Synchronization of multi-agent systems with topological interaction

Lin Wang ∗ Xiaofan Wang ∗ Xiaoming Hu ∗∗

∗ Department of Automation, Shanghai Jiao Tong University, Shanghai, China, (e-mail: wanglin@sjtu.edu.cn, zfwang@sjtu.edu.cn).
∗∗ Optimization and Systems Theory, Royal Institute of Technology (KTH), Stockholm, Sweden, (e-mail: hu@kth.se)

Abstract: The recent research by a group of European scientists shows that, each starling bird adjusts its flight in the light of topological interaction, that according to a fixed number of its nearest neighbors, rather than all agents within a fixed Euclidean distance. A theoretical study to understand such topological interaction is given in this paper, and a sufficient synchronization condition imposing only on the initial states is developed. Furthermore, the number of neighbors needed for the synchronization is discussed, which shows that for the sake of connectivity the number of neighbors should increase as the density of group increasing.

Keywords: network topologies, distributed non-linear elements, synchronization

1. INTRODUCTION

Collective behavior of large number of locally interacting agents has greatly interested scientists and researches from diverse fields, including biology (Ballerini et al., 2008), physics (Visek et al., 1995), mathematics (Cao et al. 2008), computer science (Reynolds, 1987), system and control sciences (Jadbabaie et al., 2003; Moreau, 2005; Olfati-Saber, 2006). Nowadays a well-accepted idea is that individual agents subject only to local behavioral rules can self-organize into complex coherent groups, and can perform complicated tasks.

Many local behavior rules and models have been presented to describe and discuss the mechanisms of flying, swimming, and migrating in animal aggregations (Krause and Ruxton, 2002), and many distributed algorithms have been proposed for autonomous vehicles and sensors to perform coordinated tasks in hazardous environment (Ren and Beard, 2008). To grant cohesion, models make the sound assumption that individuals align and attract each other, and that such interaction usually related with the distance between individuals. A large number of models adopt a definition of metric interaction, that each agent interacts with all agents within a fixed Euclidean distance. An example of such metric interaction is the Boid model created by Reynolds, where three simple local rules, namely separation, cohesion and alignment, have successfully imitated many behaviors of flocks (Reynolds, 1987). A number of theoretical studies related to these rules have been developed to show how simple rules can enable coherence and coordination of group (Olfati-Saber, 2006; Su et al., 2009; Wang and Wang, 2010). Another simpler model is the self propelled particle model proposed by Visek et al. (1995), for which many theoretical results have been developed as well: Jadbabaie et al. (2003) pointed out that the system will synchronize if the associated networks are infinitely-often jointly connected; Tang and Guo (2007) studied the model in a random framework, and proved that the system will synchronize with large probability as long as the density of the group is large enough. Many generalizations and variations have been devoted to similar models, where connectivity of group has been proved to be crucial for collective performances (Cao et al., 2008; Moreau, 2005; Olfati-Saber, 2006; Ren and Beard, 2008). Is the metric interaction powerful enough to keep connectivity when the group expanding, contracting, continuously changing its structure and density? What kind of interaction could maintain cohesion in such a robust way?

To answer the above questions, Ballerini et al. performed an observational study of flocking behavior in starlings. By reconstructing the three dimensional positions of individual birds and analyzing the empirical data, they show that each bird interacts on average with a fixed number of agents (six to seven), rather than with all neighbors within a fixed Euclidean distance (Ballerini et al., 2008). That is the interaction between starlings depend on the topological distance rather than the Euclidean distance. In the topological case, the strength of the interaction could remain the same at different densities. By means of empirical observations and simulations, they show that comparing with the metric interaction, the topological interaction is more suitable to keep cohesion in the face of strong density fluctuations. However, theoretical results on the topological interaction have generally been given less focus.

In this paper, we perform a theoretical study for multi-agent systems with topological interaction. A mathematical model with each agent updating its heading based on a fixed number of neighbors is proposed. A sufficient conditions for the synchronization of the group is developed, which shows the relationship between the speed,
the heading and the density of the group. Furthermore, the needed number of neighbors for the synchronization is discussed in a random framework, which shows that for the sake of connectivity, the number of neighbors should increase as the density of group increasing.

