2 & -0.3 & 0.1 \\ 0.4 & 0.3 & 0.1 \\ -0.1 & 0 & -0.9 \end{bmatrix}, \quad \omega = \begin{bmatrix} 0.9 & 0.2 \\ 0.1 & 1.1 \end{bmatrix},
\]
and let \( f_{\Delta} \) be the following (unknown) nonlinear function
\[
f_{\Delta}(x, t) = \begin{bmatrix} \frac{1}{3} x_{2}^{2} + \tanh(\frac{1}{2} x_{1}) x_{1} \\ \frac{1}{2} \cosh(x_{2}) x_{2} - \frac{1}{2} x_{3} \left(1 - e^{-0.3t}\right) \end{bmatrix}.
\]

Figure 4 shows the cases where the reference signal is the sum of steps. In both cases, the reference signal has different amplitudes in different channels, and the system output tracks the desired signal. When the quantization is coarse, the tracking error is larger. When the quantization is sufficiently dense, the tracking error is reduced and is close to the case without quantization.

These figures also show the tracking of different step signals. For different amplitudes of steps, the controller provides scaled control signals and scaled system outputs, per the properties guaranteed by \( L_{1} \) adaptive control theory Hovakimyan and Cao (2010).
Figure 5 shows the system response to sinusoidal reference signals. The uniform quantization gives similar results and the figures are omitted here. The $L_1$ adaptive controller ensures that the system output tracks the desired signal closely and smoothly. We notice that in all these cases we did not redesign or retune the $L_1$ adaptive controller.

8. CONCLUSION

This paper has extended the analysis of $L_1$ adaptive controller to a class of uncertain MIMO nonlinear systems with input quantization. We can foresee extensions to systems with both input and state quantization, as well as to underactuated systems, as directions for future work.

REFERENCES


APPENDIX

Proof of Theorem 4. (By contradiction) Assume that the bounds in (37) do not hold. Then since $\|x(t) - x(t')\|_\infty < \gamma_{u,q}(0)$, $\|u(t) - u(t')\|_\infty < \gamma_{u,q}(0)$ and $x(t)$, $x(t')$, $u(t)$, and $u(t')$ are continuous, there exists $t'$ such that

$$\|x(t') - x(t')\|_\infty \leq \gamma_{\mu,0}(t) \quad \text{and} \quad \|u(t') - u(t')\|_\infty \leq \gamma_{\mu,0}(t),$$

for $t' \in [0,T]$, which implies that

$$\|x(t') - x(t')\|_\infty \leq \gamma_{\mu,0}(t), \quad \|u(t') - u(t')\|_\infty \leq \gamma_{\mu,0}(t).$$

Then Lemma 1 implies that

$$\|x(t') - x(t')\|_\infty \leq \rho_{\mu,0}, \quad \|u(t') - u(t')\|_\infty \leq \rho_{\mu,0}.$$ (48)

Using the definitions of $\rho_{\mu,0}$ and $\rho_{\mu,0}$ in (33) and (35), together with the bounds in (47) and (48), we have

$$\|x(t') - x(t')\|_\infty \leq \rho_{\mu,0} + \gamma_{\mu,0}, \quad \|u(t') - u(t')\|_\infty \leq \rho_{\mu,0} + \gamma_{\mu,0} \leq \rho_{\mu,0}.$$ (49)

Hence, if one chooses the adaptive gain according to (26) and the projection is confined to the bounds in (27), Lemma 3 implies that $\|x(t') - x(t')\|_\infty \leq \gamma_{\mu,0}$. Next, let $\tilde{y}(t) = \tilde{w}(t)(u(t) + \theta_{u}(t)) - \tilde{w}(t)(u(t) + \theta_{u}(t))$, where $\tilde{w}(t)$ is the signal with Laplace transform $\tilde{w}(s) = H_{\mu,0}(s)H_{\mu,0}(s)\tilde{w}(s)$, and $\tilde{y}(t)$ and $\tilde{y}(t)$ are defined in (24). It follows from (13) that

$$\chi(s) = D(s)(\omega u(s) + \eta(s) + H_{\mu,0}^{-1}(s)H_{\mu,0}(s)\eta(s) - \theta_{u}(s)), \quad \tilde{y}(t) + \tilde{y}(t)$$

are the Laplace transforms of the signals $\eta(t)$, $\eta(t)$, and $\tilde{y}(t)$ are defined in (24), and $\tilde{y}(t)$. Consequently

$$\chi(s) = \left(\hat{a} + D(s)\omega K - \hat{D}(s)\eta(s) + H_{\mu,0}^{-1}(s)H_{\mu,0}(s)\eta(s) - \theta_{u}(s)\right)\tilde{y}(t) + \tilde{y}(t).$$

Next, let $\epsilon(t) = x(t) - x(t)$. From (16) and (50) we have

$$e(t) = G_{\mu,0}(s)(\eta(s) - \eta(s)) + \tilde{G}_{\mu,0}(s)(\eta(s) - \tilde{y}(t)) + \tilde{y}(t) + \tilde{y}(t) \leq \gamma_{\mu,0}(t).$$

Moreover, it follows from the error dynamics in (25) that

$$\gamma_{\mu,0}(t)C_{\mu,0}(s) \leq \gamma_{\mu,0}(t)C_{\mu,0}(s) + \tilde{y}(t) + \tilde{y}(t) \leq \gamma_{\mu,0}(t).$$

For logarithmic quantizer, the quantization error is bounded by

$$\|e(t)\|_\infty \leq \gamma_{\mu,0}(t) \quad \text{for all} \quad t \in [0,T].$$

Thus, the inequality above and (59) contradict the assumption in (46) that the tracking errors hit $\gamma_{\mu,0}(t)$ or $\gamma_{\mu,0}(t)$. Hence the error is upper bounded and (37) holds.

□