The rest of this paper is as follows: Section 2 describes the model and presents some preliminaries; The sufficient synchronization conditions are established in Section 3, and the needed number of neighbors is discussed in Section 4; Section 5 provides some simulations to show the robustness of the topological interaction by comparing with the metric interaction; Section 6 draws a conclusion.

2. MODEL DESCRIPTION AND PRELIMINARIES

2.1 Model Description

The topological distance is qualified by how many intermediate individuals separate two agents, not the Euclidean distance between them. The crucial difference between topological and metric interaction becomes obvious when the density of the group varies. In two groups of agents with different densities, the number of interacting neighbors is the same, but the metric range of interaction is different as shown in Fig. 1.

![Fig. 1. Two groups of agents with different densities.](image)

In this paper, we will analyze the topological interaction theoretically by considering a simple multi-agent system. The model consists of $n$ autonomous agents labeled by $1, 2, \ldots, n$. Each agent moves in the plane with the same absolute velocity $v$ and with the heading updated according to the average direction of its neighbors. Here, the neighbors means the nearest $m$ individuals from its recent position. Let $(x_i(t), y_i(t)) \in \mathbb{R}^2$ be the position of agent $i$, $\theta_i(t) \in (0, 2\pi)$ be the heading. The dynamics of agent $i$ can be described by

$$
\begin{cases}
x_i(t + 1) = x_i(t) + v \cos \theta_i(t) \\
y_i(t + 1) = y_i(t) + v \sin \theta_i(t),
\end{cases}
$$

(1)

$$
\theta_i(t + 1) = \frac{1}{m + 1} \left( \sum_{j \in N_i(t)} \theta_j(t) + \theta_i(t) \right),
$$

(2)

where $N_i(t)$ is the neighbor set defined as follows:

$$
N_i(t) = \{ j : \text{agent } j \text{ is one of the nearest } m \text{ individuals from agent } i, \ m \leq n - 1 \}.
$$

If at time $t$ there is more than one agent can be treated as the $m$-th nearest neighbor of agent $i$, then agent $i$ chooses the one who is the nearest at time $t - 1$. Since the topology interaction is not symmetrical, the neighbor relationship between agents is asymmetrical.

For the above model, we will investigate the topological interaction and develop sufficient conditions for the synchronization of the group. Here, *synchronization* means the headings of all agents will asymptotically move in the same direction, namely, $\lim_{t \to \infty} ||\theta_i(t) - \theta_j(t)|| = 0, \ i, j = 1, \ldots, n$.

Furthermore, we will discuss how many neighbors are sufficient for such synchronization.

2.2 Preliminary

The neighbor relationships between agents can always be represented by directed graphs. A directed graph (digraph) $G = (V, E)$ consists of a vertex set $V$ and an edge set $E = \{(i, j)\}$, where $V = \{1, 2, \ldots, n\}$ is composed of the indices of all agents, and an edge $(i, j) \in E$ means agent $i$ is a neighbor of agent $j$. A path that connects $i$ and $j$ in a digraph $G$ is a sequence of distinct vertexes $i_0, i_1, i_2, \ldots, i_k$, where $i_0 = i, i_k = j$ and $(i_l, i_{l+1}) \in E, 0 \leq l \leq k - 1$.

A digraph is called strongly connected if for every pair of distinct vertexes there is a path connecting them. A digraph is said to have a *spanning tree* if and only if there exists a vertex $i \in V$, called root, such that there is a path from $i$ to any other vertex.

Some notations from nonnegative matrix theory are also useful (Seneta, 1984),(Horn and Johnson, 1985). A matrix is nonnegative (positive) if all its entries are nonnegative (positive). Moreover, if the sum of each row satisfies $\sum_{j=1}^{n} a_{ij} = 1, i = 1, \ldots, n$, the matrix is called stochastic. A characterization of synchronization for a stochastic matrix $B = [b_{ij}]_{n \times n}$ can be defined as follows:

$$
\tau(B) = \frac{1}{2} \max_{i,j} \sum_{s=1}^{n} |b_{is} - b_{js}|,
$$

(4)

which quantifies the difference among rows of the matrix. From Seneta (1984), we know that for any two stochastic matrices $A$ and $B$,

$$
\tau(AB) \leq \tau(A)\tau(B).
$$

(5)

Let $\Delta$ be an operator that $\Delta w = \max_{i,j} |w_i - w_j|$ for vector $w = [w_1, \ldots, w_n]$. Furthermore, function $\tau(\cdot)$ has the following property:

**Lemma 1.** (Seneta, 1984) For a vector $y = [y_1, \ldots, y_n]^T \in \mathbb{R}^n$ and a stochastic matrix $B = [b_{ij}]_{n \times n}$, if $z = By$, then

$$
\Delta z \leq \tau(B)\Delta y.
$$

3. SYNCHRONIZATION CONDITIONS

In this section, we will present some sufficient synchronization conditions that only imposed on the system parameters and the initial states.

Let $\theta(t) = [\theta_1(t), \theta_2(t), \ldots, \theta_n(t)]$. Define a system matrix $P(t) = [P_{ij}(t)]$ with

$$
P_{ij}(t) = \begin{cases}
\frac{1}{m+1}, & j = i \ or \ j \in N_i(t) \\
0, & \text{otherwise}.
\end{cases}
$$

(6)

Then, we can rewrite (2) to be
\[ \theta(t + 1) = P(t)\theta(t). \]  
(7)

The main result can be described as follows:

**Theorem 2.** Consider the model (1),(2), assume the initial neighbor graph includes a spanning tree. Let \( P(0) \) denote the initial system matrix, assume \( \alpha \) is the smallest integer such that \( \tau(P(0)\alpha) < 1 \). For the initial state, let \( r_i \) be the distance between agent \( i \) and its \( m \)-th nearest neighbor, and let \( R_i(\eta) \) be the number of agents that locate in the ring \( R_i(\eta) \equiv \{(x, y) | r_i \leq ((x-x_i(0))^2 + (y-y_i(0))^2)^{1/2} \leq r_i + \eta \) except the neighbors of agent \( i \) (see Fig. 3). Then, the system will synchronize eventually if there exists a constant \( \eta \geq 0 \) such that

\[ \lambda + \alpha\epsilon < 1, \]  
(8)

\[ \max_{i,j} \{R_i(\eta)\} \geq \frac{\eta}{2}, \]  
(9)

where \( \lambda = \tau(P(0)\alpha) \), \( \Delta\theta(0) = \max_{i,j} |\theta(0)_{i} - \theta(0)_{j}| \), \( \alpha = 2 \max_{i \in \{1, \ldots, n\}} \{R_i(\eta)\}/m + 1 \).

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**Fig. 2.** Illustration of the topological interaction with seven neighbors. Each agent has seven neighbors, and different agent has different interaction range \( r_i \).

We can see that the synchronization conditions are only imposed on the initial states. As an illustration of the rationality of the above conditions, we give the following simple example.

**Example.** For the model (1), (2), assume \( m = n - 1 \), i.e. each agent treats all the other agents as neighbors. It is obvious that every agent will reach the same heading after one step. Now, we verify the conditions of Theorem 2. In this case, \( P = \frac{1}{n}1 \cdot 1^T \), \( \alpha = 1 \), \( \lambda = 0 \), and \( \epsilon = 0 \) for any \( \eta > 0 \). If we choose the constant \( \eta \) to be \( \eta = 2\epsilon\Delta\theta(0) + 1 \), then (9) is satisfied. Therefore, the group will synchronize eventually by virtue of Theorem 2.

**Claim 3.** Theorem 2 presents some relationships between the speed \( v \), the initial headings, and the density of the group. From condition (9), we can see the result comes with the following intuitions: 1) If all the agents have the same initial heading, then the speed \( v \) can be arbitrarily large; 2) If the density of the group is low, then \( \eta \) might be chosen large accordingly, which could permit a high speed for the agent; 3) If the density is high and the difference of headings is big, then high speed would easily cause isolated clusters in which agents are neighbors each other.

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In order to prove Theorem 2, we need the following lemmas.

**Lemma 4.** Consider a system

\[ z(t + 1) = A(t)z(t), \]  
(10)

where \( z \in \mathbb{R}^n \), and \( \{A(t)\}_{t=0}^{\infty} \) is a sequence of stochastic matrices. If there exist a stochastic matrix \( A \) and a constant \( \alpha \) such that

\[ \tau(A^\alpha) < 1, \quad \tau(A(t) - A) \leq \epsilon, \quad \forall t \geq 0. \]  
(11)

Then,

\[ \Delta z(k\alpha + i) \leq (\lambda + \epsilon\alpha)^k \Delta z(i), \quad \forall i \geq 0, k \geq 0, \]  
(12)

where \( \lambda = \tau(A^\alpha) \).

**Proof.** For any \( i \geq 0, k \geq 0, \)

\[ z(k\alpha + i) = A(k\alpha + i - 1)z(k\alpha + i - 1) \]  
(13)

\[ = A^{k\alpha}z(i) + \sum_{s=0}^{k\alpha} A^{k\alpha+s-1-s}(A(s) - A)z(s) \]

By Lemma 1, we have

\[ \Delta z(k\alpha + i) \leq \Delta z(i) \]  
(14)

\[ + \sum_{s=1}^{k\alpha+1} \tau(A^{k\alpha+s-1-s}) \tau(A(s) - A) \Delta z(s) \]  
(15)

\[ \leq \lambda \Delta z(i) + \epsilon \sum_{j=0}^{k\alpha} \tau(A^{k\alpha-1-j}) \Delta z(j\alpha + i), \]  
(16)

where we use the fact \( \Delta z(h+s) \leq \tau(A(h+s) \cdots A(h)) \Delta z(h) \leq \Delta z(h), \forall s \geq 0, h \geq 0. \)

Let \( y(\eta) = \Delta z(j\alpha + i) \), then (14) can be transformed into

\[ y(\eta) \leq \lambda \Delta y(0) + \epsilon \sum_{s=0}^{k\alpha} \lambda^{k\alpha-1-s} y(s). \]  
(15)

Next, two cases are considered:

Case 1: if \( \lambda \neq 0, \) i.e. \( \tau(A^\alpha) \neq 0. \)

Multiply \( \lambda^{-k} \) on both sides of (15). Denote \( \xi(\eta) = \lambda^{-\eta}y(\eta) \), then we have

\[ \xi(\eta) \leq \xi(0) + \epsilon \sum_{s=0}^{k\alpha} \lambda^{-s-1} \xi(s). \]  
(16)

Let \( \beta(\eta) = \xi(0) + \epsilon \sum_{s=0}^{k\alpha} \lambda^{-s-1} \xi(s) \), then \( \beta(0) = \xi(0) \), and

\[ \beta(\eta) = \beta(\eta - 1) + \epsilon \lambda^{-1} \xi(\eta - 1) \]  
(17)

\[ \leq (1 + \epsilon \lambda^{-1}) \beta(\eta - 1) \leq (1 + \epsilon \lambda^{-1})^k \beta(0). \]  
(18)

Thus,
\[ \xi(k) \leq \beta(k) \leq (1 + \epsilon \alpha \lambda^{-1})^{-1} \beta(0) = (1 + \epsilon \alpha \lambda^{-1})^{-1} \xi(0). \]  
(18)

Therefore,

\[ y(k) = \lambda^k \xi(k) \leq (\lambda + \epsilon \alpha) y(0). \]  
(19)

Case 2: if \( \lambda = 0 \), i.e. \( \tau(P) = 0 \).

For any \( s \geq \alpha \), from the properties of \( \tau(\cdot) \), we know that

\[ \tau(P^s) \leq \tau(A^{s-\alpha}) \tau(A^\alpha) = 0. \]

By (15), \( y(k) \leq \epsilon \alpha \cdot y(k-1) = (\lambda + \epsilon \alpha) y(k-1), \) which can deduce

\[ y(k) \leq (\lambda + \epsilon \alpha)^k y(0). \]

Therefore, in both cases we have \( y(k) \leq (\lambda + \epsilon \alpha)^k y(0). \)

That is

\[ \Delta z(ka + i) \leq (\lambda + \epsilon \alpha)^k \Delta z(i). \]

Lemma 5. Consider the model (1),(2) under the same initial conditions as in Theorem 2. Then, for any time \( t \)

\[ \tau(P(t) - P(0)) \leq \frac{2 \max \{R_i(\eta)\}}{m + 1} = \epsilon, \]

(21)

where \( P(t) \) is the system matrix defined by (6).

**Proof.** Let \( d_{ij}(t) = [(x_i(t) - x_j(t))^2 + (y_i(t) - y_j(t))^2]^{1/2} \).

From (1), we have

\[ d_{ij}^2(t + 1) = [(x_i(t + 1) - x_j(t + 1))^2 + (y_i(t + 1) - y_j(t + 1))^2] \]

\[ = [(x_i(t) - x_j(t) + v(\cos \theta_i(t) - \cos \theta_j(t))]^2 \]

\[ + [(y_i(t) - y_j(t) + v(\sin \theta_i(t) - \sin \theta_j(t))]^2 \]

\[ \leq \{d_{ij}(t) + 2v|\sin \theta_i(t) - \theta_j(t)|\}^2 \]

Therefore,

\[ d_{ij}(t + 1) \leq d_{ij}(t) + 2v|\sin \theta_i(t) - \theta_j(t)| \]

\[ \leq d_{ij}(t) + 2v \max_{i,j} |\theta_i(t) - \theta_j(t)| \]

\[ = d_{ij}(t) + 2v \Delta \theta(t) \]

\[ \leq d_{ij}(0) + v \sum_{s=0}^{t} \Delta \theta(s). \]

(22)

In a similar way, we have

\[ d_{ij}(0) \leq d_{ij}(t + 1) + v \sum_{s=0}^{t} \Delta \theta(s). \]

(23)

Thus, for \( t = 1, 2, \cdots, \alpha \),

\[ d_{ij}(t) \leq d_{ij}(0) + v \sum_{s=0}^{t-1} \Delta \theta(s) \]

\[ \leq d_{ij}(0) + v \alpha \Delta \theta(0) \]

\[ < d_{ij}(0) + \frac{\eta}{2}. \]

\[ d_{ij}(0) \leq d_{ij}(t) + v \sum_{s=0}^{t-1} \Delta \theta(s) < d_{ij}(t) + \frac{\eta}{2}. \]

(25)

From (24), we know that all the neighbors of agent \( i \) at \( t = 0 \) will locate in a circle \( C = \{(x, y)|((x - x_i(t))^2 + (y - y_i(t))^2)^{1/2} \leq r_i + \frac{\eta}{2}\} \) at time \( t, t = 1, \cdots, \alpha \). That is at time \( t \) there are at least \( m \) agents in the circle \( C \). On the other hand, from (25), we can see that all the agents with \( d_{ij}(0) \geq r_i + \eta \) will satisfy \( d_{ij}(t) > r_i + \frac{\eta}{2} \) for \( t = 1, \cdots, \alpha \). Therefore, agents satisfying \( d_{ij}(0) \geq r_i + \eta \) can not be the neighbors of agent \( i \). Since each agent only has \( m \) neighbors, based on the above analysis, we know that at time \( t, t = 1, \cdots, \alpha \), the variation of neighbors for any agent \( i \) cannot exceed the number of agents in the ring \( \{\eta \} = \{(x, y)|r_i \leq ((x - x_i(0))^2 + (y - y_i(0))^2)^{1/2} \leq r_i + \eta\} \) except the initial neighbor \( j \) with \( d_{ij}(0) = r_i \). From

\[ \tau(A) = \frac{1}{2} \max_{i,j} |A_{ij} - A_{ji}|, \]

we have for \( t = 1, \cdots, \alpha, \)

\[ \tau(P(t) - P(0)) \leq \frac{2 \max \{R_i(\eta)\}}{m + 1} = \epsilon. \]

Next, we assume that (21) is true for \( t < T \). Denote \( T - 1 = \beta \alpha + c \) with integers \( b \geq 0, c \geq 0 \). From Lemma 4, we know that

\[ \Delta \theta(\eta k) \leq (\lambda + \epsilon \alpha)^{\Delta \theta(\eta)}, \]  
\( \forall 0 \leq k \leq b. \)  
(26)

Since \( \{P(t)\} \) is stochastic matrixes, \( \{\Delta \theta(\eta)\} \) is a non-increasing sequence. For any \( j : \lambda k < j < (k + 1)\alpha \), we have

\[ \Delta \theta(j) \leq \Delta \theta(\lambda k) \leq (\lambda + \epsilon \alpha)^{\Delta \theta(\eta)}. \]

(27)

Thus, by (22), we know that

\[ d_{ij}(T) \leq d_{ij}(0) + v \sum_{s=0}^{T-1} \Delta \theta(s) \]

\[ \leq d_{ij}(0) + v \alpha \Delta \theta(0) \]

\[ \leq d_{ij}(0) + \epsilon \alpha \Delta \theta(0) \]

\[ \leq d_{ij}(0) + \frac{\eta}{2}. \]

(28)

In a similar way,

\[ d_{ij}(0) < d_{ij}(T) + \frac{\eta}{2}. \]

(29)

Therefore, at time \( T \), the variation of neighbors for any agent \( i \) cannot exceed max \( \{R_i(\eta)\} \), which implies that (21) is true for \( t = T \).

**Proof of Theorem 2**

**Proof.** Combining Lemma 5 with Lemma 4, we know that

\[ \Delta \theta(\lambda k) \leq (\lambda + \epsilon \alpha)^{\Delta \theta(\eta)}, \]  
\( \forall k \geq 0. \)  
(30)

Therefore, \( \lim_{k \to \infty} \Delta \theta(\lambda k) = 0. \) Since \( \Delta \theta(t) \) is non-increasing, we have

\[ \lim_{t \to \infty} \Delta \theta(t) = 0. \]

4. DISCUSSIONS ON THE NEEDED NEIGHBORS

In this section, we will discuss how many neighbors are sufficient for the synchronization. From the existing re-
searches on collective behavior, we know that the interaction between agents is crucial. In order to guarantee the synchronization of the group, the agents should communicate with each other constantly. If the group is divided into several small clusters that agents in one cluster do not have communication with agents in other clusters, then each cluster will move on its own. Therefore, the number of neighbors for each agent should be sufficient enough to avoid the existence of isolated cluster.

In a deterministic framework, it would be difficult to specify the number of neighbors in view of the vast varieties of agents’ states. Therefore, we consider the problem in a random framework, where the initial positions of all agents are independently and uniformly distributed in a unit square. For the initial configuration, let \( G(n, m_n) \) be the random graph formed by each agent connecting with its \( m_n \) nearest agents. \( G(n, m_n) \) is directed in view of the asymmetric neighbor relationship. Xue and Kumar (2004) investigated the needed number of neighbors for the connectivity of an undirected random graph \( G^*(n, m_n) \), where the nodes were uniformly and independently placed in a unit square, and an edge \((i, j)\) was formed by either \( j \) was one of the \( m_n \) nearest neighbors of \( i \), or \( i \) was one of the \( m_n \) nearest neighbors of \( j \). Although these two graphs are different, based on a similar analysis, we know that the result for \( G^*(n, m_n) \) is still true for graph \( G(n, m_n) \).

**Theorem 6.** For \( G(n, m_n) \) to be strongly connected asymptotically, \( \Theta(\log n) \) neighbors are necessary and sufficient. Specifically, there are two constants \( 0 < c_1 < c_2 \) such that \( G(n, c_1 \log n) \) is disconnected with probability \( 1 \) while \( G(n, c_2 \log n) \) is connected with probability \( 1 \).

**Proof.** The necessary part is easy to verified, since the underlying graph of \( G(n, m_n) \) obtained by replacing all directed edges with undirected edges is \( G^*(n, m_n) \). Therefore, the unconnectedness of \( G^*(n, c_1 \log n) \) can deduce the unconnectedness of \( G(n, c_1 \log n) \).

For the sufficient part, from the proof of Theorem 1 in Xue and Kumar (2004), we know that for any \( \delta > 0 \), digraph \( G(n, (2/\log(4/\epsilon) + \delta) \log n) \) asymptotically contains a subgraph \( G(n, r_n) \), where \( \pi r_n^2 > (1 + \kappa) \log n/n \) with \( \kappa \in (0, 1) \). Here, \( G(n, r_n) \) is a graph formed by connecting every node to its neighbors that are within distance \( r_n \).

From Theorem 3.2 in Gupta and Kumar (1998) that graph \( G(n, r_n) \) with \( \pi r_n^2 = (\log n + c(n))/n \) is connected with probability one as \( n \to \infty \) if \( c \to \infty \), we know that digraph \( G(n, (2/\log(4/\epsilon) + \delta) \log n) \) has a connected subgraph with high probability.

The above theoretical results only consider the connectivity of the initial neighbor graph. The question naturally arises as to how many neighbors are needed for the synchronization of the dynamical model. Our current theory is inadequate for this task. To investigate this we give a simulation result. In Fig. 4, all agents apply the model (1),(2) with \( v = 0.3 \). The number of neighbors \( m_n \) is equal to \( 5 \log n \), and the number of agents \( n \) adds 400 at each run. Each point represents the maximal difference of headings after 10 steps, from which we can see that all agents almost have the same heading.

Based on the above analysis, we know that for the sake of connectivity the needed number of neighbors should increase as the density increasing. Fig. 4 shows that the difference of headings will be obviously bigger if the number of neighbors is fixed, where \( m_n = 7 \) is fixed while the population adds 400 at each run.

Since each bird needs its own space to avoid collisions, the density of the flocking has an upper bound, i.e. the number of agents in the unit square has an upper bound. Therefore, from a theoretical perspective, six to seven neighbors might be sufficient for the starlings to perform coordinated movements.

5. COMPARISON WITH METRIC INTERACTION

In this section, we show the efficiency of the topological interaction by comparing with the metric interaction. A simple model with metric interaction is obtained by modifying the neighbor set in (2) with \( N^*_i = \{ j | \sqrt{(x_j - x_i)^2 + (y_j - y_i)^2} < r \} \). This metric interaction model is a linearized version of Vicsek’s model (Vicsek et al., 1995), which has been widely studied (Jadbabaie et al., 2003; Tang and Guo, 2007; Cao et al., 2008). In the following simulations, the parameters are fixed: population \( n = 1000 \), number of neighbors \( m = 15 \) for topological case, interaction distance \( r = 0.2 \) for metric case.

Fig. 5 illustrates the robustness of two models against speed variation. At each run, the speed \( v \) is chosen to be the element of vector \( [0.1, 0.5, 1, 10, 20, 30, 40, 50, 100, 150] \) sequentially. The ‘s’ point represents the maximal difference of headings of agents applying the topological interaction, while the ‘+’ point represents the maximal difference
of headings with the metric interaction. From Fig. 5, we can see that the topological interaction is much robust against the variation of speeds.

![Robust to the variation of speeds](image)

Fig. 5. Flocking with different speeds.

Fig. 6 illustrates the robustness of two models to noise. At each run, the intensity of noise is chosen to be the element of vector \([0.01, 0.05, 0.15, 0.3, 0.6, 0.9]\) successively. The ‘-‘ and ‘+‘ points represent the same items as those in Fig. 5. It is obvious that the topological interaction is much more robust to the influence of noise.

![Robust to the influence of noise](image)

Fig. 6. Flocking with different intensities of noise.

6. CONCLUSION

In this paper, we give a theoretical study for multi-agent systems with topological interaction, where each agent updates its heading according to the heading of a fixed number of neighbors. We develop some sufficient conditions for the synchronization of the group, which shows the relationship between the speed, the heading and the density of the group. Furthermore, we discuss the number of neighbors needed for the synchronization, and shows that for the sake of connectivity, the number of neighbors should increase as the density of group increasing. This result does not violate the observational study that flocking of starlings can perform coordinated movements by only interacting with six to seven neighbors, since the flocking density is bounded in view of the space needed by each bird for collision avoidance.

The theoretical result for the number of neighbors needed is only about the initial configuration. It would be more interesting and challenging to investigate the problem when the neighbor graph dynamically varies according to the states of agents.

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